COLLECTED
PAPERS OF
G. H. HARDY

INCLUDING JOINT
PAPERS WITH
J. E. LITTLEWOOD
AND OTHERS

EDITED BY
A COMMITTEE
APPOINTED BY THE
LONDON
MATHEMATICAL
SOCIETY

III

OXFORD

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VOLUME III

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The main object of this publication is to render more accessible the papers of the great mathematician, which in their original form appeared in many journals over a period of almost 60 years. The editors have kept in view a second object also; that of rendering the work useful to mathematicians generally by providing introductions to groups of papers, or comments where appropriate. These editorial additions, while not always systematic or exhaustive, will (it is hoped) assist the reader to view Hardy's papers in proper perspective.

The work will be completed in seven volumes.

Hardy and Littlewood in New Court, Trinity College, Cambridge
Taken about 1924 by Professor H. Cramér

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EDITORIAL NOTE

The work will comprise seven volumes.

For convenience of reference, papers are numbered according to years, e.g. 1912, 4. A complete list of Hardy's papers will be found at the end of this volume (pp. 729–45) and will be reproduced at the end of each volume. This list is based on that compiled by Titchmarsh (*Journal of the London Mathematical Society*, 25 (1950), 89–101).

The date of publication of a paper, where it differs from the year mentioned in the reference number, is given (for the sake of its historical interest) in the contents list of the volume containing the paper.

Where reference is made, in the corrections or comments, to the pages of a paper, the numbers used are those of the original pagination and not the consecutive page numbers of the volume. The joint papers with Ramanujan are reproduced from *The Collected Papers of S. Ramanujan* (Cambridge, 1927), and for these the 'original pagination' relates to that volume and not to the first publication in a journal.

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1. TRIGONOMETRIC SERIES

INTRODUCTION TO PAPERS ON TRIGONOMETRIC SERIES

The modern theory of Fourier series dates from the years 1904–6, which saw the publication of the pioneering work of Lebesgue, Fejér, and Fatou. The theory draws on the subjects of integration, inequalities, and the summability of series, all subjects of great interest to Hardy, and in 1913, 4 he made a substantial contribution to the early theory of the Cesàro summability of Fourier series.

During the first ten years of Hardy's collaboration with Littlewood, the major effort of the two workers was devoted to diophantine approximation, additive number theory, and the summability of series. As their work on the first two subjects came to an end, Hardy and Littlewood turned their attention more and more to the summability of Fourier series, using as tools much of the summability theory which they had already developed. This discussion of the summability of Fourier series led to their work on Fourier constants, fractional integrals, and the maximal theorem, and all these led in turn to further developments of the theory.

This section includes all the papers whose main theme is the subject of Fourier series (as opposed to inequalities and the classes H^p). The papers are divided, somewhat arbitrarily, into five groups, the papers within each group being ordered chronologically. An introduction is given to each group of papers, and further comments are given immediately after the individual papers. We refer to the two volumes of Zygmund's *Trigonometric Series* (2nd edition, Cambridge, 1959) and to the Cambridge Tract *Fourier Series* by Hardy and Rogosinski (Cambridge, 1950) by the abbreviated titles Z I, Z II, and H–R.

For an appreciation of Hardy's contribution to the theory of trigonometric series see A. C. Offord, *J. London Math. Soc.* 25 (1950), 136–8.

W. W. R.
T. M. F.

EDITORIAL NOTE

First drafts of the introductions in Section III.1 were prepared by Professor Rogosinski before his death. These were revised and completed by Professor Flett, who has also contributed the comments on individual papers.

(*a*) Convergence of a Fourier Series or its Conjugate

INTRODUCTION TO PAPERS ON THE CONVERGENCE OF A FOURIER SERIES OR ITS CONJUGATE

The six papers in this group written by Hardy in collaboration with Littlewood reflect their early interest in summability theory, for all six use summability arguments of some kind, although the main results are concerned with convergence.

The four papers 1917, 10, 1926, 8, 1932, 9, and 1934, 3 contain convergence tests of a type introduced by Fatou, in which a 'continuity' condition is combined with an order condition on the nth term of the Fourier series. The idea underlying these tests is that the 'continuity' condition implies the summability, by some method, of the Fourier series, and the order condition is a Tauberian condition for that method.

Thus, using Cesàro summability, it was proved in 1926, 8 that if (for an integrable f) the 'continuity' condition

$$\lim_{t\to 0}\frac{1}{t}\int\limits_0^t \{f(\theta+t)+f(\theta-t)\}\,dt = s$$

is satisfied, and there exists $C > 0$ such that for all n the nth term of the Fourier series of f at θ is greater than $-Cn^{-1}$, then the series is convergent to the sum s. In 1932, 9 and 1934, 3 the order condition was weakened by replacing n^{-1} by $n^{-\delta}$ for some δ satisfying $0 < \delta < 1$. The 'continuity' condition had then to be strengthened, and Valiron's generalization of Borel summability was used in place of the Cesàro method.

The papers 1928, 6 and 1928, 13, again written in collaboration with Littlewood, give sufficient conditions for the convergence and absolute convergence of a Fourier series at a point. The convergence test (1928, 6, Theorem 1) involves a 'continuity' condition combined with the condition that the function concerned belongs to the class $\mathrm{Lip}(1/p, p)$ for some $p \geqslant 1$; these conditions imply Cesàro summability of some negative order, and therefore imply convergence. The paper 1928, 6 contains also a discussion of the classes $\mathrm{Lip}(k, p)$, which was supplemented in 1932, 4.

The other five papers in this group are concerned with convergence proper rather than summability. The joint paper 1943, 3 with Rogosinski provides a counter-example to a natural conjecture concerning the Gibbs phenomenon. Of the remaining papers, 1917, 8 deals with the order of the partial sums of the Fourier series of f when f satisfies some 'continuity' condition, while 1920, 9 was the first systematic attempt to determine the relations between the various known convergence tests for Fourier series.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy*.

[Extracted from *The Messenger of Mathematics*, New Series, No. 550, Vol. xlvi., Feb., 1917].

XLV.

On a point in the theory of Fourier series.

1. In this note a_0, a_1, a_2, ..., b_1, b_2, ... are the Fourier constants of a summable function $f(x)$, and

$$s_n = \tfrac{1}{2}A_0 + \sum_1^n A_\nu = \tfrac{1}{2}a_0 + \sum_1^n (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

is the sum of the first $n+1$ terms of its Fourier series. I proved in 1913* that

$$(1.1) \qquad s_n = o(\log n),$$

and that the series†
$$\sum \frac{A_n}{\log n}$$

is convergent, for almost all values of x, and in particular those which satisfy an important criterion of Lebesgue. This criterion is obtained by writing

$$(1.2) \qquad \phi(\alpha) = f(x+\alpha) + f(x-\alpha) - 2f(x)$$

and by supposing that

$$(1.3) \qquad \Phi(\alpha) = \int_0^\alpha |\phi(t)|\, dt = o(\alpha)$$

when $\alpha \to 0$. It is often expressed by saying that $|\phi(\alpha)|$ *is the differential coefficient of its indefinite integral for* $\alpha = 0$.

2. I shall now prove that the condition (1.3) may be replaced by a slightly more general condition.

Theorem A. *If* $0 < \alpha < \delta$ *and*

$$(2.1) \qquad \Psi(\alpha) = \int_\alpha^\delta |\phi(t)| \frac{dt}{t} = o\left(\log \frac{1}{\alpha}\right)$$

when $\alpha \to 0$, *then* $s_n = o(\log n)$.

* G. H. Hardy, 'On the summability of Fourier's Series', *Proc. London Math. Soc.*, ser. 2, vol. xii. (1913), pp. 365–372.

† The series must obviously be supposed to begin with the term for which $n = 2$.

We have to prove that

$$(2.2) \qquad j_n = \int_0^\delta \phi(t) \frac{\sin nt}{t} dt = o(\log n).$$

Now

$$(2.3) \qquad j_n = \int_0^{1/n} \phi(t) \frac{\sin nt}{t} dt + \int_{1/n}^\delta \phi(t) \frac{\sin nt}{t} dt$$

$$= O\left\{n \int_0^{1/n} |\phi(t)|\, dt\right\} + O\left\{\int_{1/n}^\delta |\phi(t)| \frac{dt}{t}\right\}$$

$$= O\left\{n\, \Phi\left(\frac{1}{n}\right)\right\} + O\left\{\Psi\left(\frac{1}{n}\right)\right\}.$$

But

$$(2.4) \quad \Phi(\alpha) = -\int_0^\alpha t\Psi'(t)\, dt = -\alpha\Psi(\alpha) + \int_0^\alpha \Psi(t)\, dt$$

$$< \int_0^\alpha \Psi(t)\, dt = \int_0^\alpha o\left(\log \frac{1}{t}\right) dt$$

$$= o\left(\alpha \log \frac{1}{\alpha}\right).$$

From (2.1), (2.3), and (2.4) the theorem follows.

3. That (2.1) is more general than (1.3) may be shown as follows. Suppose that (1.3) is satisfied. Then

$$\Psi(\alpha) = \int_\alpha^\delta \frac{\Phi'(t)}{t} dt = \frac{\Phi(\delta)}{\delta} - \frac{\Phi(\alpha)}{\alpha} + \int_\alpha^\delta \frac{\Phi(t)}{t^2} dt$$

$$= O(1) + o(1) + \int_\alpha^\delta o\left(\frac{1}{t}\right) dt$$

$$= o\left(\log \frac{1}{\alpha}\right).$$

Thus (2.1) is satisfied whenever (1.3) is satisfied.

On the other hand it is not difficult to choose $\phi(x)$ so as to satisfy (2.1) and *not* (1.3). Suppose for example that

$$\phi(\alpha) = 1 \quad \left(\frac{1}{p!} \leq \alpha \leq \frac{2}{p!},\ p = 1, 2, 3, \ldots\right),$$

and that $\phi(\alpha)$ is otherwise zero. If

$$\frac{2}{(q+1)!} < \alpha < \frac{1}{q!},$$

we have

$$\Psi(\alpha) = \sum_{1}^{q} \int_{1/p!}^{2/p!} \frac{dt}{t} = q \log 2,$$

$$\log \frac{1}{\alpha} > \log q! \sim q \log q,$$

$$q = O\left\{\left(\log \frac{1}{\alpha}\right) \Big/ \left(\log \log \frac{1}{\alpha}\right)\right\} = o\left(\log \frac{1}{\alpha}\right),$$

$$\Psi(\alpha) = o\left(\log \frac{1}{\alpha}\right);$$

while if

$$\frac{1}{(q+1)!} < x < \frac{2}{(q+1)!}$$

we have

$$\Psi(x) < (q+1)\log 2 = o\left(\log \frac{1}{\alpha}\right).$$

Thus (2.1) is satisfied in either case. But (1.3) is not satisfied, for, taking

$$\alpha = \frac{1}{(q+1)!},$$

we have

$$\Phi(\alpha) = \sum_{q+1}^{\infty} \frac{1}{p!} = \frac{1}{(q+1)!}\left(1 + \frac{1}{q+2} + \dots\right)$$

$$\sim \frac{1}{(q+1)!} = \alpha.$$

4. The condition (2.1) is thus a genuine generalisation of (1.3). The interest of Theorem A lies, however, less in its superior generality than in its relations to other theorems concerning Fourier series, and in particular to Dini's classical theorem concerning their convergence. These relations are put in evidence by the following intermediate theorem.

THEOREM B. *Suppose that $\chi(x)$ is a function of x which tends steadily to infinity with x, and that*

(4.1) $$\chi(x) = o(\log x).$$

(4.2) $$\Psi(\alpha) = o\left\{\frac{\log(1/\alpha)}{\chi(1/\alpha)}\right\}.$$

Then

(4.3) $$s_n = o\left\{\frac{\log n}{\chi(n)}\right\}.$$

There is a corresponding result in which o is replaced by O in (4.2) *and* (4.3).

As in § 2, we have

$$(4.4)\qquad j_n = O\left\{n\Phi\left(\frac{1}{n}\right)\right\} + O\left\{\Psi\left(\frac{1}{n}\right)\right\},$$

and

$$(4.5)\qquad \Phi(\alpha) < \int_0^\alpha \Psi(t)\,dt$$

$$= o\int_0^\alpha \frac{\log(1/t)}{\chi(1/t)}\,dt = o\left\{\frac{1}{\chi(1/\alpha)}\int_0^\alpha \log\left(\frac{1}{t}\right)dt\right\}$$

$$= o\left\{\frac{\alpha\log(1/\alpha)}{\chi(1/\alpha)}\right\}.$$

From (4.2), (4.4), and (4.5) the theorem follows.

Theorem A and Dini's theorem correspond to the two extreme cases, just excluded by the conditions of Theorem B, the cases in which $\chi(x) = 1$ and $\chi(x) = \log x$. In the latter case (4.2) and (4.3) must be replaced by

$$(4.6)\qquad \Psi(\alpha) = H + o(1)$$

and

$$(4.7)\qquad s_n = s + o(1)$$

respectively. The superiority of Theorem A over my original result lies in the fact that it is, so to say, a member of a continuous chain of theorems connecting it with Dini's theorem. We may say, roughly, that *the order of* s_n *is at most equal to that of*

$$\Psi\left(\frac{1}{n}\right) = \int_{1/n}^{\delta} \frac{|\phi(t)|}{t}\,dt.$$

5. Suppose that $\chi(x)$ satisfies the further condition

$$\Delta\frac{n\chi(n)}{\log n} = O\frac{\chi(n)}{\log n}.$$

This condition is certainly satisfied, for example, if

$$\chi(x) = (\log x)^\alpha(\log\log x)^\beta \ldots \quad (0 < \alpha < 1).$$

Then it is easily proved, by the argument which I used in my paper already referred to, that *the series*

$$\Sigma\frac{\chi(n)}{\log n}A_n.$$

is convergent.

ANALYSE MATHÉMATIQUE. — *Sur la convergence des séries de Fourier et des séries de Taylor.* Note de MM. G.-H. HARDY et J.-E. LITTLEWOOD, présentée par M. Hadamard.

1. M. Fatou a considéré dans sa Thèse [1] les séries trigonométriques

$$\tfrac{1}{2}A_0 + \sum A_n = \tfrac{1}{2}a_0 + \sum (a_n \cos n\theta + b_n \sin n\theta) \tag{1}$$

satisfaisant aux conditions

$$\lim n a_n = 0, \qquad \lim n b_n = 0; \tag{2}$$

et il a donné une condition nécessaire et suffisante pour qu'une telle série, qui est évidemment la série de Fourier d'une fonction sommable $f(\theta)$, soit convergente. Nous allons généraliser le résultat de M. Fatou en remplaçant les conditions (2) par les conditions plus générales que $|na_n|$ et $|nb_n|$ soient bornées.

On a, en effet, le théorème suivant:

THÉORÈME I.—*Pour que la série*

$$a_0 + \sum (a_n \cos n\theta + b_n \sin n\theta) \sim f(\theta),$$

où

$$|na_n| < 1, \qquad |nb_n| < 1, \tag{3}$$

soit convergente pour une valeur donnée de θ, *il faut et il suffit que*

$$\Phi(\alpha) = \frac{1}{2\alpha} \int_{\theta-\alpha}^{\theta+\alpha} f(t)\, dt \tag{4}$$

tende vers une limite déterminée quand α *tend vers zéro.*

a. La condition est suffisante. C'est une conséquence presque immédiate des théorèmes connus concernant la sommabilité des séries divergentes. D'après un théorème de M. Lebesgue [2], l'existence de la limite (4) entraîne la sommabilité de la série $\sum A_n$ par les moyennes de Cesàro d'ordre 2.

---

[1] *Acta mathematica*, t. 30.
[2] *Mathematische Annalen*, t. 61.

1917, 10 (with J. E. Littlewood) *Comptes Rendus*, 165, 1047–9.

Mais, $|n\mathrm{A}_n|$ étant borné, la sommabilité de la série entraîne sa convergence (1). On pourrait l'établir autrement en démontrant (au moyen de l'intégrale de Poisson, comme le fait M. Fatou) l'existence de la limite

$$\lim_{r=1} \sum \mathrm{A}_n r^n,$$

et en faisant application d'un théorème de Littlewood (2).

b. La condition est nécessaire. La démonstration se fonde sur le lemme suivant: *La série* $\sum \mathrm{A}_n$ *étant convergente, on a*

$$\lim_{m=\infty} m^{-p} \sum_1^m n^p \mathrm{A}_n = 0,$$

UNIFORMÉMENT POUR TOUTES LES VALEURS ENTIÈRES POSITIVES DE p. Cela étant, nous posons

$$m = k\left[\frac{1}{\alpha}\right] \qquad (k > 1)$$

(d'où

$$\tfrac{1}{2}k < m\alpha \leqq k,$$

pour les valeurs suffisamment petites de α), et nous supposons, pour simplifier l'écriture, que $a_0 = 0$. Alors on a

$$\Phi(\alpha) = \sum_1^\infty \mathrm{A}_n \frac{\sin n\alpha}{n\alpha} = \sum_1^m + \sum_{m+1}^\infty = \Phi_1 + \Phi_2;$$

$$|\Phi_2| < \frac{1}{\alpha} \sum_{m+1}^\infty \frac{1}{n^2} < \frac{1}{m\alpha} < \frac{2}{k}$$

et

$$\Phi_1 = \sum_1^m \mathrm{A}_n \sum_0^\infty \frac{(-1)^p (n\alpha)^{2p}}{(2p+1)!} = \sum_0^\infty \frac{(-1)^p \alpha^{2p}}{(2p+1)!} \sum_1^m n^{2p} \mathrm{A}_n,$$

d'où

$$\left|\Phi_1 - \sum_1^m \mathrm{A}_n\right| < \epsilon \sum_1^\infty \frac{(m\alpha)^{2p}}{(2p+1)!} < \epsilon\, e^{m\alpha} \leqq \epsilon\, e^k$$

(1) HARDY, *Proc. London Math. Soc.*, t. 8.
(2) *Proc. London Math. Soc.*, t. 9.

pour $m > m_0(\epsilon)$. Donc
$$\left|\Phi - \sum A_n\right| < \frac{2}{k} + \epsilon e^k + \epsilon$$
pour $m > m_0(\epsilon)$. Le nombre positif δ étant donné, on peut prendre
$$k = \frac{4}{\delta}, \qquad \epsilon = \frac{\delta}{3} e^{-k};$$
et l'on en tire
$$|\Phi - \sum A_n| < \delta$$
pour $m > m_0(\delta)$, c'est-à-dire pour $0 < \alpha < \alpha_0(\delta)$; ce qui achève la démonstration du théorème.

2. Dans le même ordre d'idées, nous avons démontré un théorème analogue concernant les séries de Taylor.

THÉORÈME II.—*Soit* $|na_n| < 1$. *Alors, pour que la série* $\sum a_n$ *soit convergente, il faut et il suffit que*
$$\Phi(x) = \frac{1}{1-x} \sum \frac{a_n}{n+1} (1 - x^{n+1}) \quad (|x| < 1)$$
tende vers une limite déterminée quand x *tend vers un suivant un chemin simple continu quelconque* C.

C'est une généralisation des théorèmes 49 et 50 de notre Mémoire: *Contributions to the arithmetic theory of series* (1). Là, nous avons supposé que na_n tend vers zéro.

Nous dirons que le chemin C est *régulier* s'il s'approche du point $x = 1$ dans une direction déterminée (qui peut d'ailleurs être tangente au cercle $|x| = 1$). Cela étant, il suffit, en particulier, pour la convergence de la série $\sum a_n$, que $f(x) = \sum a_n x^n$ tende vers une limite déterminée quand x tend vers un suivant un chemin régulier quelconque C. Mais cette condition n'est pas nécessaire. Si l'on pose
$$a_n = \frac{1}{n \log n} \sin \frac{n\pi}{r} \quad (n_r = e^{e^r},\ n_r < n < n_{r+1}),$$
la série $\sum a_n$ est convergente. Mais $f(x)$ ne tend vers aucune limite déterminée quand C a un ordre de contact assez élevé avec le cercle.

(1) *Proc. London Math. Soc.*, t. 2.

COMMENTS

A simpler proof of the theorem of this paper is given in 1920, 7, pp. 228–32. A further simplification of the proof of necessity, which was suggested by M. Riesz, is given in 1924, 3. The theorem is contained in Theorem 1 of 1926, 8.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

[Extracted from the *Messenger of Mathematics*, No. 586, Vol. xlix., Feb. 1920.]

LIII.

On certain criteria for the convergence of the Fourier series of a continuous function.

1. In this note I propose to examine the logical relations of certain modern criteria for the truth of the equation

$$(1) \qquad \lim_{n\to\infty} \frac{2}{\pi}\int_0^\delta \frac{\sin nx}{x} f(x)\,dx = f(0),$$

where $f(x)$ is a function integrable in the sense of Lebesgue and continuous for $x=0$. It is well known that any test for the convergence of the Fourier series of a continuous* function may be reduced to a set of sufficient conditions for the truth of (1).

I suppose, as we may do without loss of generality, that $f(0)=0$. This being so, there are five tests in question, two 'classical' and three modern. The two classical tests are

* More generally, of a function all of whose discontinuities are regular, *i.e.* such that $\phi(x-0)+\phi(x+0)=2\phi(x)$.

1920, 9 *Messenger of Mathematics*, 49, 149–55.

(D) *Dini's test: the integral*

$$\int_0^\delta \frac{|f(x)|}{x}\,dx$$

is convergent:

(J) *Jordan's test:* $f(x)$ *is of bounded variation.*

The three modern tests are

(V) *de la Vallée-Poussin's test*: the mean value*

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

is of bounded variation:

(Y) *Young's test†:* $\int_0^x |d\{tf(t)\}| = O(x)$:

(L) *Lebesgue's test‡:* $\int_x^\delta \frac{|f(t+x)-f(t)|}{t}\,dt = o(1)$.

The relations between these tests are of considerable interest, and have not all been stated explicitly. I proceed to consider them more closely.

2. I. *Neither* (D) *nor* (J) *includes the other.*

This is well known. Thus

$$f(x) = \left(\log\frac{1}{x}\right)^{-1}$$

satisfies (J), but not (D); and

$$f(x) = x^\rho \sin\frac{1}{x} \quad (0 < \rho \leq 1) \tag{2}$$

satisfies (D), but not (J).

II. (V) *includes both* (D) *and* (J).

This is proved by de la Vallée-Poussin. For the sake of completeness, and a slight simplification, I repeat the proofs.

* Ch. J. de la Vallée-Poussin, 'Un nouveau cas de convergence des séries de Fourier', *Rendiconti del Circ. Mat. di Palermo*, vol. xxxi. (1911), pp. 296–299. See also his *Cours d'Analyse*, vol. ii., ed. 2, p. 149.

† W. H. Young, 'Sur la convergence des séries de Fourier', *Comptes Rendus*, 21 Aug. 1916.

‡ H. Lebesgue, 'Recherches sur la convergence des séries de Fourier', *Math. Annalen*, vol. lxi. (1905), pp. 251–280. See also his *Leçons sur les séries trigonométriques*, pp. 59 and 64.

(*a*) If a function is monotonic, its mean value is also monotonic. Hence, if $f(x)$ is of bounded variation, so also is $F(x)$. Thus (V) includes (J).

(*b*) Suppose (D) satisfied. We have

$$(3) \qquad |F'| = \left|\frac{f}{x} - \frac{1}{x^2}\int_0^x f\,dt\right| \leq \frac{|f|}{x} + \frac{1}{x^2}\int_0^x |f|\,dt$$

for $x > 0$, at any rate with the exception of a set of values of x of measure zero. The first term on the right-hand side is integrable in $(0, \delta)$. Also

$$\int_\epsilon^\delta \frac{dx}{x^2}\int_0^x |f|\,dt = \frac{1}{\epsilon}\int_0^\epsilon |f|\,dt - \frac{1}{\delta}\int_0^\delta |f|\,dt + \int_\epsilon^\delta \frac{|f|}{x}\,dx$$

tends to a limit when $\epsilon \to 0$. Hence the second term on the right-hand side of (3) is also integrable in $(0, \delta)$. Therefore $|F'|$ is integrable in $(0, \delta)$; and F is of bounded variation.

III. (Y) *includes* (J).

For
$$\int_0^x |d(tf)| \leq \int_0^x |f|\,dt + \int_0^x t\,|df|$$
$$= o(x) + O\left(x\int_0^x |df|\right) = O(x)$$

if (J) is satisfied.

IV. (Y) *does not include* (D).

This may be shown by means of the function (2). Here we have

$$d(xf) = \left\{(\rho+1)\,x^\rho \sin\frac{1}{x} - x^{\rho-1}\cos\frac{1}{x}\right\} dx;$$

and (Y) demands that

$$\int_0^x |t|^{\rho-1}\left|\cos\frac{1}{t}\right| dt = O(x).$$

This is only true if $\rho \geq 1$, whereas (D) is satisfied whenever $\rho > 0$.

A fortiori it follows that (Y) *does not include* (V).

3. V. (V) *does not include* (Y).

Write $\log(1/x) = lx$, and consider the function

$$f(x) = \frac{\sin lx}{lx}.$$

Here (Y) demands that

$$\int_0^x \left| \frac{\sin lt}{lt} - \frac{\cos lt}{lt} + \frac{\sin lt}{(lt)^2} \right| dt = O(x),$$

which is plainly true. On the other hand it is easily verified that

$$F = \frac{1}{x} \int_0^x \frac{\sin lt}{lt}\, dt = \frac{\cos lx + \sin lx}{2lx} + O\left\{\frac{1}{(lx)^2}\right\},$$

$$F' = \frac{f - F}{x} = \frac{\sin lx - \cos lx}{2x\, lx} + O\left\{\frac{1}{x\,(lx)^2}\right\};$$

so that

$$\int_0^\delta |F'|\, dx$$

is divergent. Thus (V) is not satisfied.

4. VI. (L) *includes* (V).

This is stated by de la Vallée-Poussin*, on the ground of a communication of Lebesgue; but no proof seems to have been printed.

It is useful to observe first that (L) may be stated also in the form†

$$\text{(L')} \qquad \int_x^\delta |\psi(t+x) - \psi(t)|\, dt = o(1),$$

where $\psi = f/x$. To see this, we observe that the difference of the integrals which occur in (L) and (L') does not exceed

$$\int_\eta^\delta \frac{|f(t+x) - f(t)|}{t}\, dt + \int_\eta^\delta |\psi(t+x) - \psi(t)|\, dt + \int_x^\eta \frac{|f(t+x)|}{t(t+x)}\, dt,$$

where $x < \eta < \delta$. We can choose η so that the last integral is less than

$$\epsilon \int_x^\eta \frac{dt}{t(t+x)} < \epsilon \log 2.$$

When η is fixed, the first two integrals tend to zero, by a fundamental theorem in the theory of Lebesgue.‡ The proposition is thus established.

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* *Cours d'Analyse*, vol. ii., ed. 2, p. 150.

† This is remarked by Lebesgue, *loc. cit.*

‡ If ϕ is integrable in (a, b), then

$$\int_a^b |\phi(t+x) - \phi(t)|\, dt$$

tends to zero with x. See for example Lebesgue, *Leçons sur les séries trigonométriques*, p. 15.

Suppose now that (V) is satisfied. Then $f=F+xF'$ for almost all values of x, and (L′) will certainly be satisfied if

$$(4)\qquad \int_x^\delta |F'(t+x)-F'(t)|\,dt=o(1)$$

and

$$(5)\qquad \int_x^\delta \left|\frac{F(t+x)}{t+x}-\frac{F(t)}{t}\right| dt=o(1).$$

Of these equations, (4) is satisfied in virtue of the integrability of $F'(x)$, and (5) may (by the same argument that was used above*) be replaced by

$$(6)\qquad \int_x^\delta \frac{|F(t+x)-F(t)|}{t}\,dt=o(1).$$

We write

$$\int_x^\delta \frac{|F(t+x)-F(t)|}{t}\,dt=\int_x^\eta+\int_\eta^\delta=J_1+J_2.$$

Then

$$J_1=\int_x^\eta \frac{dt}{t}\left|\int_t^{t+x}F'(u)\,du\right| \leq \int_x^\eta \frac{dt}{t}\int_t^{t+x}|F'(u)|\,du$$

$$=\left[\log t\int_t^{t+x}|F'(u)|\,du\right]_x^\eta-\int_x^\eta \log t\,\{|F'(t+x)|-|F'(t)|\}\,dt$$

$$=\log\eta\int_\eta^{\eta+x}|F'|\,du-\log x\int_x^{2x}|F'|\,du-\int_{2x}^{\eta+x}\log(u-x)\,|F'|\,du$$

$$+\int_x^\eta \log u\,|F'|\,du$$

$$=\int_x^{2x}\log\left(\frac{u}{x}\right)|F'|\,du+\int_{2x}^\eta \log\left(\frac{u}{u-x}\right)|F'|\,du$$

$$+\int_\eta^{\eta+x}\log\left(\frac{\eta}{u-x}\right)|F'|\,du.$$

In each of these integrals the logarithmic factor is bounded. Hence J_1 is less than a constant multiple of

$$\int_x^{\eta+x}|F'|\,du,$$

and may therefore be made less than ϵ by choice of η. Finally, when η is fixed, J_2 tends to zero with x. Thus (6), and therefore (L′), is satisfied.

* Note that the continuity of f for $x=0$ involves that of F.

5. VII. (L) *includes* (Y).

Suppose (Y) satisfied, and write $xf=g$; and denote by $V(g)$ the variation of g in $(0, x)$. Then

$$g=o(x), \quad V(g)=O(x);$$

and we can write $$g=g_1-g_2,$$

where g_1 and g_2 are steadily increasing functions of the form $O(x)$. And

$$(7) \quad \int_x^\delta \frac{|f(t+x)-f(t)|}{t}\,dt \leq \int_x^{mx} \frac{|f(t+x)-f(t)|}{t}\,dt$$

$$+\int_{mx}^\delta \left|\frac{g_1(t+x)}{t+x}-\frac{g_1(t)}{t}\right|\frac{dt}{t}+\int_{mx}^\delta \left|\frac{g_2(t+x)}{t+x}-\frac{g_2(t)}{t}\right|\frac{dt}{t}$$

$$=J+J_1+J_2,$$

say. In the first place we have

$$(8) \quad J\leq \int_x^{mx} |f(t+x)|\frac{dt}{t}+\int_x^{mx} |f(t)|\frac{dt}{t}\leq \mu\log m,$$

where μ is the upper bound of $|f|$ in the interval

$$0\leq t\leq (m+1)x.$$

Next, we have

$$(9) \quad |J_1|\leq \int_{mx}^\delta \frac{|g_1(t+x)-g_1(t)|}{t(t+x)}\,dt+x\int_{mx}^\delta \frac{|g_1(t)|}{t^2(t+x)}\,dt.$$

The second term is

$$(10) \quad O\left\{x\int_{mx}^\delta \frac{dt}{t(t+x)}\right\}=O\left(\log\frac{m+1}{m}\right)=O(\epsilon_m),$$

where ϵ_m is a function of m only which tends to zero when $m\to\infty$. The first term, since g_1 is monotone, is

$$\int_{mx}^\delta \frac{g_1(t+x)-g_1(t)}{t(t+x)}\,dt=\int_{(m+1)x}^{\delta+x} \frac{g_1(u)}{(u-x)u}\,du-\int_m^\delta \frac{g_1(u)}{u(u+x)}\,du.$$

In the first of these integrals we may replace $u-x$ by $u+x$; for the error thus introduced is not greater than

$$2x\int_{(m+1)x}^{\delta+x} \frac{g_1(u)}{(u^2-x^2)u}\,du=O\left\{x\int_{(m+1)x}^{\delta+x}\frac{du}{u^2}\right\}$$

$$=O\left\{\frac{x}{(m+1)x}\right\}=O(\epsilon_m),$$

which may be absorbed into (10). When we make this simplification, we are left with

$$\left(\int_{(m+1)x}^{\delta+x} - \int_{mx}^{\delta}\right) \frac{g_1(u)}{u(u+x)}\, du = \left(\int_{\delta}^{\delta+x} - \int_{mx}^{(m+1)x}\right) \frac{g_1(u)}{u(u+x)}\, du$$

$$= O\left(\int_{\delta}^{\delta+x} \frac{du}{u+x}\right) + O\left(\int_{mx}^{(m+1)x} \frac{du}{u+x}\right)$$

$$= O\left(\log \frac{\delta+2x}{\delta+x}\right) + O\left(\log \frac{m+2}{m+1}\right) = O(\epsilon_x) + O(\epsilon_m),$$

where ϵ_x is a positive function of x alone which tends to zero with x. Combining this result with (9) and (10), we obtain

$$J_1 \leq O(\epsilon_x) + O(\epsilon_m). \tag{11}$$

Plainly J_2 can be treated in the same manner; and so we have, from (7), (8), and (11),

$$\int_x^{\delta} \frac{|f(t+x) - f(t)|}{t}\, dt \leq \mu \log m + O(\epsilon_x) + O(\epsilon_m).$$

The right-hand side may be made as small as we please by choice first of m and then of x; which proves the theorem.

6. Our general conclusion is thus that (L) includes all the remaining tests. The peculiar interest of (V) lies in the fact that so direct a generalisation of (J) should include (D), a test on the face of it of an absolutely different character.

Some of these tests may be extended in such a manner as to apply to functions discontinuous at $x = 0$, the condition

$$f(x) - f(0) = o(1)$$

being replaced by

$$\int_0^x |f(t) - f(0)|\, dt = o(x).$$

But the logical interdependence of the tests, which is all that I am concerned with at the moment, appears most clearly in the simplest case.

COMMENTS

A more extensive discussion of the relations between the known convergence tests for Fourier series, which incorporates the work of Hardy in this paper, was given by J. J. Gergen, *Quart. J. of Math.* 1 (1930), 252–75.

TWO THEOREMS CONCERNING FOURIER SERIES

G. H. HARDY *and* J. E. LITTLEWOOD.*

1. The theorems which we prove here are developments of results contained in our papers **1–5**.†

It was proved by Fatou‡ that, if

$$\tfrac{1}{2}a_0+\Sigma(a_n\cos nt+b_n\sin nt) = \tfrac{1}{2}A_0+\Sigma A_n\text{§} \tag{1.1}$$

is the Fourier series of an integrable function $f(t)$, and

$$a_n = o\left(\frac{1}{n}\right), \quad b_n = o\left(\frac{1}{n}\right), \tag{1.2}$$

then the necessary and sufficient condition that the series should converge, when $t = x$, to the sum s, is that

$$\frac{1}{t}\int_0^t \phi(u)\,du, \tag{1.3}$$

where

$$\phi(u) = \tfrac{1}{2}\{f(x+u)+f(x-u)-2s\}, \tag{1.4}$$

* Received 16 November, 1925; read 10 December, 1925.

† See the bibliographical note at the end.

‡ Fatou, 1, 345–7, 385–7.

§ Σ denotes always a sum from 1 to infinity, unless the contrary is expressly stated.

should tend to zero when $t \to 0$. In our papers **1**, **2*** we proved that the conclusion is valid under the more general conditions

$$a_n = O\left(\frac{1}{n}\right), \quad b_n = O\left(\frac{1}{n}\right). \tag{1.5}$$

Our original proof, however, was unnecessarily elaborate, and in our note **3** we indicated a considerable simplification of the argument, suggested to us by Dr. Marcel Riesz.

Our objects here are (1) to generalise this theorem still further, replacing the conditions (1.5) by the still more general condition

$$nA_n > -C, \tag{1.6}$$

where C is a positive constant, and (2) to establish the corresponding theorem for the series conjugate to (1.1). In each case a part of the proof is most naturally presented as an inference from the general theorems of our papers **4**, **5**; but it is not really necessary to appeal to these theorems in their full generality, and we show shortly how what is actually required may be proved independently.

2. We shall require two lemmas.

Lemma α.—*If* (i) Σa_n *is any series which converges to zero and* (ii) $na_n > -C$, *then*

$$\sigma_n = \sum_1^n \nu\,|a_\nu| < Cn + o(n) < Bn, \tag{2.1}$$

$$\tau_n = \sum_n^\infty \frac{|a_\nu|}{\nu} < \frac{B}{n}. \tag{2.2}$$

Here, and in the sequel, B is a positive number depending only on the sequence (a_n) or the function ϕ. In particular it is independent of n and of the numbers h and k below. But, subject to this condition, its value will in general vary from one occurrence to another.

Suppose that Σ' and Σ'' indicate summations over those values of ν, in the range $(1, n)$, for which $a_\nu > 0$ and $a_\nu \leqslant 0$ respectively. Then

$$\Sigma' \nu a_\nu + \Sigma'' \nu a_\nu = \sum_1^n \nu a_\nu = o(n),$$

$$0 \leqslant -\Sigma'' \nu a_\nu \leqslant Cn,$$

* See, in particular, Hardy and Littlewood, **2**, 228–232.

and so $$\sigma_n = \Sigma' \nu a_\nu - \Sigma'' \nu a_\nu < Cn + o(n) < Bn.$$

Also $$\tau_{n,N} = \sum_n^N \frac{\sigma_\nu - \sigma_{\nu-1}}{\nu^2} = \sum_n^{N-1} \sigma_\nu \Delta \frac{1}{\nu^2} - \frac{\sigma_{n-1}}{n^2} + \frac{\sigma_N}{N^2}$$

plainly tends to a limit when $N \to \infty$, and

$$\tau_n = \lim_{N\to\infty} \tau_{n,N} \leqslant \sum_n^\infty \sigma_\nu \Delta \frac{1}{\nu^2} < B \sum_n^\infty \frac{1}{\nu^2} < \frac{B}{n}.$$

LEMMA β.—*If h and k are independent of n, Σa_n is convergent, and $\chi(t)$ is either of the functions*

$$\frac{\sin t}{t}, \quad \int_t^\infty \frac{\sin u}{u}\, du, \tag{2. 3}$$

then

$$\sum_{h/t}^{k/t} n^p a_n \chi(nt) = o(t^{-p}) \tag{2. 4}$$

when $t \to 0$, for any real value of p.

The factor $\chi(nt)$ is bounded, and the range of summation may be divided into B ranges in each of which it is monotonic. If M, N is such a range, we have

$$\begin{aligned}\left|\sum_M^N n^p a_n \chi(nt)\right| &\leqslant B \operatorname*{Max}_{M \leqslant \mu < \nu \leqslant N} \left|\sum_\mu^\nu n^p a_n\right| \\ &\leqslant B N^p \operatorname*{Max}_{M \leqslant \mu < \nu \leqslant N} \left|\sum_\mu^\nu a_n\right| = o(N^p) = o(t^{-p}),\end{aligned}$$

which proves the lemma.

3. THEOREM 1.—*If the Fourier series* (1. 1) *satisfies the condition* (1. 6) *when $t = x$, then the necessary and sufficient condition that it should converge to s, for $t = x$, is that*

$$\frac{1}{t}\int_0^t \phi(u)\, du \to 0. \tag{3. 1}$$

The series $\frac{1}{2}A_0 + \Sigma A_n \cos nt$, which reduces to (1. 1) when $t = 0$, is the Fourier series of $\phi(t)$. And we may plainly suppose, without real loss

of generality, that $A_0 = 0$ and $s = 0$.* It is therefore only necessary to prove

THEOREM 1 *a.—If $\phi(t)$ is even, periodic, and integrable, and*

(3. 2) $$\phi(t) \sim \Sigma a_n \cos nt,$$

where

(3. 3) $$na_n > -C,$$

then (3.1) *is the necessary and sufficient condition that Σa_n should converge to zero.*†

(i) *The condition is necessary.* We have

(3. 4) $$\frac{1}{t}\int_0^t \phi(u)\,du = \Sigma a_n \frac{\sin nt}{nt} = \Sigma a_n \chi(nt) = S$$
$$= \sum_1^{h/t} + \sum_{h/t}^{k/t} + \sum_{k/t}^{\infty} = S_1 + S_2 + S_3,$$

say.‡ Also

$$0 \leqslant 1-\chi(t) < Bt, \quad |\chi(t)| < \frac{B}{t} \quad (t > 0).$$

Hence

(3. 5) $$|S_1| \leqslant \left|\sum_1^{h/t} a_n\right| + \left|\sum_1^{h/t} a_n\{1-\chi(nt)\}\right|$$
$$< o(1) + Bt\sum_1^{h/t} n|a_n| < Bh + o(1),$$

by Lemma α (2.1); and

(3. 6) $$|S_3| \leqslant \frac{B}{t}\sum_{k/t}^{\infty} \frac{|a_n|}{n} < \frac{B}{k},$$

by Lemma α (2. 2). From (3. 4), (3. 5), and (3. 6), it follows that

$$|S| < Bh + \frac{B}{k} + o(1) + |S_2| = Bh + \frac{B}{k} + o(1),$$

by Lemma β. We choose h and k so that the sum of the first two terms is less than $\frac{1}{2}\epsilon$, and then t_0 so that the third is numerically less than ϵ for $0 < t \leqslant t_0$. Then $|S| < \epsilon$ for $0 < t \leqslant t_0$, which proves (3. 1).

* Adding an appropriate $\gamma + \delta \cos t$ to $f(t)$.

† We now write a_n for A_n.

‡ If h/t or k/t is an integer, we suppose the corresponding term of S included in the earlier of the two possible sums.

(ii) *The condition is sufficient.* For (3. 1) is a sufficient condition for the summability of the series,* and therefore, when (3. 3) is satisfied, for its convergence.

4. We consider now the series

$$\Sigma\,(b_n \cos nt - a_n \sin nt) = \Sigma\, B_n \tag{4. 1}$$

allied or conjugate to (1. 1). This series is not necessarily a Fourier series. But the series

$$\Sigma\, B_n \sin nt \tag{4. 2}$$

is the Fourier series of the function

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\}. \tag{4. 3}$$

THEOREM 2.—*If the allied series* (4. 1) *satisfies the condition*

$$nB_n > -C \tag{4. 4}$$

when $t = x$, *then the necessary and sufficient condition that it should converge, for* $t = x$, *to the sum* s, *is that the integral*

$$\frac{2}{\pi}\int_0^\infty \frac{\psi(t)}{t}\,dt \tag{4. 5}$$

should be convergent, as a Cauchy integral,† and have the value s.

The condition (4. 4) is certainly satisfied, for every x, if a_n and b_n satisfy (1. 5).

We may suppose $s = 0$, and it is enough to prove

THEOREM 2 *a*.—*If* $\psi(t)$ *is odd, periodic, and integrable, and*

$$\psi(t) \sim \Sigma\, a_n \sin nt,$$

where $na_n > -C$, *then the necessary and sufficient condition that* $\Sigma\, a_n$ *should converge to zero is that* (4. 5) *should converge to zero.*‡

The proof of the necessity of the condition is much like the corresponding part of the proof of Theorem 1 *a*. Taking for $\chi(t)$ the second of the two forms of Lemma β, we have

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* Hardy and Littlewood, **4**, 70 (Theorem C). It is not necessary, however, to use this rather difficult theorem in its general form. The theorem of Lebesgue (**1**, 278), that (3.1) is a sufficient condition for summability (C, 2), is enough for the purpose.

† That is to say as $\lim\limits_{\epsilon \to 0} \int_\epsilon^\infty$.

‡ We now write a_n for B_n.

$$0 \leqslant \frac{\pi}{2} - \chi(t) < Bt, \quad |\chi(t)| < \frac{B}{t}$$

for small and large values of t respectively. We write now

$$\int_t^\infty \frac{\psi(u)}{u}\, du = \Sigma a_n \int_t^\infty \frac{\sin nu}{u}\, du = \Sigma a_n \chi(nt) = S,*$$

and divide S into sums S_1, S_2, S_3 as before. Since Lemma β is valid for both forms of $\chi(t)$, our former proof is in essentials unchanged.

The sufficiency of the criterion is a corollary from the main theorem† of our paper **5**, since the existence of (4. 5) ensures the summability, and so, by (3. 3), the convergence of the series. It is, however, not necessary to appeal to this difficult theorem in its general form. It is sufficient to know that the existence of (4.5) involves the summability $(C, 2)$ of the series. For this, it is sufficient that Σb_n, where

$$b_n = \sum_n^\infty \frac{a_\nu}{\nu},$$

should be summable $(C, 1)$‡; and for this that the function

$$\psi_1(t) = \tfrac{1}{2} \cot \tfrac{1}{2}t \int_0^t \psi(u)\, du$$

should tend to zero when $t \to 0$, or that

$$\chi_1(t) = \frac{1}{t} \int_0^t \psi(u)\, du$$

should tend to zero.§ If now we write

$$\omega(t) = \int_t^\infty \frac{\psi(u)}{u}\, du,$$

so that $\omega(t) \to 0$ when $t \to 0$, we have

$$\int_\epsilon^t \psi(u)\, du = -\int_\epsilon^t u\omega'(u)\, du = -t\omega(t) + \epsilon\omega(\epsilon) + \int_\epsilon^t \omega(u)\, du,$$

$$\frac{1}{t} \int_0^t \psi(u)\, du = -\omega(t) + \frac{1}{t} \int_0^t \omega(u)\, du,$$

which tends to zero.

* See **5**, 221, f.n. †, for the term-by-term integration.
† Theorem 3, 219.
‡ See **4**, 75 (Theorem A1).
§ See **5**, 224–225.

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A convergence criterion for Fourier series.

Von

G. H. Hardy in Oxford und J. E. Littlewood in Cambridge.

1. Our first object in this note is to establish the criterion for the convergence of Fourier series, in the classical sense, which is embodied in Theorem 1 below, and which we stated without proof in our memoir 7.

We suppose throughout that $f(\theta)$ is periodic and integrable, that $0 < \alpha \leqq 1$, and that $p \geqq 1$. We denote by

$$\Delta = \Delta f = \Delta_h f \tag{1.1}$$

one of the three differences

$$f(\theta) - f(\theta - h), \quad f(\theta + h) - f(\theta), \quad f(\theta + h) - f(\theta - h),$$

where $h > 0$. We say that $f(\theta)$ belongs to the Lipschitz class Lip (α, p) if

$$\left(\int_{-\pi}^{\pi} |\Delta f|^p \, d\theta\right)^{\frac{1}{p}} = O(h^\alpha) \tag{1.2}$$

when $h \to 0$, and to the class Lip* (α, p) if (1.2) still holds when O is replaced by o. It is indifferent which form of Δ we select[1]. A function of Lip (α, p) necessarily belongs to the Lebesgue class L^p.[2]

If $\Delta f = O(h^\alpha)$, or $o(h^\alpha)$, we say that $f(\theta)$ belongs to Lip α, or Lip* α. A function of Lip α belongs to Lip (α, p) for every positive p, and the properties of such functions may be regarded as limiting cases (for $p = \infty$) of those of functions of Lip (α, p).

In particular we call the class Lip $\left(\frac{1}{p}, p\right)$, *i. e.* the class of functions satisfying

$$\int_{-\pi}^{\pi} |\Delta f|^p \, d\theta = O(h), \tag{1.3}$$

the class Λ_p. Our main theorem is then as follows.

[1]) Hardy and Littlewood, 7, 597.

[2]) Hardy and Littlewood, 7, 566.

1928, 6 (with J. E. Littlewood) *Mathematische Zeitschrift*, 28, 612–34.

Theorem 1. *The Fourier series of a function of Λ_p is convergent, and indeed summable $\left(C, -\frac{1}{p}+\delta\right)$ for any positive δ, whenever it is summable by any Cesàro mean. The necessary and sufficient condition for summability to sum s, for a particular value of θ, is*

$$\int_0^t \varphi(u)\,du = o(t), \tag{1.4}$$

where

$$\varphi(t) = \frac{1}{2}\left\{f(\theta+t)+f(\theta-t)-2s\right\}. \tag{1.5}$$

The theorem does not include any of the recognised criteria except Jordan's[3]), nor is it included by any. Its origin is however to be found in the well known criterion of Lebesgue[4]). The conditions of Lebesgue are

$$\int_0^t |\varphi(u)|\,du = o(t) \tag{1.6}$$

and

$$\int_t^\pi \frac{|\varphi(u+t)-\varphi(u)|}{u}\,du = o(1). \tag{1.7}$$

If now $f(\theta)$ belongs to $\mathrm{Lip}^*\left(\frac{1}{p}, p\right)$, so that (1.3) holds with o for O, then $\varphi(t)$ belongs to the same class, and

$$\int_t^\pi \frac{|\Delta_t\varphi|}{u}\,du \leqq \left(\int_0^\pi |\Delta_t\varphi|^p du\right)^{\frac{1}{p}} \left(\int_t^\pi \frac{du}{u^{p'}}\right)^{\frac{1}{p'}} = o\left(t^{\frac{1}{p}+\frac{1}{p'}-1}\right) = o(1),\ ^{5)}$$

so that Lebesgue's second criterion is satisfied uniformly; and it was this that suggested the investigations leading to Theorem 1[6]).

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[3]) See Lemma 9 below.

[4]) See Lebesgue, **10**; de la Vallée-Poussin, **15**; Hobson, 8.

[5]) We write $p' = \frac{p}{p-1}$ when $p > 1$, and similarly for other letters. Our argument supposes $p > 1$. If $p = 1$ we have

$$\int_t^\pi \frac{|\Delta_t\varphi|}{u}\,du \leqq \frac{1}{t}\int_0^\pi |\Delta_t\varphi|\,du = o(1).$$

[6]) Pollard (**14**) has generalised Lebesgue's criterion in two directions. He starts not with (1.7) but with Lebesgue's alternative condition

$$\int_t^\pi \left|\frac{\varphi(u+t)}{u+t} - \frac{\varphi(u)}{u}\right| du = o(1), \tag{1.7a}$$

(Fortsetzung der Fußnote [6]) auf nächster Seite.

Our main purpose then is to define a class of functions for which the convergence problem admits a complete solution; but we also prove some other theorems which are suggested by Theorem 1 or help to elucidate its relations to theorems proved by other writers. We prove in Theorem 7, for example, generalising a theorem of Zygmund[7]), that the Fourier series of a function of Lip (α, p), where $\alpha p > 1$, is uniformly summable $(C, -\alpha+\delta)$, and in Theorem 8 (after Titchmarsh) that it is absolutely convergent when also $p \leqq 2$. In Theorem 5 we show that a function of Lip (α, p) belongs also to

$$\mathrm{Lip}\left(\alpha - \frac{1}{p} + \frac{1}{q},\, q\right)$$

for a certain range of values of q, and in particular that, when $\alpha p > 1$, it is equivalent to a function of Lip $\left(\alpha - \frac{1}{p}\right)$.

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a condition equivalent to (1. 7) when (1. 6) is satisfied. He shows that (1. 7a) may be replaced by

$$(1.7\mathrm{a}') \qquad \varlimsup_{t\to 0} \int_{kt}^{\pi} \left| \frac{\varphi(u+t)}{u+t} - \frac{\varphi(u)}{u} \right| du < \eta(k),$$

where $\eta(k) \to 0$ when $k \to \infty$, and (1. 6) by (1. 4).

The situation is different if we state Lebesgue's criterion as we have stated it in the text. In this case (1. 7) may be replaced by

$$(1.7') \qquad \varlimsup_{t\to 0} \int_{kt}^{\pi} \frac{|\varphi(u+t)-\varphi(u)|}{u}\, du < \eta(k),$$

but it does not seem to be possible to replace (1. 6) by (1. 4).

The condition (1. 7′) is a consequence of (1. 3), since

$$\int_{kt}^{\pi} \frac{|\Delta_t \varphi|}{u}\, du \leqq \left(\int_0^{\pi} |\Delta_t \varphi|^p du\right)^{\frac{1}{p}} \left(\int_{kt}^{\pi} \frac{du}{u^{p'}}\right)^{\frac{1}{p'}} \leqq K t^{\frac{1}{p}} (kt)^{\frac{1}{p'}-1} = K k^{-\frac{1}{p}},$$

where K is independent of t, θ and k. Hence any function of Λ_p satisfies (1. 7′) uniformly in θ. In order to deduce Theorem 1 from this, it would be necessary to show that (1. 7′) involves the convergence of the series wherever it is summable, *i. e.* (by Lemma 6 below) wherever

$$(1.4') \qquad \varphi_r = o(t^r)$$

for some r, φ_r being the r-th integral of φ over $(0, t)$. That this should be true seems to us very improbable.

The conditions (1. 7) and (1. 7a) are no longer equivalent when (1. 6) is replaced by (1. 4), which is (1. 4′) with $r = 1$. It has never been shown even that (1. 7) and (1. 4) are sufficient conditions for convergence.

[7]) Zygmund, 25. Zygmund's theorem is the special case in which $p = \infty$, so that $f(\theta)$ belongs to Lip α.

We use freely the idea of summability by Cesàro means of negative order. Such means may not appear to be of great intrinsic interest, but our methods of proof compel us to use them[8]), and, if they are to be used at all, results involving them should be stated with the greatest precision attainable.

Generalities concerning Fourier series and analytic functions.

2. The complex Fourier series of an arbitrary function $f(\theta)$ of the class L is defined by

$$(2.1)\qquad f(\theta) \sim \sum_{-\infty}^{\infty} c_m e^{mi\theta} = \sum_{-\infty}^{\infty} C_m, \qquad c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-mi\theta}\,d\theta.$$

If we write

$$(2.2)\qquad c_n = \frac{1}{2}(a_n - i\,b_n)$$

for all n, then a_n and b_n are respectively even and odd functions of n, and the formal relations between $f(\theta)$, a_n, and b_n are the classical formulae

$$(2.3)\qquad f(\theta) \sim \frac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) = \frac{1}{2}A_0 + \sum_{1}^{\infty} A_n,$$

$$(2.4)\qquad \begin{matrix} a_n \\ b_n \end{matrix} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \begin{matrix} \cos \\ \sin \end{matrix}\, n\theta\,d\theta.$$

In these $f(\theta)$ is usually supposed to be real, though this is naturally unnecessary.

In what follows we use the 'complex' or 'real' system of formulae as may be most convenient, and call the series the complex and real Fourier series of $f(\theta)$, the use of the word 'real' not implying that $f(\theta)$ is real. We shall generally use the suffix m when we are concerned with the complex formulae, and n when we use the formulae of the classical type.

If $c_m = 0$ for all negative or for all positive m, then we call (2.1) a Fourier series *of power series type*[9]) or a *Fourier power series.* If

$$f(\theta) \sim \sum_{0}^{\infty} c_m e^{mi\theta}$$

[8]) We make repeated use of Lemma 2 (the 'convexity theorem' for Cesàro means). It is obvious that, if we are to use this lemma to prove the convergence of a series in the classical sense, the range of values of r considered must strictly include $r = 0$ and therefore negative values.

[9]) This phrase was introduced by F. Riesz, 18.

is a Fourier power series, then the analytic function $\Sigma c_m g^m$ has $f(\theta)$ for its 'boundary function'[10]).

The series

$$\Sigma(b_n \cos n\theta - a_n \sin n\theta) \tag{2.5}$$

is called the series conjugate or allied to (2.3). It is not necessarily a Fourier series. It was however proved by M. Riesz[11]) that, if $f(\theta)$ belongs to L^p, where $p > 1$, then (2.5) is also a Fourier series, and that of a function $-g(\theta)$ also belonging to L^p. The function $g(\theta)$ is defined, for almost all θ, by

$$g(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\cot\frac{1}{2}(\varphi - \theta)\,d\varphi, \tag{2.6}$$

where the integral is a 'principal value' in Cauchy's sense. In these circumstances $h(\theta) = f(\theta) - i\,g(\theta)$ is the boundary function of an analytic function $H(z)$; and this function belongs to the complex class L^p, *i. e.*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |H(re^{i\theta})|^p\,d\theta \tag{2.7}$$

is bounded for $r < 1$.

Riesz's theorem may be stated in various forms. The form which is most important here is that[12]) *if the series* (2.1) *is the Fourier series of a function of* L^p, *then the series*

$$\sum_0^\infty c_m e^{mi\theta}, \quad \sum_{-\infty}^{-1} c_m e^{mi\theta} \tag{2.8}$$

are the Fourier (*power*) *series of functions of* L^p. We call the series (2.8) the (positive and negative) *power series components* of (2.1).

Preliminary lemmas.

3.1. In this section we collect a number of propositions, for the most part known, which will be required in the proofs of our principal theorems. In one or two cases, where the results have an independent interest, we have developed them a little further than is really necessary.

Lemmas concerning Cesàro summability.

3.2. In these lemmas $\sum u_n$ or U is a series

$$u_0 + u_1 + u_2 + \ldots$$

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10) For all this see F. and M. Riesz, **19**.

11) M. Riesz, **20**.

12) See Hardy and Littlewood, **5**, and M. Riesz, **21**, for fuller explanations.

and $\sum v_m$ or V a series

$$\dots + v_{-2} + v_{-1} + v_0 + v_1 + v_2 + \cdots .$$

We write

$$s_n^{(r)} = \sum_{\nu=0}^{n} \binom{n-\nu+r}{r} u_\nu, \qquad \sigma_n^{(r)} = \sum_{\nu=0}^{n} \binom{n-\nu+r}{r} \nu u_\nu,$$

$$t_m^{(r)} = \sum_{\mu=-m}^{m} \binom{m-|\mu|+r}{r} v_\mu, \qquad \tau_m^{(r)} = \sum_{\mu=-m}^{m} \binom{m-|\mu|+r}{r} |\mu| v_\mu,$$

where $r > -1$. The series are *summable* (C, r), to sums U or V, if

$$s_n^{(r)} \sim U\binom{n+r}{r} \sim U\frac{n^r}{\Gamma(r+1)}$$

or

$$t_m^{(r)} \sim V\binom{m+r}{r} \sim V\frac{m^r}{\Gamma(r+1)};$$

and *bounded* (C, r) if $s_n^{(r)} = O(n^r)$ or $t_m^{(r)} = O(m^r)$.

Lemma 1. *If U is bounded $(C, r+1)$, then the necessary and sufficient condition that it should be bounded (C, r) is that*

$$\sigma_n^{(r)} = O(n^{r+1}). \tag{3.21}$$

The corresponding condition for V is

$$\tau_m^{(r)} = O(m^{r+1}). \tag{3.22}$$

In either case we may replace 'bounded' every where by 'summable' if we replace O by o.

These results are immediate corollaries of the identities

$$\sigma_n^{(r)} = (n+r+1)s_n^{(r)} - (r+1)s_n^{(r+1)},$$
$$\tau_m^{(r)} = (m+r+1)t_m^{(r)} - (r+1)t_m^{(r+1)}.$$

Lemma 2[13]. *If U (or V) is summable (C), i. e. summable (C, r) for some r, and bounded (C, r) for $r > \gamma \geqq -1$, then it is summable (C, r) for $r > \gamma$.*

Lemma 3. *If U or V is summable (C), and (3.21) or (3.22) holds for $r > \gamma \geqq -1$, then the series is summable (C, r) for $r > \gamma$.*

This follows at once from Lemmas 1 and 2.

Lemma 4. *If U is summable (C), and*

$$\sum_{\nu=1}^{n} |\nu u_\nu|^p = O(n),$$

where $p \geqq 1$, then U is summable $\left(C, -1+\frac{1}{p}+\delta\right)$ for every positive δ.

[13]) For U see Andersen, **1**: the proof applies to V with trivial alterations.

We have shown already[14]) that the series is convergent. There is an obvious analogue for V which we leave to the reader.

After Lemma 3, we have only to verify that $\sigma_n^{(r)} = O(n^{r+1})$ for $r > -1 + \frac{1}{p}$. But then

$$\sigma_n^{(r)} = O\left(\sum_{\nu=1}^{n}(n-\nu+1)^r\,|\nu u_\nu|\right) = O\left(\sum_{\nu=1}^{n}|\nu u_\nu|^p\right)^{\frac{1}{p}}\left(\sum_{\nu=1}^{n}(n-\nu+1)^{p'r}\right)^{\frac{1}{p'}}$$

$$= O\left(n^{\frac{1}{p}+\frac{1}{p'}+r}\right) = O(n^{r+1}),$$

since $p'r > -1$.

Lemmas concerning Fourier series in general.

3.3. Lemma 5. *If c_m, a_n, b_n are the Fourier constants of $f(\theta)$, defined as in § 2, and $u_0 = \frac{1}{2}A_0$, $u_n = A_n$ $(n > 0)$, $v_m = C_m$, then $s_n^{(r)} = t_n^{(r)}$ for all n and r, so that the convergence and summability properties of the real and complex Fourier series are identical.*

This results immediately from the definitions. It follows that, in the succeeding lemmas, we need not distinguish the two forms.

Lemma 6. *The necessary and sufficient condition for the summability (C) of the series is that*

$$\varphi_r = o(t^r) \tag{3.31}$$

for some r, φ_r being the r-th integral of φ over $(0, t)$.

We may express the condition by saying that

$$\varphi \to 0 \quad (C, r)$$

for some r. The lemma is the principal theorem of our memoir **3**. It is most natural to quote the theorem in its general form, but it may be worth while (since the proof is not entirely easy) to observe that special cases, such as Lebesgue's theorem[15]) that $\varphi_1 = o(t)$ implies summability $(C, 2)$, would suffice for our applications.

Lemma 7.[16]) *If the series is summable by a mean of negative order, then $\varphi_1 = o(t)$.*

Lemma 8. *If $-1 < \gamma < 1$, and $s_n^{(\gamma)}$ is the γ-th Cesàro mean of the Fourier series, then*

$$s_n^{(\gamma)} - s = \frac{1}{\pi}\int_0^\pi \varphi(t)\,\Omega(t)\,dt,$$

[14]) Hardy and Littlewood, **4**.

[15]) Lebesgue, **10**.

[16]) Hardy and Littlewood, **6**.

where

$$\Omega = \Omega_1 + \Omega_2, \qquad |\Omega| \leqq Kn, \qquad |\Omega_2| \leqq \frac{K}{nt^2},$$

$$\Omega_1 = \frac{\Gamma(\gamma+1)\,\Gamma(n+1)}{\Gamma(n+\gamma+1)} \frac{\sin\left(\left(n+\frac{1}{2}\gamma+\frac{1}{2}\right)t - \frac{1}{2}\gamma\pi\right)}{2^\gamma\left(\sin\frac{1}{2}t\right)^{\gamma+1}},$$

and the K's are independent of n, t, *and* θ.

See Zygmund, **25**; the results are due to Kogbetliantz, **9**. The simplest and most natural proof is by complex integration, but Szegö (**23**) has given a very elegant proof by elementary methods.

Lemmas concerning the classes Lip (α, p).

3. 4. Lemma 9. *Any function of bounded variation belongs to* Lip (1, 1), *and any function of* Lip (1, 1) *is equivalent to a function of bounded variation.*

If $f(\theta)$ is of bounded variation, then

$$\int_{-\pi}^{\pi} |\Delta f|\, d\theta = \int_{-\pi}^{\pi} d\theta \left| \int_{\theta}^{\theta+h} df(t) \right| \leqq \int_{-\pi}^{\pi} d\theta \int_{\theta}^{\theta+h} |df(t)| \leqq \int_{-\pi}^{\pi+h} |df(t)| \int_{t-h}^{t} d\theta = O(h).$$

For the converse, see Theorem 24 of our memoir **7**.

Lemma 10[17]. *If* $f(\theta)$ *is the* α*-th integral of a function of* L^p, *then* $f(\theta)$ *belongs to* Lip* (α, p).

Lemma 11. *If* $f(\theta)$ *belongs to* Lip (α, p), *then* $c_m = O(|m|^{-\alpha})$, *and if to* Lip* (α, p), *then* $c_m = o(|m|^{-\alpha})$.

For, supposing for example $m > 0$, we have

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-mi\theta}\, d\theta = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\theta + \frac{\pi}{m}\right) e^{-mi\theta}\, d\theta$$

$$= -\frac{1}{4\pi}\int_{-\pi}^{\pi} \left(f\left(\theta+\frac{\pi}{m}\right) - f(\theta)\right) e^{-mi\theta}\, d\theta$$

$$= O\left(\int_{-\pi}^{\pi} \left| f\left(\theta+\frac{\pi}{m}\right) - f(\theta)\right|^p d\theta\right)^{\frac{1}{p}} = O(m^{-\alpha}),$$

or $o(|m|^{-\alpha})$, according to the hypothesis.

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17) Hardy and Littlewood, **7**.

Lemma 12. *If* $p \geqq 2$ *and*

$$\gamma_n = \sum_{-n}^{n} | m c_m |^{p'} = O(n),$$

and in particular if $p \geqq 2$ *and* $c_m = O(|m|^{-1})$, *then* $f(\theta)$ *belongs to* Λ_p.

We have

$$\Delta f = f(\theta + h) - f(\theta - h) \sim 2i \sum c_m \sin m h \, e^{mi\theta}.$$

Hence, by Hausdorff's theorem,

$$(3.41)\quad \begin{aligned} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta f|^p d\theta\right)^{\frac{1}{p}} &\leqq 2\left(\sum (|c_m| \, |\sin m h|)^{p'}\right)^{\frac{1}{p'}} \\ &= O\left(h^{p'} \sum_{|m| \leqq \frac{1}{h}} |m c_m|^{p'}\right)^{\frac{1}{p'}} + O\left(\sum_{|m| > \frac{1}{h}} |c_m|^{p'}\right)^{\frac{1}{p'}}. \end{aligned}$$

But

$$\begin{aligned} \sum_{|m| > \frac{1}{h}} |c_m|^{p'} &= \sum_{n > \frac{1}{h}} n^{-p'} (\gamma_n - \gamma_{n-1}) \leqq \sum_{n > \frac{1}{h}} \gamma_n (n^{-p'} - (n+1)^{-p'}) \\ &= O\left(\sum_{n > \frac{1}{h}} n^{-p'}\right) = O(h^{p'-1}). \end{aligned}$$

Hence (3.41) gives

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta f|^p d\theta\right)^{\frac{1}{p}} = O\left(h^{\frac{p'-1}{p'}}\right) = O\left(h^{\frac{1}{p}}\right).$$

The result ceases to be true when $p < 2$. Suppose, for example, that the series is a (positive) Fourier power series, that $c_m = O(m^{-1})$, and that $F(z)$ is the corresponding analytic function, so that the coefficients in $F'(z) = g(z)$ are bounded. If the result of the lemma is true it follows, if we anticipate for a moment the result of Theorem 3, that

$$\int |g(re^{i\theta})|^p d\theta = O\left(\frac{1}{(1-r)^{p-1}}\right)$$

for all $g(z)$ with bounded coefficients. This is true when $p \geqq 2$, since

$$g = O\left(\frac{1}{1-r}\right), \quad \int |g|^2 d\theta = O\left(\sum |c_m|^2 r^{2m}\right) = O\left(\frac{1}{1-r}\right),$$

$$\int |g|^p d\theta = O\left(\operatorname{Max} |g|^{p-2} \int |g|^2 d\theta\right) = O\left(\frac{1}{(1-r)^{p-1}}\right).$$

That it is false when $p < 2$ may be shown by the example

$$g(z) = \sum e^{ain\log n} z^n \qquad (a > 0).$$

Here $g(z)$ is (to put it roughly) of order $\frac{1}{2}$ for all θ, and the integral is of order $\frac{1}{2}p > p-1$.

Incidentally we see that a function of Δ_p is not necessarily bounded. This is easily proved directly. For example, if $f(\theta) = \log\frac{1}{|\theta|}$, we have

$$\int_{-\pi}^{\pi} |\Delta f|^p \, d\theta = O\left(\int_{-\infty}^{\infty}\left|\log\left|\frac{\theta+h}{\theta-h}\right|\right|^p d\theta\right) = O\left(h\int_{-\infty}^{\infty}\left|\log\left|\frac{x+1}{x-1}\right|\right|^p dx\right) = O(h)$$

for all $p > 1$.

3.5. Lemma 13. *If* $p > 1$, $0 < \alpha \leqq 1$ *or* $p = 1$, $0 < \alpha < 1$, *and* $f(\theta)$ *belongs to* Lip (α, p), *then the conjugate function* $-g(\theta)$ *also belongs to* Lip(α, p).

This is an immediate corollary of Riesz's theorem when $p > 1$, since Δf and $-\Delta g$ are conjugate, and

$$\int |\Delta g|^p \, d\theta \leqq K(p) \int |\Delta f|^p \, d\theta,$$

where $K(p)$ depends only on p. This proof fails when $p = 1$. We therefore give an alternative proof valid when $p \geqq 1$, $\alpha < 1$[18]. When $p = 1$, $\alpha = 1$, the result is false, since the conjugate of a function of bounded variation is not necessarily of bounded variation.

After (2.6), we have

$$g(\theta+h) - g(\theta-h) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi+\theta)\left(\cot\frac{1}{2}(\varphi-h) - \cot\frac{1}{2}(\varphi+h)\,d\varphi\right.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(\varphi+\theta) - f(\theta))\left(\cot\frac{1}{2}(\varphi-h) - \cot\frac{1}{2}(\varphi+h)\right)d\varphi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-2h} + \frac{1}{2\pi}\int_{2h}^{\pi} + \frac{1}{2\pi}\int_{-2h}^{2h}(f(\varphi+\theta) - f(\theta))\cot\frac{1}{2}(\varphi-h)\,d\varphi$$

$$-\frac{1}{2\pi}\int_{-2h}^{2h}(f(\varphi+\theta) - f(\theta))\cot\frac{1}{2}(\varphi+h)\,d\varphi = J_1 + J_2 + J_3 + J_4,$$

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[18]) Modelled on Priwaloff's for the case $p = \infty$: see Priwaloff, 17. It should be observed that the theorem also fails for $p = \infty$, $\alpha = 1$.

say; and it is sufficient to show that

$$(3.51)\qquad \left(\int |J_i|^p\, d\theta\right)^{\frac{1}{p}} = O(h^\alpha) \qquad (i = 1, 2, 3, 4).$$

In J_1 and J_2 we have

$$\cot\frac{1}{2}(\varphi - h) - \cot\frac{1}{2}(\varphi + h) = O\left(\frac{h}{\varphi^2 - h^2}\right)$$

and so

$$\left(\int_{-\pi}^{\pi} |J_2|^p\, d\theta\right)^{\frac{1}{p}} = O\left(\int_{2h}^{\pi} \frac{h\, d\varphi}{\varphi^2 - h^2}\left(\int_{-\pi}^{\pi} |f(\varphi+\theta) - f(\theta)|^p\, d\theta\right)^{\frac{1}{p}}\right)$$

$$= O\left(h\int_{2h}^{\infty} \frac{\varphi^\alpha\, d\varphi}{\varphi^2 - h^2}\right) = O(h^\alpha)$$

(since $\alpha < 1$), and similarly for J_1.

Of J_3 and J_4 we need only consider the first. We write

$$J_3 = \frac{1}{2\pi}\int_{-2h}^{2h} (f(\varphi+\theta) - f(h+\theta)) \cot\frac{1}{2}(\varphi - h)\, d\varphi$$

$$+ \frac{1}{2\pi}(f(h+\theta) - f(\theta)) \int_{-2h}^{2h} \cot\frac{1}{2}(\varphi - h)\, d\varphi = J_3' + J_3'',$$

say. Then

$$\left(\int_{-\pi}^{\pi} |J_3'|^p\, d\theta\right)^{\frac{1}{p}} \leqq \frac{1}{2\pi}\left(\int_{-\pi} d\theta \left(\int_{-2h}^{2h} |f(\varphi+\theta) - f(h+\theta)| \left|\cot\frac{1}{2}(\varphi - h)\right| d\varphi\right)^p\right)^{\frac{1}{p}}$$

$$\leqq \frac{1}{2\pi}\int_{-2h}^{2h} \left|\cot\frac{1}{2}(\varphi - h)\right| d\varphi \left(\int_{-\pi}^{\pi} f|(\varphi+\theta) - f(h+\theta)|^p\, d\theta\right)^{\frac{1}{p}}$$

$$= O\left(\int_{-2h}^{2h} |\varphi - h|^{\alpha-1}\, d\varphi\right) = O(h^\alpha).$$

Also

$$\int_{-2h}^{2h} \cot\frac{1}{2}(\varphi - h)\, d\varphi = 2\log\left|\frac{\sin\frac{1}{2}h}{\sin\frac{3}{2}h}\right| = O(1),$$

$$\left(\int_{-\pi}^{\pi} |J_3''|^p\, d\theta\right)^{\frac{1}{p}} = O\left(\int_{-\pi}^{\pi} |f(\theta+h) - f(\theta)|^p\, d\theta\right)^{\frac{1}{p}} = O(h^\alpha).$$

Hence (3.51) holds for $i = 3$.

Lemma 14. *If p and α satisfy the conditions of Lemma 13, and $f(\theta)$ belongs to* Lip(α, p), *then the power series components of $f(\theta)$ are the Fourier power series of functions of* Lip(α, p).

This is merely a restatement of Lemma 13 in different language.

Preliminary theorems concerning analytic functions.

4. 1. Suppose that

$$F(z) = F(r e^{i\theta}) = \sum_{n=0}^{\infty} c_n z^n$$

is regular for $r < 1$. We write

$$M_p = M_p(r) = M_p(r, F) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(z)|^p \, d\theta\right)^{\frac{1}{p}}$$

(for any $p > 0$). It is known that M_p is an increasing function of r for every p and an increasing function of p for every r, and that $M_p(r) \to M(r)$, the maximum modulus of $F(z)$, when $p \to \infty$.[19] If $p > 1$, then the necessary and sufficient condition that $F(z)$ should have a boundary value $F(e^{i\theta}) = f(\theta)$ belonging to L^p is that $M_p(r)$ should be bounded[20].

Theorem 2. *If $p \geqq 1$, $\beta \geqq 0$, and*

$$M_p(r, F) = O((1-r)^{-\beta}), \tag{4. 11}$$

then

$$M_q(r, F) = O\left((1-r)^{-\beta-\frac{1}{p}+\frac{1}{q}}\right) \tag{4. 12}$$

for $q > p$. In particular, for $q = \infty$,

$$M(r) = O\left((1-r)^{-\beta-\frac{1}{p}}\right). \tag{4. 13}$$

If $\beta = 0$, the O's in (4. 12) *and* (4. 13) *may be replaced by o's.*

The results are still true when $0 < p < 1$, but it would carry us too far to prove them here. We need only prove (4. 13), as it stands or in the sharpened form; for if (4. 13) is proved, we have

$$M_q \leqq M_p^{\frac{p}{q}} M^{\frac{q-p}{q}} = O((1-r)^{-\lambda}),$$

where

$$\lambda = \frac{p}{q}\beta + \frac{q-p}{q}\left(\beta + \frac{1}{p}\right) = \beta + \frac{1}{p} - \frac{1}{q}.$$

If $p > 1$ and $\varrho = \frac{1}{2}(1+r)$, we have

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[19]) See Littlewood, **12** and **13**, for these and other properties of M_p.

[20]) F. and M. Riesz, **19**.

$$| F(re^{i\theta}) | = \left| \frac{\varrho}{2\pi} \int_{-\pi}^{\pi} \frac{F(\varrho e^{i\varphi}) e^{i\varphi} d\varphi}{\varrho e^{i\varphi} - r e^{i\theta}} \right|$$

$$\leqq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} | F(\varrho e^{i\varphi}) |^p d\varphi \right)^{\frac{1}{p}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\varphi}{| \varrho e^{i\varphi} - r e^{i\theta} |^{p'}} \right)^{\frac{1}{p'}}$$

$$= O\left((1-\varrho)^{-\beta} \left(\int_0^{\infty} \frac{d\psi}{((\varrho - r)^2 + \psi^2)^{\frac{1}{2}p'}} \right)^{\frac{1}{p'}} \right)$$

$$= O((1-r)^{-\beta} (\varrho - r))^{-\frac{1}{p}} = O((1-r)^{-\beta - \frac{1}{p}}).$$

If $p = 1$, we have

$$| F(r e^{i\theta}) | \leqq \frac{1}{2\pi} \int_{-\pi}^{\pi} | F(\varrho e^{i\varphi}) | \, d\varphi \cdot \operatorname*{Max}_{(\varphi)} \frac{1}{| \varrho e^{i\varphi} - r e^{i\theta} |}$$

$$= O((1-\varrho)^{-\beta} (\varrho - r)^{-1}) = O((1-r)^{-\beta - 1}).$$

If $\beta = 0$, $F(e^{i\theta}) = f(\theta)$ belongs to L^p, and we can choose δ so that

$$\frac{1}{2\pi} \int_{-\delta}^{\delta} | f(\theta + \varphi) |^p d\varphi < \varepsilon^p$$

for all θ. Also[21])

$$F(r e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\varphi) e^{i\varphi} d\varphi}{e^{i\varphi} - r e^{i\theta}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\theta + \varphi) e^{i\varphi} d\varphi}{e^{i\varphi} - r}$$

$$= \frac{1}{2\pi} \int_{-\delta}^{\delta} + \frac{1}{2\pi} \int_{\delta}^{\pi} + \frac{1}{2\pi} \int_{-\pi}^{-\delta} = J_1 + J_2 + J_3,$$

say. If $p > 1$,

$$| J_1 | \leqq \left(\frac{1}{2\pi} \int_{-\delta}^{\delta} | f(\theta + \varphi) |^p d\varphi \right)^{\frac{1}{p}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\varphi}{| e^{i\varphi} - r |^{p'}} \right)^{\frac{1}{p'}} < K \varepsilon (1 - r)^{-\frac{1}{p}},$$

where K is independent of r and δ; and J_2 and J_3 are bounded when δ is fixed. Hence

$$F(r e^{i\theta}) = o((1-r)^{-\frac{1}{p}}).$$

The proof when $p = 1$ may obviously be modified in the same manner.

It may be observed that when $p \leqq 2$ we can assert more, viz. that

$$F(r) = \sum | c_n | r^n = O((1-r)^{-\beta - \frac{1}{p}}),$$

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[21]) F. and M. Riesz, 19.

or the corresponding equation with o. Thus if $p>1$, $\beta>0$, we have, by Hausdorff's theorem,

$$\left(\sum |c_n|^{p'} r^{p'n}\right)^{\frac{1}{p'}} \leqq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}} = O((1-r)^{-\beta}),$$

and so

$$F(r) = \sum (|c_n| r^{\frac{1}{2}n} \cdot r^{\frac{1}{2}n}) \leqq \left(\sum |c_n|^{p'} r^{\frac{1}{2}p'n}\right)^{\frac{1}{p'}} \left(\sum r^{\frac{1}{2}pn}\right)^{\frac{1}{p}}$$
$$= O((1-r^{\frac{1}{2}})^{-\beta}(1-r^{\frac{1}{2}p})^{-\frac{1}{p}}) = O((1-r)^{-\beta-\frac{1}{p}}).$$

When $\beta=0$, we must use the boundary function $f(\theta)$. Since $f(\theta)$ belongs to L^p, $\sum |c_n|^{p'}$ is convergent, and we can choose N so that

$$\left(\sum_{N+1}^{\infty} |c_n|^{p'}\right)^{\frac{1}{p'}} < \varepsilon.$$

Hence $F(r)$ is the sum of two parts, of which one is a polynomial in r of degree N, while the other is

$$\sum_{N+1}^{\infty} |c_n| r^n \leqq \left(\sum_{N+1}^{\infty} |c_n|^{p'}\right)^{\frac{1}{p'}} \left(\sum_{N+1}^{\infty} r^{pn}\right)^{\frac{1}{p}} < \varepsilon (1-r)^{-\frac{1}{p}}.$$

Hence $F(r) = o((1-r)^{-\frac{1}{p}})$. The case $p=1$ is trivial.

These results are false for $p>2$, a *Gegenbeispiel* being given by the function

$$F(z) = \sum n^{-\frac{1}{2}-\delta} e^{ain\log n} z^n \qquad (a>0, \delta>0)$$

continuous for $|z| \leqq 1$.

4.2. Theorem 3. *Suppose that* $p \geqq 1$, $0 < \alpha \leqq 1$, *that*

$$\sum_0^{\infty} c_n e^{ni\theta}$$

is the Fourier power series of a function $F(e^{i\theta}) = f(\theta)$, *and that* $F(z)$ *is the corresponding analytic function. Then the necessary and sufficient condition that* $f(\theta)$ *should belong to* Lip (α, p) *is that*

$$M_p(r, F') = O((1-r)^{-1+\alpha}). \tag{4.21}$$

We suppose first that $p>1$. The case $\alpha=1$ may be disposed of at once, since hypothesis and condition are each necessary and sufficient for $f(\theta)$ to be equivalent to the integral of a function of L^p [22]. We may therefore suppose that $0<\alpha<1$.

[22]) See Hardy and Littlewood, 7, (Theorems 22 and 24).

Suppose first that $f(\theta)$ belongs to Lip (α, p). Then

$$F'(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\varphi)\, e^{i\varphi} d\varphi}{(e^{i\varphi} - re^{i\theta})^2} = \frac{e^{-i\theta}}{2\pi}\int_{-\pi}^{\pi} \frac{f(\theta+\varphi) e^{i\varphi}\, d\varphi}{(e^{i\varphi}-r)^2}$$

$$= \frac{e^{-i\theta}}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\varphi}}{(e^{i\varphi}-r)^2}(f(\theta+\varphi) - f(\theta))\, d\varphi.$$

Hence

$$\left(\int_{-\pi}^{\pi} |F'|^p\, d\theta\right)^{\frac{1}{p}} \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\varphi}{|e^{i\varphi}-r|^2}\left(\int_{-\pi}^{\pi} |f(\theta+\varphi) - f(\theta)|^p\, d\theta\right)^{\frac{1}{p}}$$

$$= O\left(\int_{-\pi}^{\pi} \frac{|\varphi|^\alpha\, d\varphi}{|e^{i\varphi}-r|^2}\right) = O\left(\int_0^\infty \frac{\varphi^\alpha\, d\varphi}{(1-r)^2+\varphi^2}\right) = O\left(\frac{1}{(1-r)^{1-\alpha}}\right),$$

which is (4.21).

Next, suppose that (4.21) is satisfied. In order to prove that $f(\theta)$ belongs to Lip (α, p), it is sufficient to prove that

$$(4.22)\qquad r\int_{-\pi}^{\pi} |F(re^{i\theta+ih}) - F(re^{i\theta})|^p\, d\theta = r\int_{-\pi}^{\pi} |\varDelta|^p\, d\theta < Kh^{p\alpha},$$

for $0 < h \leqq \frac{1}{2}$, $r \geqq \frac{1}{2}$, K being independent of h and r. Now

$$(4.23)\qquad |\varDelta| \leqq \left(\int_1 + \int_2 + \int_3\right) |F'(z)|\, |dz| = \varDelta_1 + \varDelta_2 + \varDelta_3,$$

where the path 2 is the arc $(\theta, \theta+h)$ of the circle $|z| = r - h = \varrho$, and 1 and 3 are the lines joining the ends of the arc to the corresponding ends on the circle $|z| = r$.

Now, in the first place,

$$(4.24)\qquad \left(\int_{-\pi}^{\pi} \varDelta_2^p\, d\theta\right)^{\frac{1}{p}} \leqq \left(\int_{-\pi}^{\pi} d\theta \left(\int_0^h |F'(\varrho e^{i\theta+i\varphi})|\, d\varphi\right)^p\right)^{\frac{1}{p}}$$

$$\leqq \int_0^h d\varphi \left(\int_{-\pi}^{\pi} |F'|^p\, d\theta\right)^{\frac{1}{p}} = O(h(1-\varrho)^{\alpha-1}) = O(h^\alpha),$$

uniformly in r. Secondly,

$$\left(\int_{-\pi}^{\pi} \varDelta_1^p\, d\theta\right)^{\frac{1}{p}} = \left(\int_{-\pi}^{\pi} d\theta \left(\int_\varrho^{\varrho+h} |F'(te^{i\theta})|\, dt\right)^p\right)^{\frac{1}{p}}$$

$$\leqq \int_\varrho^{\varrho+h} dt \left(\int_{-\pi}^{\pi} |F'(te^{i\theta})|^p\, d\theta\right)^{\frac{1}{p}} = O\left(\int_\varrho^{\varrho+h} (1-t)^{\alpha-1}\, dt\right),$$

uniformly in h and ϱ (or r). If $1-r>h$ or $1-\varrho>2h$, this is

$$O(h(1-\varrho-h)^{\alpha-1})=O(h^{\alpha}),$$

while if $1-r\leqq h$ or $1-\varrho\leqq 2h$ it is

$$O(\int_{\varrho}^{1}(1-t)^{\alpha-1}dt)=O((1-\varrho)^{\alpha})=O(h^{\alpha}).$$

Hence

$$\left(\int_{-\pi}^{\pi}\Delta_1^p\,d\theta\right)^{\frac{1}{p}}=O(h^{\alpha}), \tag{4.25}$$

and the same argument applies to Δ_3. The conclusion follows from (4.23), (4.24), and (4.25).

We have now to consider the case $p=1$. The arguments which precede remain valid when $p=1$, $\alpha<1$, so that we may suppose $p=1$, $\alpha=1$. By Lemma 9, $f(\theta)$ belongs to Lip (1, 1) if, and only if, it is of bounded variation. But it is known[23] that $f(\theta)$ is of bounded variation if, and only if, $M_1(r, F')$ is bounded. This completes the proof of the theorem.

It is to be observed that Theorem 3 remains true in the limiting case $p=\infty$. We have in fact

Theorem 4. *The necessary and sufficient condition that $f(\theta)$ should be equivalent to a function of* Lip α, *where* $0<\alpha<1$, *is that*

$$F'(re^{i\theta})=O((1-r)^{-1+\alpha}).$$

The proof is a trivial simplification of that of Theorem 3.

A theorem concerning the Lipschitz classes.

5. Theorem 5. *Suppose that $f(\theta)$ belongs to* Lip (α, p), *where* $p\geqq 1$, $0<\alpha\leqq 1$. *Then*

(i) *if* $\alpha p\leqq 1$ *and* $p<q<\frac{p}{1-\alpha p}$, $f(\theta)$ *belongs to* Lip $\left(\alpha-\frac{1}{p}+\frac{1}{q}, q\right)$;

(ii) *if* $\alpha p>1$, *then this is true for all* $q>p$, *and* $f(\theta)$ *is equivalent to a function of* Lip $\left(\alpha-\frac{1}{p}\right)$.

We exclude for the moment the case $p=1$, $\alpha=1$. If then $f(\theta)$ belongs to Lip (α, p), so, by Lemma 14, do its power series components. There is therefore no loss of generality in supposing $f(\theta)=F(e^{i\theta})$ to be the boundary function of an analytic function $F(z)$.

By Theorem 3,

$$M_p(r, F')=O((1-r)^{-1+\alpha}),$$

[23]) F. and M. Riesz, 19.

and therefore, by Theorem 2,

$$M_q(r, F') = O\left((1-r)^{-1+\alpha-\frac{1}{p}+\frac{1}{q}}\right)$$

for $q > p$. Now $\alpha - \frac{1}{p} + \frac{1}{q} > 0$ for $q < \frac{p}{1-\alpha p}$ if $\alpha p \leqq 1$, and for all positive q if $\alpha p > 1$. In either case our conclusion follows from Theorem 3.

We have also

$$F' = O\left((1-r)^{-1+\alpha-\frac{1}{p}}\right),$$

by Theorem 2. It follows from Theorem 4 that, when $\alpha p > 1$, $f(\theta)$ is equivalent to a function of Lip $\left(\alpha - \frac{1}{p}\right)$.

In the excluded case $p = 1$, $\alpha = 1$ of (i), $f(\theta)$ is equivalent to a function of bounded variation (*a fortiori* a bounded function), and

$$\int |\Delta f|^q d\theta = O(\int |\Delta f| d\theta) = O(h),$$

so that the conclusion still holds.

The case $\alpha p = 1$ is relevant to Theorem 1, and we state it separately.

Theorem 6. *If $f(\theta)$ belongs to Λ_p, for a $p \geqq 1$, then it belongs to Λ_q for $q > p$.*

In concluding this section we should remark that the first results of this character were found by Titchmarsh[24]), and that ours were suggested by his. In his work the orders of the Lipschitz conditions are affected with an ε, and his method of proof introduces an unnecessary limitation[25]).

Proof of Theorem 1.

6. If $p = 1$, $f(\theta)$ is equivalent to a function of bounded variation. The series converges, and (1.4) is satisfied, for all θ. Also a_n and b_n are $O\left(\frac{1}{n}\right)$, so that the series satisfies the condition of Lemma 4, and is summable $(C, -1+\delta)$, for all θ and all positive δ. In this case then there is nothing to prove, and we may suppose $p > 1$.

Assuming therefore that $p > 1$, we write the series in the complex form, and denote its positive and negative power series components by $f_1(\theta) = F_1(e^{i\theta})$ and $f_2(\theta) = F_2(e^{i\theta})$. By Lemma 14, each of these belongs to Λ_p. We shall prove that

$$\sum_0^\infty C_n = \sum_0^\infty c_n e^{ni\theta},$$

[24]) Titchmarsh, 24.

[25]) Arising from the unsymmetrical role of the number 2 in the Hausdorff theorems.

the Fourier power series of $f_1(\theta)$, satisfies (3.21) for $r > -\frac{1}{p}$ and all θ. An analogous argument will apply to the second component, and it will then follow that $\sum C_m$ satisfies (3.22), and is therefore summable $\left(C, -\frac{1}{p}+\delta\right)$ whenever it is summable (C).

We write $z = \varrho e^{i\psi}$ and

$$G(z) = F_1(ze^{i\theta}) \doteq \sum C_n z^n.$$

Then $g(\psi) = G(e^{i\psi})$ belongs to Λ_p, and so, by Theorem 3,

$$\int_{-\pi}^{\pi} |G'(\varrho e^{i\psi})|^p \, d\psi = O((1-\varrho)^{1-p}). \tag{6.1}$$

Now $S_n^{(r)}$, the sum formed from C_n as $\sigma_n^{(r)}$ is formed from u_n, is the coefficient of z^n in $(1-z)^{-r-1} G'(z)$, so that

$$S_n^{(r)} = \frac{1}{2\pi i} \int \frac{G'(z)\, dz}{(1-z)^{r+1} z^{n+1}},$$

the contour of integration being $|z| = \varrho = 1 - \frac{1}{n}$. Hence

$$S_n^{(r)} = O\left(\left(\int |G'|^p \, d\psi\right)^{\frac{1}{p}} \left(\int \frac{d\psi}{|1-z|^{(r+1)p'}}\right)^{\frac{1}{p'}}\right)$$

$$= O\left((1-\varrho)^{-1+\frac{1}{p}} (1-\varrho)^{\frac{1}{p'}-r-1}\right) = O(n^{r+1}),$$

provided that $(r+1)p' > 1$ or $r > -\frac{1}{p}$. This is (3.21).

Finally, (1.4) is sufficient for summability by Lemma 6, and necessary by Lemma 7. This completes the proof of the theorem.

It may be worth observing that when $p \leqq 2$, and the series is summable (C), much more is true than is expressed by (1.4). Thus when $1 < p \leqq 2$ it follows from (6.1), by Hausdorff's theorem, that

$$\sum |n C_n|^{p'} \varrho^{p'n} = O((1-\varrho)^{-1})$$

and so that

$$\sum_{1}^{n} |\nu C_\nu|^{p'} = O(n). \tag{6.2}$$

From (6.2), the convergence of $\sum C_n$, and Theorem 4 of our note 4, it results that

$$\int_0^t |\varphi(u)|^q \, du = o(t)$$

for all positive q. In particular (1.6) is true. These results are obviously trivial when $p = 1$.

An extension of a theorem of Zygmund.

7.1. Theorem 7. *If $f(\theta)$ belongs to* Lip (α, p), *where* $0 < \alpha \leqq 1$, $\alpha p > 1$, *then the Fourier series is uniformly summable* $(C, -\alpha + \delta)$ *for every positive* δ. *If $f(\theta)$ belongs to* Lip* (α, p), *then the δ may be omitted.*

Since $\alpha p > 1$, $f(\theta)$ is equivalent to a continuous function[26]. We may therefore suppose $f(\theta)$ continuous, and take $s = f(\theta)$.

We take $\gamma = -\alpha + \delta$, $\gamma = -\alpha$ in the two cases, and write, in the notation of Lemma 8,

$$J = \int_0^\pi \varphi \Omega \, dt = \int_0^h \varphi \Omega \, dt + \int_h^\pi \varphi \Omega_1 \, dt + \int_h^\pi \varphi \Omega_2 \, dt = J_1 + J_2 + J_3,$$

where $h = \frac{\pi}{n}$. Then since $\varphi(t)$ tends to zero with t, uniformly in θ, we have

$$J_1 = O\left(n \int_0^h |\varphi| \, dt\right) = o(1), \qquad J_3 = O\left(\frac{1}{n} \int_h^\pi \frac{|\varphi|}{t^2} \, dt\right) = o(1),$$

each uniformly in θ.

We can express J_2 as the sum of four terms each of which is the product of a factor $O(n^{-\gamma})$ by an integral

$$L = \int_h^\pi \psi(t) \frac{\cos}{\sin} nt \frac{dt}{\left(\sin \frac{1}{2} t\right)^{\gamma+1}}, \tag{7.11}$$

where $\psi(t) = \varphi(t) \frac{\cos}{\sin} \left(\frac{1}{2}\gamma + \frac{1}{2}\right) t$. It is plain that $\psi(t)$ belongs to Lip (α, p)[27], uniformly in θ, and that everything is reduced to proving that

$$L = o(n^\gamma)$$

uniformly in θ.

We have

$$L = -\int_0^{\pi-h} \psi(t+h) \frac{\cos}{\sin} nt \frac{dt}{\left(\sin \frac{1}{2}(t+h)\right)^{\gamma+1}}, \tag{7.12}$$

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[26]) Indeed to a function of Lip $\left(\alpha - \frac{1}{p}\right)$, by Theorem 5. This, combined with Zygmund's theorem, gives summability $\left(C, -\alpha + \frac{1}{p} + \delta\right)$, and in particular convergence, but not the full result.

[27]) The product of two bounded functions of Lip (α, p) belongs to Lip (α, p).

and here we may replace the limits by h and π with an error $o(h\cdot h^{-\gamma-1})=o(n^{\gamma})$. If we ignore errors of this form, and add (7.11) and (7. 12), we obtain

$$L=-\frac{1}{2}\int_h^\pi \frac{\cos}{\sin} nt\left(\frac{\psi(t+h)}{\left(\sin\frac{1}{2}(t+h)\right)^{\gamma+1}}-\frac{\psi(t)}{\left(\sin\frac{1}{2}t\right)^{\gamma+1}}\right)dt$$

$$=O\left(\int_h^\pi \frac{|\psi(t+h)-\psi(t)|}{t^{\gamma+1}}dt\right)$$

$$+O\left(\int_h^\pi |\psi(t)|\left|\frac{1}{\left(\sin\frac{1}{2}t\right)^{\gamma+1}}-\frac{1}{\left(\sin\frac{1}{2}(t+h)\right)^{\gamma+1}}\right|dt\right).$$

The second term here is

$$o\left(h\int_h^\pi \frac{dt}{t^{\gamma+2}}\right)=o(n^{\gamma}).$$

The first is

$$O\left(\left(\int_0^\pi |\psi(t+h)-\psi(t)|^p dt\right)^{\frac{1}{p}}\left(\int_h^\pi t^{-(\gamma+1)p'}dt\right)^{\frac{1}{p'}}\right).$$

In the first case this is

$$O(h^{\alpha})=o(n^{\gamma}),\qquad O\left(h^{\alpha}\left(\log\frac{1}{h}\right)^{\frac{1}{p'}}\right)=o(n^{\gamma}),$$

or

$$O(h^{\alpha}\cdot h^{\frac{1}{p'}-1-\gamma})=O(n^{\gamma+\frac{1}{p}-\alpha})=o(n^{\gamma}),$$

according as $-\alpha<\gamma<-\frac{1}{p}$, $\gamma=-\frac{1}{p}$, or $\gamma>-\frac{1}{p}$. In the second case, when $f(\theta)$, and so $\psi(t)$, belongs to Lip* (α, p), and $\gamma=-\alpha$, we obtain $o(h^{\alpha})=o(n^{\gamma})$. This completes the proof. It should be observed that the series is not necessarily summable $(C, -\alpha)$ in the general case, Weierstrass's function $\sum a^{-n\alpha}\cos a^n\theta$ giving a *Gegenbeispiel.*

7.2. If $p\leqq 2$ (but not otherwise) there is a stronger theorem, viz.

Theorem 8. *If (in addition to the conditions of Theorem 7), $p\leqq 2$, then the series is absolutely convergent. In fact $\sum|c_m|^{\varkappa}$ is convergent for*

$$\varkappa>\frac{p}{p+p\alpha-1}$$

whenever $0<\alpha\leqq 1$, $1<p\leqq 2$.

This theorem, in its general form, is due to Titchmarsh[28]), and is a generalisation of a theorem of Szász[29]). The first result in this direction

[28]) Titchmarsh, 24 (the corresponding theorem for transforms).

[29]) Szász, 22.

was the well known theorem of S. Bernstein[30]), that the Fourier series of a function of Lip α, where $\alpha > \frac{1}{2}$, is absolutely convergent. The general theorem may be proved very simply as follows[31]).

Since $p > 1$, we may suppose $f(\theta)$ the boundary function of an analytic $F(z) = \sum c_n e^{ni\theta}$. By Theorem 3

$$\int |F'(re^{i\theta})|^p \, d\theta = O\left(\frac{1}{(1-r)^{p-\alpha p}}\right).$$

Hence, by Hausdorff's theorem,

$$\sum |n c_n|^{p'} r^{p'n} = O\left(\frac{1}{(1-r)^{p'-\alpha p'}}\right),$$

and, taking $r = 1 - \frac{1}{n}$,

$$\sum_1^n |\nu c_\nu|^{p'} = O(n^{p'-\alpha p'}).$$

Now we may obviously suppose that $\varkappa < p'$, and then Hölder's inequality gives

$$\sum_1^n |\nu c_\nu|^{\varkappa} = O\left(n^{\varkappa - \alpha\varkappa} \cdot n^{\frac{p+\varkappa-p\varkappa}{p}}\right) = O\left(n^{1+\frac{\varkappa}{p}-\alpha\varkappa}\right).$$

But (7.21) implies $\varkappa > 1 + \frac{\varkappa}{p} - \alpha\varkappa$, and so our conclusion follows by partial summation.

It should be observed that $\sum |c_m|^{\varkappa}$ is not necessarily convergent when

$$\varkappa = \frac{p}{p + \alpha p - 1}.$$

Suppose for example that $f(\theta) = |\theta|^{-a}$, where $(a+1)p > 1$. Then

$$\int_{-\pi}^{\pi} |f(\theta + h) - f(\theta - h)|^p \, d\theta = O\left(\int_{-\infty}^{\infty} \big||\theta + h|^{-a} - |\theta - h|^{-a}\big|^p \, d\theta\right)$$

$$= O\left(h^{1-ap} \int_{-\infty}^{\infty} \big||w+1|^{-a} - |w-1|^{-a}\big|^p \, dw\right) = O(h^{1-ap}),$$

so that $f(\theta)$ belongs to Lip (α, p) with $\alpha = \frac{1}{p} - a$. But the Fourier constants of $f(\theta)$ behave like multiples of $|m|^{a-1}$, and

$$(a-1)\frac{p}{p + \alpha p - 1} = -1,$$

so that $\sum |c_m|^{-\varkappa}$ is divergent.

The result of Theorem 8 is false when $p > 2$. In fact

$$(7.22) \qquad f(\theta) = \sum_1^\infty n^{-\frac{1}{2}-\alpha} e^{ain \log n} e^{ni\theta} \qquad (a > 0,\ \alpha > 0)$$

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[30]) Bernstein 2.

[31]) We have corrected an obscurity pointed out to us by Professor O. Szász, who has proved Theorem 8 independently.

belongs to Lip α, and *a fortiori* to Lip (α, p) for all p. In this case $\sum |c_m|^{\varkappa}$ is convergent if

$$\varkappa > \frac{2}{1+2\alpha}$$

and otherwise divergent, and the result of Theorem 8 fails if

$$\frac{2}{1+2\alpha} > \frac{p}{p+p\alpha-1},$$

i. e. if $p > 2$.

The same example settles a question suggested to us by Professor M. Riesz. It is known that, in Bernstein's theorem, the number $\frac{1}{2}$ cannot be replaced by any smaller number, but it has apparently not been proved that the theorem is false when $\alpha = \frac{1}{2}$. The function (7.22), with $\alpha = \frac{1}{2}$, plainly gives the requisite *Gegenbeispiel*[32]).

We add in conclusion that the class Lip (α, p) is identical with *the class of functions $f(\theta)$ approximable in mean p-th power, with error $O(n^{-\alpha})$, by trigonometrical polynomials $\varphi_n(\theta)$ of degree n*: *i. e.* approximable so that

$$\left(\int |f-\varphi_n|^p \, d\theta\right)^{\frac{1}{p}} = O(n^{-\alpha}).$$

This approximation may be made, in general, by the Fourier polynomials of $f(\theta)$; the case $p = \infty$, in which this is not true[33]), is exceptional.

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[32]) It fails to show that *the series is not necessarily absolutely convergent when $f(\theta)$ is the integral of order $\frac{1}{2}$ of a bounded function* (in which case it necessarily belongs to Lip $\frac{1}{2}$), or, in other words, that *there are bounded functions for which*

$$\sum n^{-\frac{1}{2}}(|a_n| + |b_n|)$$

is divergent.

Mr. A. E. Ingham has however proved this by the example of the (continuous) function

$$\sum n^{-\frac{1}{2}} (\log n)^{-\frac{1}{2}-\delta} e^{ain(\log n)^{\frac{1}{2}}} e^{ni\theta}.$$

[33]) See de la Vallée-Poussin, **15**, and Lebesgue, **11**.

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CORRECTIONS

p. 624, *line* 4. For $O((1-r)^{-\beta}(\rho-r))^{-1/p}$ read $O((1-r)^{-\beta}(\rho-r)^{-1/p})$.

§ 4.2. There is a correction to the proof of Theorem 3 in **1932**, 4, p. 437, footnote.

§ 7.2. The displayed inequality in the statement of Theorem 8 should be labelled (7.21).

p. 633, *last line of footnote*. For $(\log n)^{\frac{1}{2}}$ in the index of the exponential factor read $(\log n)^{\gamma}$, where $0 < \gamma < 2\delta$. See A. E. Ingham, *Annals of Math.* 31 (1930), 241–5.

COMMENTS

The results of this paper concerning the convergence of Fourier series are in fact stronger than is indicated by the title of the paper, since they give sufficient conditions for the Cesàro summability of the Fourier series to a negative order. More complete results in this direction are now known, and there are similar extensions of Theorem 8 in which absolute convergence is replaced by absolute Cesàro summability $|C,\gamma|$. For the real Lipschitz classes the complete results are as follows.

(A) *Let $f \in \mathrm{Lip}(\alpha, p)$, where $p \geqslant 1$, $0 < \alpha \leqslant 1$, and let $\gamma > -\alpha$. Then the Fourier series of f is summable (C,γ) whenever it is summable (C) (and therefore p.p.). Moreover, if $p > 1$ and $1/p < \alpha \leqslant 1$, the summability (C,γ) is uniform in $[-\pi,\pi]$.*

(B) *Let $f \in \mathrm{Lip}(\alpha, p)$, where $p \geqslant 1$, $0 < \alpha \leqslant 1$, and let $\gamma > \max\{1/p-\alpha, 1/2-\alpha\}$. Then the Fourier series of f is summable $|C,\gamma|$ p.p. Moreover, if $p > 1$ and $1/p < \alpha \leqslant 1$, the series is summable $|C,\gamma|$ everywhere.*

Here the case $p \geqslant 1$, $\alpha = 1/p$ of (A) is Theorem 1 of the present paper, and the second part of (A) is Theorem 7. The remaining cases of the first part of (A) were proved by H. C. Chow, *J. London Math. Soc.* 26 (1951), 290–4. The case $p > 1$, $1/p < \alpha \leqslant 1, \gamma = 0$ of (B) is Theorem 8, the case $p = \alpha = 1$ is due to L. S. Bosanquet, *J. London Math. Soc.* 11 (1936), 11–15, and the remaining cases to H. C. Chow and N. Matsuyama (for references see Chow, loc. cit.). A unified treatment of (A) and (B) is given by T. M. Flett, *Proc. London Math. Soc.* (3), 8 (1958), 357–87. Chow's and Flett's results are proved for power series ϕ belonging to the complex class $\mathrm{Lip}(\alpha, p)$ rather than for real-valued functions f, but, by Lemma 14, there is complete equivalence between the power series and Fourier series results except in the case $\alpha = p = 1$. For the case $\alpha = p = 1$ there is not equivalence, since the property that $f \in \mathrm{Lip}(1,1)$ does not imply that the associated power series ϕ belongs to the complex class $\mathrm{Lip}(1,1)$ (see § 3.5). For the complex class $\mathrm{Lip}(1,1)$ we have the following result.

(C) *If ϕ belongs to the complex class $\mathrm{Lip}(1,1)$, and $\phi(z) = \sum c_n z^n$ $(|z| < 1)$, then $\sum c_n e^{ni\theta}$ is absolutely convergent.*

This is a consequence of Theorem 3 (which shows that $\phi \in \mathrm{Lip}(1,1)$ if and only if ϕ' belongs to the Hardy class H^1) combined with Theorem 16 of 1926, 7.

There are extensions of (A), (B), and (C) for power series valid for index $p < 1$ (see Chow, loc. cit.).

§ 1. It should be noted that the O in the definition (1.2) of a function of class $\mathrm{Lip}(\alpha, p)$ is uniform in θ. The same applies to the o in the case of $\mathrm{Lip}^*(\alpha, p)$.

§ 4.1. The result of Theorem 2 is extended to the case $0 < p < 1$ in 1932, 4, Theorem 27. The result for the majorant $\sum |c_n| r^n$ given at the end of § 4.1 can also be extended to $p < 1$, and a proof is given in 1932, 4, Theorem 29.

§ 4.2. The result of Theorem 3 has been extended to the case $0 < p < 1$ by A. E. Gwilliam, *Proc. London Math. Soc.* (2), 40 (1936), 353–64.

The proof of Theorem 4, which is omitted here, is given in 1932, 4, Theorem 40.

§ 5. An alternative proof of Theorem 5 has been given by V. P. Il'in, *Trudy Mat. Inst. Steklov*, 53 (1959), 128–44.

NOTES ON THE THEORY OF SERIES (IX): ON THE ABSOLUTE CONVERGENCE OF FOURIER SERIES

G. H. HARDY *and* J. E. LITTLEWOOD*.

[*Extracted from the Journal of the London Mathematical Society, Vol.* 3, *Part* 4.]

1. The Fourier series of $f(x)$ is uniformly convergent if either

(*a*) $f(x)$ is continuous and of bounded variation, or

(*b*) $f(x)$ satisfies a Lipschitz condition

$$f(x+h)-f(x) = O(|h|^{\alpha}) \quad (\alpha > 0)$$

* Received 25 July, 1928; read 8 November, 1928.

1928, 13 (with J. E. Littlewood) *Journal of the London Mathematical Society*, 3, 250–3.

uniformly in x; but neither of these conditions necessitates the absolute convergence of the series. Thus

$$f(x) \sim \Sigma \frac{\sin nx}{n \log n}$$

is continuous and of bounded variation (indeed absolutely continuous), and

$$f(x) \sim \Sigma \frac{e^{ain \log n}}{n} e^{nix} \quad (a > 0)$$

satisfies a Lipschitz condition of order $\frac{1}{2}$, but neither series is absolutely convergent*.

It is known that the series is absolutely convergent (i) if $f(x)$ *and its conjugate* $g(x)$ are of bounded variation†, or (ii) if $f(x)$ satisfies a Lipschitz condition of order greater than $\frac{1}{2}$‡; and the last condition has been generalized by Szász, Titchmarsh, and ourselves§. There are, however, few simple criteria for absolute convergence.

In a recent note in the *Journal*‖ Zygmund proves that the series is absolutely convergent if $f(x)$ satisfies *both* of the conditions (a) and (b). Our object here is to extend this very curious theorem and to show its relations to the theorems which we have proved in our paper **4** in the *Mathematische Zeitschrift*¶.

2. We say in the paper just referred to that $f(x)$ belongs to Lip (α, p), where $0 < \alpha \leqslant 1$, $p \geqslant 1$, if

$$\left(\int_{-\pi}^{\pi} |f(x+h)-f(x)|^p \, dx\right)^{1/p} = O(|h|^\alpha),$$

and to Lip α if it satisfies an ordinary Lipschitz condition of order α. A function of Lip (α, p) belongs to Lip (α, q) for $1 \leqslant q < p$, and a function of Lip α belongs to Lip (α, p) for all $p \geqslant 1$. The class Lip α may be regarded roughly as the limit of Lip (α, p) for $p = \infty$. The class Lip $(1, p)$, where $p > 1$, is that of integrals of functions of the Lebesgue class L^p, and the class Lip $(1, 1)$ is that of functions equivalent to functions of bounded variation**.

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* The sums extend over 2, 3, ... and 1, 2, ... respectively.

† Hardy and Littlewood, **2**, 163, 208.

‡ S. Bernstein, **1**.

§ Szász, **5**, **6**; Titchmarsh, **7**; Hardy and Littlewood, **4**, 631.

‖ Zygmund, **8**.

¶ Hardy and Littlewood, **4**.

** See Hardy and Littlewood, **3**, 599.

It is known that the Fourier series of $f(x)$ is uniformly convergent* if $f(x)$ belongs to Lip (a, p) and $ap > 1$, and absolutely convergent if also $p \leqslant 2$. In order to deduce Zygmund's theorem from these results we require a lemma embodying the "convexity" property of the class Lip (α, p).

LEMMA. *If $f(x)$ belongs to* Lip (α_1, p_1) *and to* Lip (α_2, p_2), *where* $p_1 < p_2$, *it belongs to* Lip (α, p), *where*

$$a = a_1 \frac{p_1(p_2-p)}{p(p_2-p_1)} + a_2 \frac{p_2(p-p_1)}{p(p_2-p_1)},$$

for $p_1 \leqslant p \leqslant p_2$. The result is still true when $p_2 = \infty$; if $f(x)$ belongs to Lip (α_1, p_1) *and to* Lip α_2, *it belongs to* Lip (a, p), *where*

$$a = a_2 + (a_1 - a_2)\frac{p_1}{p},$$

for $p \geqslant p_1$.

In fact, writing Δ for $f(x+h)-f(x)$, we have, by Hölder's inequality,

$$\int |\Delta|^p\,dx \leqslant \left(\int |\Delta|^{p_1}dx\right)^{(p_2-p)/(p_2-p_1)} \left(\int |\Delta|^{p_2}dx\right)^{(p-p_1)/(p_2-p_1)} = O(|h|^{ap}).$$

If $p_2 = \infty$,

$$\int |\Delta|^p\,dx = O(|h|^{a_2(p-p_1)}\,.\,|h|^{a_1 p_1}) = O(|h|^{ap}).$$

These equations prove the lemma.

In order to deduce Zygmund's theorem, we take $a_1 = 1$, $p_1 = 1$, $a_2 = \delta > 0$, $p_2 = \infty$. It follows that $f(x)$ belongs to Lip (a, p), where

$$pa = 1 + \delta(p-1) > 1,$$

for any $p > 1$. Hence the series is absolutely convergent.

3. It is obvious that we can prove more general results of the same character, since the series will be absolutely convergent whenever

$$a_1 p_1(p_2-p) + a_2 p_2(p-p_1) > p_2 - p_1$$

* Indeed summable $(C, -a+\delta)$ for every $\delta > 0$. When $ap = 1$, the series is convergent whenever summable by any Cesàro mean. See Hardy and Littlewood, **4**, 631.

for a $p \leqslant 2$. The most interesting cases are

(i) *the series is absolutely convergent if* $f(x)$ *is of bounded variation and belongs to any class* Lip (α, p) *for which* $\alpha p > 1$;

(ii) *the series is absolutely convergent if* $f(x)$ *belongs to any class* Lip (α, p) *for which* $\alpha p = 1$, $p < 2$, *and also satisfies any ordinary Lipschitz condition.*

The first of these reduces to Zygmund's theorem when $\alpha = \delta > 0$, $p = \infty$, and the second when $\alpha = 1$, $p = 1$. The second theorem becomes false for $p = 2$; the second special series mentioned in § 1 belongs to Lip $\frac{1}{2}$ and *a fortiori* to Lip $(\frac{1}{2}, 2)$.

We may observe in conclusion that the "absolute convergence problem" for Fourier series admits, in a sense, a complete solution : *the necessary and sufficient condition that the Fourier series of* $f(x)$ *should be absolutely convergent is that* $f(x)$ *should be the "Fourier Faltung"*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\, h(x-t)\, dt$$

of two functions $g(x)$ *and* $h(x)$ *of the Lebesgue class* L^2. In fact, if the complex Fourier series of $g(x)$ and $h(x)$ are $\Sigma a_m e^{mix}$ and $\Sigma b_m e^{mix}$, then that of $f(x)$ is $\Sigma a_m b_m e^{mix}$, which is absolutely convergent. And if the Fourier series $\Sigma c_m e^{mix}$ of $f(x)$ is absolutely convergent, we may take $a_m = |c_m|^{\frac{1}{2}}$, $b_m = |c_m|^{\frac{1}{2}} \operatorname{sgn} c_m$. We owe the substance of this remark to Prof. M. Riesz. There is, however, no obvious criterion for deciding directly whether a given function $f(x)$ is of the form prescribed.

References.

1. S. Bernstein, "Sur la convergence absolue des séries trigonométriques", *Comptes rendus*, 158 (1914), 1661–1663.

2. G. H. Hardy and J. E. Littlewood, "Some new properties of Fourier constants", *Math. Annalen*, 97 (1926), 159–209.

3. G. H. Hardy and J. E. Littlewood, "Some properties of fractional integrals (I)", *Math. Zeitschrift*, 27 (1928), 563–606.

4. G. H. Hardy and J. E. Littlewood, "A convergence criterion for Fourier series", *ibid.*, 28 (1928), 612–634.

5. O. Szász, "Über den Konvergenzexponenten der Fourierschen Reihen gewisser Funktionenklassen", *Münchener Sitzungsberichte*, 1922, 135–150.

6. O. Szász, "Über die Fourierschen Reihen gewisser Funktionenklassen", *Math. Annalen*, not yet published.

7. E. C. Titchmarsh, "A note on Fourier transforms", *Journal London Math. Soc.*, 2 (1927), 148–150.

8. A. Zygmund, "Remarque sur la convergence absolue des séries de Fourier", *Journal London Math. Soc.*, 3 (1928), 194–196.

COMMENTS

There are extensions of the results of this paper, similar to the extension of Theorem 8 of 1928, 6, in which absolute convergence is replaced by absolute Cesàro summability $|C, \alpha|$. For instance, by combining the argument of this paper with the case $p = 2$ of Theorem (B) stated in the comments on 1928, 6, we obtain the following extension of Zygmund's theorem: *If f is of bounded variation and belongs to* Lip α, *where* $0 < \alpha \leqslant 1$, *then the Fourier series of f is summable $|C, \gamma|$ for $\gamma > -\frac{1}{2}\alpha$* (see e.g. K. K. Chen, *Amer. J. of Math.* 66 (1944), 299–312, where this and other extensions are given).

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

LXIX.

On asymptotic values of Fourier constants.

1. THE theorems proved here were suggested by Mr. Haslam-Jones' note 'On the Fourier coefficients of unbounded functions' published recently*, and by the earlier paper of W. H. Young† to which Haslam-Jones' refers; but they are not of quite the same type as the results of these writers, and the method of proof is quite different.

2. THEOREM 1. *Suppose that $\phi(\theta)$ is periodic and integrable, that*

$$\phi(\theta) \sim \tfrac{1}{2}a_0 + \Sigma(a_n \cos n\theta + b_n \sin n\theta),$$

that

$$a_n = O\left(\frac{1}{n}\right), \quad b_n = O\left(\frac{1}{n}\right),$$

and that $\tfrac{1}{2}a_0 + \Sigma a_n$ is convergent and equal to s. Finally suppose that $-\pi < \alpha < \pi$ and $0 < \rho < 1$. Then

(2.1)

$$k_n = \int_{-\pi}^{\pi} \frac{\phi(\theta-\alpha)}{|\theta-\alpha|^\rho} \cos n(\theta-\alpha)\, d\theta \sim 2s\Gamma(1-\rho)\sin\tfrac{1}{2}\rho\pi\, |n|^{\rho-1}.$$

Supposing n positive, we have

$$k_n = \tfrac{1}{2}(J_n + J_{-n}),$$

where

$$J_n = \int_{-\pi}^{\pi} \frac{\phi(\theta-\alpha)}{|\theta-\alpha|^\rho} e^{ni|\theta-\alpha|}\, d\theta = \int_{-\pi-\alpha}^{\pi-\alpha} \frac{\phi(x)}{|x|^\rho} e^{nix}\, dx.$$

Now we may write

$$\phi(x) \sim \sum_{-\infty}^{\infty} c_\mu e^{\mu i x},$$

where

(2.2) $\quad c_\mu = \tfrac{1}{2}(a_\mu - ib_\mu),\ a_\mu = a_{-\mu},\ b_\mu = -b_{-\mu},\ b_0 = 0$;‡

* U. S. Haslam-Jones, 'A note on the Fourier coefficients of unbounded functions', *Journal London Math. Soc.*, 2 (1927), 151–154.

† W. H. Young, 'On the order of magnitude of the coefficients of a Fourier series', *Proc. Roy. Soc.* (A), 93 (1917), 42–55.

‡ Cf. G. H. Hardy and J. E. Littlewood, 'On Parseval's Theorem', *Proc. London Math. Soc.* (2), 26 (1927), 287–294.

and $$\frac{1}{|x|^\rho} = \sum_{-\infty}^{\infty} l_\nu e^{-\nu ix} \quad (-\pi-\alpha \leqq x \leqq \pi-\alpha,\ x \neq 0);$$

and we may calculate J_n by substituting these Fourier series under the integral sign and carrying out the formal process of integration*. If we do this, we obtain

$$J_n = 2\pi \sum_{-\infty}^{\infty} c_\mu l_{\mu+n},$$

the series being absolutely convergent, since

$$c_\mu = O(|\mu|^{-1}), \quad l_\mu = O(|\mu|^{\rho-1});$$

and so

$$(2.3) \qquad k_n = \tfrac{1}{2}(J_n + J_{-n}) = \pi \sum_{-\infty}^{\infty} c_\mu (l_{\mu+n} + l_{\mu-n}).$$

Collecting the terms corresponding to pairs of positive and negative values of μ, we find

$$(2.4) \qquad k_n = \pi (\tfrac{1}{2}a_0 P_0 + \sum_1^{\infty} a_\mu P_\mu - i \sum_1^{\infty} b_\mu Q_\mu,$$

where

$$(2.5) \qquad \begin{cases} P_\mu = \tfrac{1}{2}(l_{\mu+n} + l_{\mu-n} + l_{-\mu+n} + l_{-\mu-n}), \\ Q_\mu = \tfrac{1}{2}(l_{\mu+n} + l_{\mu-n} - l_{-\mu+n} - l_{-\mu-n}). \end{cases}$$

3. The term l_0 occurs in (2.4) only when $\mu = n$, and its contribution is $O(1/n)$ and may be neglected.

If $\nu \neq 0$ we have

$$l_\nu = \frac{1}{2\pi} \int_{-\pi-\alpha}^{\pi-\alpha} \frac{e^{-\nu ix}}{|x|^\rho} dx = l_\nu' + l_\nu'',$$

where

$$l_\nu' = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\nu ix}}{|x|^\rho} dx = \frac{1}{\pi} \int_0^{\infty} \frac{\cos \nu x}{x^\rho} dx$$
$$= \frac{\Gamma(1-\rho)}{\pi} \sin \tfrac{1}{2}\rho\pi\, |\nu|^{\rho-1},$$

and

$$l_\nu'' = -\frac{1}{2\pi} \int_{-\infty}^{-\pi-\alpha} - \frac{1}{2\pi} \int_{\pi-\alpha}^{\infty} = O\left(\frac{1}{|\nu|}\right)$$

* See W. H. Young, 'Sur la généralisation du théorème de Parseval', *Comptes Rendus*, 1 July 1912.

(on integrating by parts). Substituting in (2.4) and (2.5), we consider first the contribution of the terms l_ν''; and this is plainly less than a constant multiple of

$$\sum_{-\infty}^{\infty}{}' \frac{1}{\mu\,|\mu+n|} \leq 2 \sum_{1}^{\infty}{}' \frac{1}{\mu\,|n-\mu|} \text{ *}$$

$$= 2 \sum_{1}^{n-1} \frac{1}{\mu\,(n-\mu)} + 2 \sum_{n+1}^{\infty} \frac{1}{\mu\,(\mu-n)} = O\left(\frac{\log n}{n}\right).$$

If now we observe that l_ν' is an even function of ν, we obtain, from (2.4) and (2.5),

$$k_n = \pi\,\{\tfrac{1}{2}a_0(l_n' + l_{-n}') + \sum_{1}^{\infty}{}' a_\mu (l'_{\mu+n} + l'_{\mu-n})\} + O\left(\frac{\log n}{n}\right),$$

$$(3.1)\quad k_n = \Gamma(1-\rho)\sin\tfrac{1}{2}\rho\pi\,\{a_0 n^{\rho-1}$$

$$+ \sum_{1}^{\infty}{}' a_n \left[(n+\mu)^{\rho-1} + |\mu-n|^{\rho-1}\right]\} + O\left(\frac{\log n}{n}\right),$$

where the dash excludes $\mu = n$.

4. It is now plain that Theorem 1 will follow from

THEOREM 2. *If $a_1 + a_2 + \ldots$ converges to t, then*

$$(4.1)\qquad \sum_{1}^{\infty} a_\mu (\mu+n)^{\rho-1} \sim t n^{\rho-1}.$$

If also $a_n = O(1/n)$, then

$$(4.2)\qquad \sum_{1}^{\infty}{}' a_\mu\,|n-\mu|^{\rho-1} \sim t n^{\rho-1}.\dagger$$

The first part of the theorem is trivial. We suppose, as plainly we may without real loss of generality, that $t = 0$. We choose $N = N(\epsilon)$ so that

$$\left|\sum_{N+1}^{\nu} a_\mu\right| < \epsilon \quad (\nu > N).$$

Then

$$S = \sum_{1}^{\infty} a_\mu \left(\frac{n}{\mu+n}\right)^{1-\rho} = \sum_{1}^{N} + \sum_{N+1}^{\infty} = S_1 + S_2,$$

* The dashes, as usual, excluding from the sums terms whose denominators vanish.

† The second half of the theorem is very like the theorem that the hypotheses imply the summability of Σa_n by Cesàro means of any order greater than -1. See G. H. Hardy and J. E. Littlewood, 'Contributions to the arithmetic theory of series', *Proc. Lond. Math. Soc.* (2), 11 (1912), 411–478 (462, Theorem 37).

say. Here

$$|S_2| \leqq \left(\frac{n}{N+n}\right)^{1-\rho} \underset{\nu>N}{\operatorname{Max}} \left|\sum_{N+1}^{\nu} a_\mu\right| < \epsilon,$$

and $S_1 \to a_1 + a_2 + \ldots + a$ when N is fixed. Hence

$$|S-t| < \epsilon + \left|\sum_1^N a_\mu\right| + \left|S_1 - \sum_1^N a_\mu\right| < 3\epsilon$$

for $n > n_0(\epsilon, N) = n_0(\epsilon)$, which proves (4.1).

To prove (4.2) we write

$$S = \Sigma' a_\mu |\mu - n|^{\rho-1} = \sum_1^N + \sum_{N+1}^{n-\delta n} + \sum_{n-\delta n}^{n-1} + \sum_{n+1}^{n+\delta n} + \sum_{n+\delta n}^{\infty}$$
$$= S_1 + S_2 + S_3 + S_4 + S_5,$$

say, where $1 < N < n - \delta n < n$. We may conveniently suppose δ irrational, so as to avoid ambiguities about the limits. We also suppose $t = 0$, as before.

We have

$$|S_4| = O\left\{\sum_{n+1}^{n+\delta n} \frac{1}{\mu(\mu-n)^{1-\rho}}\right\} = O\left\{\int_n^{n+\delta n} \frac{dy}{y(y-n)^{1-\rho}}\right\}$$
$$= O\left\{n^{\rho-1}\int_1^{1+\delta} \frac{dw}{w(w-1)^{1-\rho}}\right\},$$

the constants implied by the O's being independent of n and δ. The integral here tends to zero with δ, and we can choose $\delta = \delta(\epsilon)$ so that

$$|S_4| < \epsilon n^{\rho-1}.$$

Similarly $$|S_3| < \epsilon n^{\rho-1}.$$

When δ is fixed, we can choose $N = N(\epsilon, \delta) = N(\epsilon)$ so that

$$\left|\sum_1^N a_\mu\right| < \epsilon\delta^{1-\rho}, \quad \underset{\nu'>\nu>N}{\operatorname{Max}} \left|\sum_\nu^{\nu'} a_\mu\right| < \epsilon\delta^{1-\rho},$$

the first inequality being a corollary of the second, since $t=0$. We have then

$$|S_2| \leqq (\delta n)^{\rho-1} \underset{N<\nu<n-\delta n}{\operatorname{Max}} \left|\sum_\nu^{n-\delta n} a_\mu\right| < \epsilon n^{\rho-1};$$

and a similar argument shows that

$$|S_5| < \epsilon n^{\rho-1}.$$

Finally, when N is fixed, and $n \to \infty$,

$$S_1 \sim n^{\rho-1} \sum_1^N a_\mu,$$

so that
$$|S_1| < n^{\rho-1} \left(\left|\sum_1^N a_\mu\right| + \epsilon\right) < 2\epsilon n^{\rho-1}$$

for $n \geqq n_0 = n_0(\epsilon, N) = n_0(\epsilon)$. Collecting our results, we have

$$|S| < 6\epsilon n^{\rho-1} \quad (n \geqq n_0);$$

which proves the theorem.

5. Theorem 2 implies Theorem 1, as we saw in § 3. We may prove similarly

THEOREM 3. *If the conditions of Theorem 1 are satisfied, except that it is Σb_n which is convergent, then*

$$k_n = \int_{-\pi}^{\pi} \frac{\phi(\theta - \alpha)}{|\theta - \alpha|^\rho} \sin n(\theta - \alpha)\, d\theta = o(|n|^{\rho-1}).$$

If $\phi(\theta) = \psi(\theta - \alpha)$, and the Fourier coefficients of $\psi(\theta)$ are A_n and B_n, then the Fourier series of $\phi(\theta)$ is

$$\tfrac{1}{2}A_0 + \Sigma\{(A_n \cos n\alpha - B_n \sin n\alpha)\cos n\theta$$
$$+ (B_n \cos n\alpha + A_n \sin n\alpha)\sin n\theta\},$$

which reduces to $\frac{1}{2}A_0 + \Sigma A_n$ for $\theta = \alpha$. Similarly the allied series reduces to ΣB_n. Hence, combining Theorems 1 and 3, we find

THEOREM 4. *If $\phi(\theta)$ is periodic and integrable, its Fourier constants are $O(1/n)$, and both its Fourier series and the allied series are convergent for $\theta = \alpha$, then*

$$\int_{-\pi}^{\pi} \frac{\phi(\theta)}{|\theta - \alpha|^\rho} \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta\, d\theta$$
$$= 2s\Gamma(1-\rho) \sin \tfrac{1}{2}\rho\pi \begin{matrix}\cos\\ \sin\end{matrix}\, n\alpha\, |n|^{\rho-1} + o(|n|^{\rho-1}),$$

where s is the sum of the Fourier series.

There is a similar theorem concerning the integrals

$$\int_{-\pi}^{\pi} \phi(\theta) \frac{\operatorname{sgn}(\theta - \alpha)}{|\theta - \alpha|^\rho} \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta . d\theta,$$

in which the roles of the Fourier series and the allied series are interchanged.

It should be observed that Theorem 4 becomes false if the condition that the *allied* series is convergent be omitted. If for example we take $\alpha = 0$ and

$$\phi(\theta) = \Sigma \frac{\sin n\theta}{n} = \tfrac{1}{2}(\pi - \theta),\ 0,\ -\tfrac{1}{2}(\pi + \theta)$$

according as θ is positive, zero, or negative, then the Fourier series converges to 0 but the allied series diverges. Here

$$\int_{-\pi}^{\pi} \frac{\phi(\theta)}{|\theta|^{\rho}} \cos n\theta \, d\theta = 0 = o(|n|^{\rho-1}),$$

in agreement with Theorem 1; but

$$\int_{-\pi}^{\pi} \frac{\phi(\theta)}{|\theta|^{\rho}} \sin n\theta \, d\theta = \int_{0}^{\pi} \frac{\sin n\theta}{\theta^{\rho}} (\pi - \theta) \, d\theta \sim \pi \int_{0}^{\infty} \frac{\sin n\theta}{\theta^{\rho}} \, d\theta$$

$$= \pi \Gamma(1-\rho) \cos \tfrac{1}{2}\rho\pi \, |n|^{\rho-1} \operatorname{sgn} n,$$

and is not $o(|n|^{\rho-1})$.

5. If $\alpha = 0$ and $\phi(\theta)$ is of bounded variation, then its Fourier constants are $O(1/n)$ and its Fourier series convergent, so that the result of Theorem 4 holds so far as the cosine integral is concerned. We thus obtain one of Haslam-Jones' results, viz. his Theorem 1 with (in his notation) $\alpha = 0$ and for a_n only. The corresponding result for b_n is not included in any of the preceding theorems.*

Mr. Haslam-Jones has pointed out to me an alternative form of Theorem 3: *the result of Theorem* 3 *holds if*

$$a_n = O(1/n), \quad b_n = o(1/n),$$

without any assumption of the convergence of Σb_n; *and if*

$$nb_n \to B \neq 0,$$

then $$k_n \sim \pi B \Gamma(1-\rho) \cos \tfrac{1}{2}\rho\pi \, |n|^{\rho-1} \operatorname{sgn} n.$$

This does not include Theorem 3, but the hypotheses are in some ways more natural.

* These results are those for which Haslam-Jones refers to Bromwich.

NOTES ON THE THEORY OF SERIES (XVII): SOME NEW CONVERGENCE CRITERIA FOR FOURIER SERIES

G. H. Hardy *and* J. E. Littlewood*.

[*Extracted from the Journal of the London Mathematical Society, Vol.* 7, *Part* 4.]

1. In this note we are concerned with the convergence of a Fourier series in the classical sense. We make the usual formal simplifications; we consider an integrable, periodic, and even function $\phi(t)$, and investigate conditions under which its Fourier series $\Sigma a_n \cos nt$ converges to zero when $t = 0$. It is convenient to suppose also that the mean value of $\phi(t)$ over a period is zero, so that $a_0 = 0$†.

2. Theorem 1. *If* (i)

$$\phi(t) = o\left\{\frac{1}{\log|1/t|}\right\}, \tag{2.1}$$

and (ii)

$$a_n = O(n^{-\delta}) \tag{2.2}$$

for some positive δ, *then*

$$\Sigma a_n = 0. \tag{2.3}$$

We suppose, as we may, that

$$|a_n| < n^{-\delta}, \tag{2.4}$$

and that $\delta < 1$. We choose a positive c, and take $r = \frac{1}{2}\delta$. Then it is necessary and sufficient for convergence that

$$S(\lambda) = \int_0^c \phi(t)\frac{\sin \lambda t}{t}\,dt = \int_0^{\lambda^{-1}} + \int_{\lambda^{-1}}^{\lambda^{-r}} + \int_{\lambda^{-r}}^{c} = S_1(\lambda) + S_2(\lambda) + S_3(\lambda) \tag{2.5}$$

should tend to zero when $\lambda \to \infty$.

In the first place,

$$S_1(\lambda) = o\left(\int_0^{\lambda^{-1}} \lambda\,dt\right) \to 0. \tag{2.6}$$

Next,

$$|S_2(\lambda)| < \epsilon \int_{\lambda^{-1}}^{\lambda^{-r}} \frac{dt}{t\log(1/t)} = -\epsilon\left[\log\log\frac{1}{t}\right]_{\lambda^{-1}}^{\lambda^{-r}} = \epsilon\log\frac{1}{r} = \epsilon\log\frac{2}{\delta}$$

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* Received 13 August, 1932; read 10 November, 1932.

† We can secure this by adding an appropriate $A + B\cos t$ to $\phi(t)$.

if $\lambda > \lambda_0(\epsilon)$, and so

(2.7) $$S_2(\lambda) \to 0.$$

In $S_3(\lambda)$, we replace $\phi(t)$ by its Fourier series and integrate term by term. We thus obtain

(2.8) $$S_3(\lambda) = \Sigma a_n \int_{\lambda^{-r}}^{c} \cos nt \frac{\sin \lambda t}{t} dt$$

$$= \tfrac{1}{2}\Sigma a_n u_n + \tfrac{1}{2}\Sigma a_n v_n = \tfrac{1}{2}U(\lambda) + \tfrac{1}{2}V(\lambda),$$

where

(2.81) $$u_n = \int_{(\lambda+n)\lambda^{-r}}^{(\lambda+n)c} \frac{\sin w}{w} dw = Si \frac{\lambda+n}{\lambda^r} - Si(\lambda+n)c,$$

(2.82) $$v_n = \operatorname{sgn}(\lambda-n) \int_{|\lambda-n|\lambda^{-r}}^{|\lambda-n|c} \frac{\sin w}{w} dw$$

$$= \operatorname{sgn}(\lambda-n)\left(Si \frac{|\lambda-n|}{\lambda^r} - Si|\lambda-n|c\right),$$

(2.83) $$Si\, x = \int_x^\infty \frac{\sin w}{w} dw.$$

3. We denote by A a positive constant, by C a positive number depending on δ only. We have

$$|Si\, x| < A \quad (x \geqslant 0), \qquad |Si\, x| < \frac{A}{x} \quad (x \geqslant 1).$$

Hence

(3.1) $$|U(\lambda)| \leqslant \Sigma |a_n u_n| \leqslant A\lambda^r \Sigma \frac{|a_n|}{\lambda+n} < A\lambda^r \Sigma \frac{n^{-\delta}}{\lambda+n}$$

$$< A\lambda^r \int_0^\infty \frac{x^{-\delta}}{\lambda+x} dx = AC\lambda^{r-\delta} = AC\lambda^{-\frac{1}{2}\delta} \to 0.$$

We write $V(\lambda)$ in the form

(3.2) $$V = \Sigma a_n v_n = \sum_{n<\lambda-\lambda^r} a_n v_n + \sum_{|\lambda-n|\leqslant\lambda^r} a_n v_n + \sum_{n>\lambda+\lambda^r} a_n v_n = V_1 + V_2 + V_3.$$

Here, first,

(3.3) $$|V_2| < (2\lambda^r+1)(\lambda-\lambda^r)^{-\delta} < C\lambda^{-\frac{1}{2}\delta} \to 0.$$

Secondly,

$$|V_1| < A\lambda^r \sum_{n<\lambda-\lambda^r} \frac{n^{-\delta}}{\lambda-n}. \tag{3.4}$$

Since $x^\delta(\lambda-x)$ has one maximum, given by $x = C\lambda$, between 0 and λ, (3.4) gives

$$|V_1| < C\lambda^{r-\delta-1} + C\lambda^r \int_0^{\lambda-\lambda^r} \frac{x^{-\delta}}{\lambda-x}\,dx.$$

The first term tends to zero, and the second is less than

$$C\lambda^r \int_{\lambda^r}^{\lambda} (\lambda-w)^{-\delta}\frac{dw}{w} = C\lambda^{r-\delta}\int_{\lambda^{r-1}}^{1}(1-y)^{-\delta}\frac{dy}{y} < C\lambda^{-\frac{1}{2}\delta}\log\lambda \to 0.$$

Hence $V_1 \to 0$; and a similar, but rather simpler, argument shows that $V_3 \to 0$. From these results, together with (3.2) and (3.3), we conclude that

$$S_3(\lambda) \to 0. \tag{3.5}$$

The theorem follows from (2.5), (2.6), (2.7), and (3.5).

4. The criterion contained in Theorem 1 involves (i) a "continuity condition", viz. (2.1), and an "order condition" on the Fourier constant of $\phi(t)$, viz. (2.2). One criterion of this character is familiar: *if* $\phi(t) \to 0$, *and* $a_n = O(n^{-1})$, *then* $\Sigma a_n = 0$. In this case Σa_n is summable $(C, 1)$, by Fejér's theorem, and convergent by a known "Tauberian" theorem. Here the minimum is required of $\phi(t)$ in the way of continuity, but the order condition on a_n is stringent. In Theorem 1 the continuity condition is strengthened and the order condition greatly relaxed.

It should be observed that (2.1) is not in itself a sufficient condition for convergence, although

$$\phi(t+h)-\phi(t) = o\left\{\frac{1}{\log(1/h)}\right\},$$

for $h > 0$, and uniformly in t,* is a sufficient condition for uniform convergence. It may in fact be shown, by an appropriate modification of Fejér's well-known construction for divergent Fourier series, that Theorem 1 is a best possible theorem in two different senses. If we keep (2.1) unaltered,

* The criterion of "Dini-Lipschitz".

(2.2) cannot be replaced by any condition of the type

$$a_n = O(n^{-\eta_n}),$$

where $\eta_n \to 0$ when $n \to \infty$; and if we keep (2.2) unaltered, (2.1) cannot be replaced by any condition of the type

$$\phi(t) = o\left\{\frac{\eta(t)}{\log|1/t|}\right\},$$

where $\eta(t) \to \infty$ when $t \to 0$.

5. There are, however, other directions in which Theorem 1 may be extended. In the first place, we may replace the "continuity condition" (2.1) by some form of "average continuity" condition. This generalization, which corresponds to Lebesgue's generalization of Fejér's theorem, is straightforward.

There is another generalization, perhaps more interesting, which we can only prove indirectly. The first step is to prove a theorem concerning the summability of Fourier series by Borel's exponential method.

THEOREM 2. *If $\phi(t)$ satisfies* (2.1), *then Σa_n is summable* (B) *to sum* 0.

We can combine this theorem with known Tauberian theorems. It has been proved by various writers that, if $a_n = O(n^{-\frac{1}{2}})$, or, more generally, if $a_n > -An^{-\frac{1}{2}}$, then Σa_n, if summable (B), is necessarily convergent*. From this we see that either of these conditions, together with (2.1), is sufficient for convergence.

This argument will not, as it stands, prove Theorem 1; for this we must use more general methods of summation. The "Tauberian index" of Borel's method is $\frac{1}{2}$, and we require a method whose Tauberian index is δ. The appropriate generalizations of Borel's method have been studied by Valiron†. If $1 \leqslant k \leqslant 2$ and

$$s_n = a_0 + a_1 + \ldots + a_n \quad (n \geqslant 0), \qquad s_n = 0 \quad (n < 0),$$

and if
$$\lim_{n\to\infty} \frac{n^{\frac{1}{2}k-1}}{\sqrt{(2\pi)}} \sum_{m=-\infty}^{\infty} e^{-\frac{1}{2}m^2 n^{k-2}} s_{m+n} = s,$$

then we say that Σa_n is summable (V, k) to s. When $k = 1$ the method is (at any rate for Fourier series) equivalent to Borel's.

It is not difficult to show that the result of Theorem 2 applies also to summability (V, k). Further, the Tauberian index of the method is

* Hardy and Littlewood (1), R. Schmidt (2), Vijayaraghavan (4), Wiener (5).

† Valiron (3).

$1-\frac{1}{2}k$, which is δ if $k=2-2\delta$*. In this way we are led to a theorem which includes and generalizes Theorem 1.

THEOREM 3. *It is sufficient for convergence that $\phi(t)$ should satisfy* (2.1) *and that*

(5.1) $$a_n > -An^{-\delta}.$$

6. Theorems 1–3 belong to a group which we intend to discuss more systematically elsewhere. We state here one further theorem of the group.

THEOREM 4. *If* (i) *$\phi(t)$ satisfies* (2.1), (ii) *$\phi(t)$ is an integral except at $t=0$, and* (iii)

$$\phi'(t) > -At^{-\Delta}$$

for some Δ, then $\Sigma a_n = 0$.

Here there is no "order condition", but an additional condition on $\phi(t)$. The theorem corresponds to Young's convergence criterion (or rather to a particular case of this criterion) as Theorem 1 corresponds to the theorem quoted at the beginning of § 4.

References.

1. G. H. Hardy and J. E. Littlewood, "Theorems concerning the summability of series by Borel's exponential method", *Rend. di Palermo*, 41 (1916), 36-53.
2. R. Schmidt, "Über das Borelsche Summierungsverfahren", *Schriften der Königsberger gelehrten Gesellschaft*, 1 (1925), 202–256.
3. G. Valiron, "Remarques sur la sommation des séries divergentes par les méthodes de M. Borel", *Rend. di Palermo*, 42 (1917), 267–284.
4. T. Vijayaraghavan, "A theorem concerning the summability of series by Borel's method", *Proc. London Math. Soc.* (2), 27 (1928), 316–326.
5. N. Wiener, "Tauberian theorems", *Annals of Math.*, 33 (1932), 1–100.

* Valiron proves that $a_n = O(n^{\frac{1}{2}k-1})$ is a sufficient Tauberian condition. The condition may be generalized to $a_n > -An^{\frac{1}{2}k-1}$ by the methods of Vijayaraghavan or Wiener.

COMMENTS

This paper is essentially a preliminary account of results published in full in 1934, 3. References to later work are given in the comments on the latter paper.

SOME NEW CONVERGENCE CRITERIA FOR FOURIER SERIES

by GODFREY HAROLD HARDY and JOHN EDENSOR LITTLEWOOD (Cambridge).

1. - In this paper [1] we are concerned with the convergence, in the classical sense, of the FOURIER series of an integrable function $\varphi(t)$. We suppose that $\varphi(t)$ is periodic, with period 2π, and even; that the fundamental interval is $(-\pi, \pi)$; that the mean value of $\varphi(t)$ over a period is 0; that the special point to be considered is the origin; and that the sum of the series is to be 0. In these circumstances

(1.1) $$\varphi(t) \sim \sum_{1}^{\infty} a_n \cos nt = \sum a_n \cos nt,$$

and our conclusion is to be

(1.2) $$\sum a_n = 0.$$

It is familiar that these formal simplifications do not impair the generality of the problem.

Since $\varphi(t)$ is even, any conditions which it is to satisfy may be stated for $t \geqq 0$.

Criteria containing a condition on the order of magnitude of a_n.

2. - Our main theorem (Theorem 2) involves (i) a « continuity » condition on $\varphi(t)$ and (ii) an « order » condition on a_n. One theorem of this character is known already.

THEOREM 1. - *It is sufficient that* (i)

(2.1) $$\varphi(t) \to 0$$

when $t \to 0$, *and* (ii)

(2.2) $$a_n = O(n^{-1}).$$

In fact (i) implies the summability $(C, 1)$ of the series, and (ii) then implies its convergence. As is well known, the theorem covers the classical case in which $\varphi(t)$ is of bounded variation.

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[1] A short account of some of the principal results has appeared in the Journal of the London Math. Soc. (HARDY and LITTLEWOOD, 6).

1934, 3 (with J. E. Littlewood) *Annali Pisa* (2), 3, 43–62.

It is natural to ask whether, if we strengthen the continuity condition (i), we may correspondingly relax the order condition (ii). If we replace (i) by any of

$$\varphi(t)=O(t^{a}),\quad \varphi(t)=O\left\{\left(\log\frac{1}{t}\right)^{-1-a}\right\},\quad \varphi(t)=O\left\{\left(\log\frac{1}{t}\right)^{-1}\left(\log\log\frac{1}{t}\right)^{-1-a}\right\},\dots \tag{2.3}$$

where $a>0$, we may drop (ii) entirely, these conditions being sufficient in themselves. A natural intermediate hypothesis is

$$\varphi(t)=o\left\{\left(\log\frac{1}{t}\right)^{-1}\right\}. \tag{2.4}$$

and it will be found that this hypothesis leads to very interesting results.

It should be observed first that (2.4) is not itself a sufficient condition for convergence ([2]). This is no doubt well known, though we have not met with any explicit proof; a more precise result is contained in Theorem 4 below. There is a distinction here between convergence at a point and uniform convergence, since

$$\varphi(t+h)-\varphi(t)=o\left\{\left(\log\frac{1}{|h|}\right)^{-1}\right\}, \tag{2.5}$$

uniformly in t, is a sufficient condition for the uniform convergence of the series ([3]).

3. - THEOREM 2. - *It is sufficient that* (i) $\varphi(t)$ *should satisfy* (2.4) *and that* (ii)

$$a_n=O(n^{-\delta}) \tag{3.1}$$

for some positive δ.

We suppose, as we may, that $\delta<1$ and

$$|a_n|<n^{-\delta}. \tag{3.2}$$

We choose a positive c and take

$$r=\frac{1}{2}\delta. \tag{3.3}$$

It is necessary and sufficient for convergence that

$$S(\lambda)=\int_0^c \varphi(t)\frac{\sin\lambda t}{t}\,dt\to 0 \tag{3.4}$$

when $\lambda\to\infty$.

We write

$$S(\lambda)=\int_0^{\lambda^{-1}}+\int_{\lambda^{-1}}^{\lambda^{-r}}+\int_{\lambda^{-r}}^{c}=S_1(\lambda)+S_2(\lambda)+S_3(\lambda). \tag{3.5}$$

([2]) Indeed no condition $\varphi=o(\chi)$, with

$$\int_0 \frac{\chi(t)}{t}\,dt=\infty.$$

is sufficient (in other words, the classical test of DINI is the best possible of its kind).

([3]) This is the « Dini-Lipschitz » criterion; see for example HOBSON, 7, p. 537.

Then, in the first place

$$S_1(\lambda)=o\left(\int_0^{\lambda^{-1}}\lambda dt\right)=o(1). \tag{3.6}$$

Here we do not require the full force of (2.4).

Next

$$|S_2(\lambda)|<\varepsilon\int_{\lambda^{-1}}^{\lambda^{-r}}\frac{dt}{t\log\frac{1}{t}}=-\varepsilon\left[\log\log\frac{1}{t}\right]_{\lambda^{-1}}^{\lambda^{-r}}=\varepsilon\log\frac{1}{r}=\varepsilon\log\frac{2}{\delta} \tag{3.7}$$

if $\lambda>\lambda_0(\varepsilon)$; and so

$$S_2(\lambda)=o(1). \tag{3.8}$$

It remain to consider $S_3(\lambda)$. Here we replace $\varphi(t)$ by its FOURIER series and integrate term by term. We thus obtain

$$S_3(\lambda)=\sum a_n\int_{\lambda^{-r}}^{c}\cos nt\frac{\sin\lambda t}{t}dt=\frac{1}{2}\sum a_n\left\{\int_{\lambda^{-r}}^{c}\frac{\sin(\lambda+n)t}{t}dt+\int_{\lambda^{-r}}^{c}\frac{\sin(\lambda-n)t}{t}dt\right\}= \tag{3.9}$$

$$=\frac{1}{2}\sum a_n u_n+\frac{1}{2}\sum a_n v_n=\frac{1}{2}U(\lambda)+\frac{1}{2}V(\lambda),$$

say; where

$$u_n=\int_{(\lambda+n)\lambda^{-r}}^{(\lambda+n)c}\frac{\sin w}{w}dw=Si\frac{\lambda+n}{\lambda^r}-Si(\lambda+n)c, \tag{3.10.1}$$

$$v_n=\operatorname{sgn}(\lambda-n)\int_{|\lambda-n|\lambda^{-r}}^{|\lambda-n|c}\frac{\sin w}{w}dw=\operatorname{sgn}(\lambda-n)\left(Si\frac{|\lambda-n|}{\lambda^r}-Si|\lambda-n|c\right)\ \ (^4), \tag{3.10.2}$$

$$Si\,x=\int_x^\infty\frac{\sin w}{w}dw. \tag{3.10.3}$$

4. - The function $Si\,x$ satisfies the inequalities

$$|Si\,x|<A\quad(x>0),\qquad |Si\,x|<\frac{A}{x}\quad(x>1), \tag{4.1}$$

in which A is an absolute constant. Hence

$$|u_n|<A\frac{\lambda^r}{\lambda+n},$$

$$|U(\lambda)|\leqq A\lambda^r\sum\frac{|a_n|}{\lambda+n}<A\lambda^r\sum\frac{n^{-\delta}}{\lambda+n}<A\lambda^r\int_0^\infty\frac{x^{-\delta}}{\lambda+x}dx=C\lambda^{r-\delta}.$$

Here, and in the sequel, $C=C(\delta)$ denotes a number depending only on δ. It follows that

$$U=O(\lambda^{-\frac{1}{2}\delta})=o(1). \tag{4.2}$$

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(4) $v_n=0$ if $n=\lambda$.

We write V in the form

(4.3) $$V=\sum v_n=\sum_{n<\lambda-\lambda^r} v_n+\sum_{|n-\lambda|\leqq\lambda^r} v_n+\sum_{n>\lambda+\lambda^r} v_n=V_1+V_2+V_3,$$

say. Here, first,

(4.4) $$|V_2|<A(2\lambda^r+1)\lambda^{-\delta}=O(\lambda^{-\frac{1}{2}\delta})=o(1),$$

by (3.2), (3.3), (3.10.2) and (4.1). Next, in V_1,

$$|v_n|<A\frac{\lambda^r}{\lambda-n},$$

and so

(4.5) $$|V_1|<A\lambda^r\sum_{n<\lambda-\lambda^r}\frac{n^{-\delta}}{\lambda-n}.$$

Since $x^\delta(\lambda-x)$ has one maximum, at

$$x=\frac{\delta\lambda}{\delta+1},$$

between 0 and λ, and increases to this maximum and then decreases, (4.5) gives

$$|V_1|<C\lambda^{r-\delta-1}+C\lambda^r\int_0^{\lambda-\lambda^r}\frac{x^{-\delta}}{\lambda-x}dx.$$

The first term is $o(1)$, and the second is

$$O\left\{\lambda^r\int_{\lambda^r}^{\lambda}(\lambda-w)^{-\delta}\frac{dw}{w}\right\}=O\left\{\lambda^{r-\delta}\int_{\lambda^{r-1}}^{1}(1-y)^{-\delta}\frac{dy}{y}\right\}=O(\lambda^{-\frac{1}{2}\delta}\log\lambda)=o(1).$$

Hence $V_1=o(1)$, and a similar, but rather simpler (5), argument shows that $V_3=o(1)$. Combining these results with (4.3) and (4.4), we find that $V=o(1)$. It then follows from (3.9) and (4.2) that $S_3(\lambda)=o(1)$; and this completes the proof of the theorem.

A particular case of some interest is that in which

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi(t+h)-\varphi(t-h)|^p dt=O(|h|^{p\delta})$$

for some $p\geqq 1$ and small $|h|$, i. e. when φ belongs to what we have called the class Lip (δ, p) (6). In this case (3.1) is certainly satisfied.

5. - There is a generalisation of Theorem 2 corresponding partly to LEBESGUE's generalisation of FEJÉR's theorem.

THEOREM 3. - *In Theorem* 2, *condition* (ii) *may be replaced by*

(5.1) $$\Phi^*(t)=\int_0^t|\varphi(u)|\,du=o\left(\frac{t}{\log\frac{1}{t}}\right).$$

(5) Because $x^\delta(x-\lambda)$ is monotonic in (λ,∞).

(6) See HARDY and LITTLEWOOD (4, 5).

In this case

$$S_1(\lambda)=o\left\{\lambda\int_0^{\lambda^{-1}}|\varphi(t)|\,dt\right\}=o\left(\frac{1}{\log\lambda}\right)=o(1);$$

while

$$|S_2(\lambda)|\leqq\int_{\lambda^{-1}}^{\lambda^{-r}}\frac{|\varphi(t)|}{t}\,dt\leqq\lambda^r\Phi^*\left(\frac{1}{\lambda^r}\right)+\int_{\lambda^{-1}}^{\lambda^{-r}}\frac{\Phi^*(t)}{t^2}\,dt,$$

which may be shown to tend to zero as in § 3. The discussion of $S_3(\lambda)$ is unaltered.

We have not been able to replace (5.1) by

$$\Phi(t)=\int_0^t\varphi(u)du=o\left(\frac{t}{\log\frac{1}{t}}\right).$$

Negative theorems.

6. - We prove next that Theorem 2 is a best possible theorem, in that the condition (3.1) cannot be replaced by any wider condition on the order of a_n.

THEOREM 4. - *Suppose that η_n decreases steadily to zero when $n\to\infty$. Then there is a function $\varphi(t)$ such that* (i) *$\varphi(t)$ satisfies* (2.4), (ii)

$$a_n=O(n^{-\eta_n}),\tag{6.1}$$

and (iii) *$\sum a_n$ is divergent.*

We prove this by a modification of FEJÉR's well known method for the construction of divergent FOURIER series. We require

Lemma *a.* - *There is a constant A such that*

$$\left|\sum_M^N\frac{\sin nt}{n\log n}\right|<\frac{A}{\log\frac{1}{t}}\tag{6.2}$$

for $1<M\leqq N,\ 0<t<\frac{1}{2}$.

To prove the lemma, let $\tau=[t^{-1}]$ and write

$$S=\sum_M^N=\sum_M^\tau+\sum_{\tau+1}^N=S_1+S_2,$$

when τ falls between M and N. Then

$$|S_1|\leqq t\sum_M^\tau\frac{1}{\log n}<\frac{At\tau}{\log\tau}<\frac{A}{\log\frac{1}{t}},$$

$$|S_2|\leqq\frac{1}{\tau\log\tau}\operatorname*{Max}_{\nu>\tau}\left|\sum_{\tau+1}^{\nu}\sin nt\right|<\frac{A}{t\tau\log\tau}<\frac{A}{\log\frac{1}{t}}.$$

This proves the lemma when τ falls in (M, N), and in the contrary case the proof is simpler.

We now define $\varphi(t)$ by

(6.3) $$\varphi(t)=\sum h_r C(m_r, n_r, q_r, t)=\sum h_r C_r,$$

where

(6.4) $$C_r=2\sin q_r t\sum_{m_r}^{n_r}\frac{\sin nt}{n\log n}.$$

Here

(6.5) $$h_r>0, \qquad \sum h_r<\infty, \qquad q_r=2n_r,$$

and m_r and n_r increase rapidly with r, in a manner to be specified more precisely later.

We prove first that $\varphi(t)$ satisfies (2.4). We choose R so that

$$\sum_{R+1}^{\infty} h_r<\varepsilon.$$

Then, by (6.2),

$$|\varphi(t)|<\sum_{1}^{R} h_r|C_r|+\frac{A\varepsilon}{\log\frac{1}{t}}=\frac{A\varepsilon}{\log\frac{1}{t}}+O(t^2)<\frac{2A\varepsilon}{\log\frac{1}{t}}$$

for $0<t\leqq t_0(\varepsilon, R)=t_0(\varepsilon)$.

Next

(6.6) $$C_r=\frac{\cos(q_r-n_r)t}{n_r\log n_r}+\ldots+\frac{\cos(q_r-m_r)t}{m_r\log m_r}-\frac{\cos(q_r+m_r)t}{m_r\log m_r}-\ldots-\frac{\cos(q_r+n_r)t}{n_r\log n_r}.$$

If

(6.7) $$n_r>3n_{r-1}$$

then, by (6.5),

$$q_r-n_r=n_r>3n_{r-1}=q_{r-1}+n_{r-1},$$

and there is no overlapping between the cosines in different C_r, so that the FOURIER series of $\varphi(t)$ is $\sum h_r C_r$ written out at length in conformity with (6.6). When $t=0$, the series contains blocks of terms of the type

$$h_r\sum_{n_r}^{m_r}\frac{1}{n\log n},$$

and will certainly diverge if

(6.8) $$h_r s_r=h_r(\log\log n_r-\log\log m_r)\to\infty.$$

We have finally to consider the order of a_n as a function of n. The largest coefficient in C_r is $(m_r\log m_r)^{-1}$, and the highest and lowest ranks of a cosine are $q_r+n_r=3n_r$ and $q_r-n_r=n_r$. Also $h_r\to 0$. Hence condition (6.1) will certainly be satisfied if

(6.9) $$\frac{1}{m_r\log m_r}=O\{(3n_r)^{-\eta_{n_r}}\},$$

and *a fortiori* if

$$m_r^{-1}=O\{(3n_r)^{-\eta_{n_r}}\},$$

or if

$$\log m_r - \eta_{n_r} \log 3n_r \to \infty,$$

or if

(6.10) $$-s_r - \log \eta_{n_r} = \log\log m_r - \log\log n_r - \log \eta_{n_r} \to \infty.$$

7. - A moment's reflection will show that we can always choose our sequences so as to satisfy (6.5), (6.7), (6.8) and (6.10). Suppose, for example, that

$$\eta_n = (l_3 n)^{-1}:$$

we write $l_3 n$ for $\log\log\log n$ and use a similar notation for repeated exponentials. Take

$$h_r = e_1(-r), \qquad s_r = e_1(2r).$$

Then (6.5) and (6.8) are satisfied, and (6.7) and (6.10) will be satisfied if

$$l_4 n_r - e_1(2r) \to \infty.$$

We may for example take

$$n_r = e_5(3r),$$

and then m_r is given by

$$m_r = e_2 \left\{ \frac{e_3(3r)}{e_1(2r)} \right\}.$$

8. - There are two other theorems, of the same character as Theorem 4, whose proofs we leave to the reader.

THEOREM 5. - *Suppose that χ_n tends steadily to infinity with n. Then there is a continuous function $\varphi(t)$ such that*

$$a_n = O\left(\frac{\chi_n}{n}\right)$$

and $\sum a_n$ is divergent.

THEOREM 6. - *There is a function $\varphi(t)$ such that*

(i) $$\varphi(t) = O\left(\frac{1}{\log \frac{1}{t}}\right),$$

(ii) $$a_n = O(n^{-\delta}) \qquad (\delta > 0),$$

(iii) *$\sum a_n$ is divergent.*

Theorem 5 shows that the condition (ii) of Theorem 1 is the best possible, while Theorem 6 shows that Theorem 2 is a best possible theorem in a second sense, viz. that condition (i) cannot be relaxed if condition (ii) is left unaltered.

An analogue of Theorem 2.

9. - It is natural to ask what happens to Theorem 2 when the o of (2.4) is replaced by O. The answer is that the order condition must then be strengthened considerably; roughly, a_n must be « very nearly $O(n^{-1})$ ».

THEOREM 7. - *It is sufficient that*

(i)
$$\varphi(t) = O\left(\frac{1}{\log \frac{1}{t}}\right)$$

and

(ii)
$$a_n = O(n^{-1+\delta})$$

for every positive δ.

The proof is very like that of Theorem 2, and we do not give it in full. We split up $S(\lambda)$ as in (3.5), but suppose now that $r = 1 - \eta$, where η is small. $S_1(\lambda)$ is $o(1)$ as before; and $S_2(\lambda)$ is bounded, and numerically less than $\varepsilon(\eta)$, tending to zero with η. Finally

$$S_3(\lambda) = O(\lambda^{r-1+\delta} \log \lambda) = O(\lambda^{\delta-\eta} \log \lambda),$$

and tends to zero if $\delta < \eta$.

The conditions (i) and (ii) are again the best possible of their kind.

Tauberian proofs and one-sided conditions.

10. - The proof of Theorem 1 is « Tauberian », and we have no direct proof corresponding to that of Theorem 2. It is natural to look for a Tauberian proof of the latter theorem, and the argument thus suggested is interesting in itself and leads to a generalisation which we cannot prove directly.

It will be convenient to introduce the notion of the « Tauberian index » of any method of summation of divergent series. Suppose that S is a method of summation, and that the hypotheses (a) $\sum a_n$ is summable (S), and (b) $a_n = O(n^{-k})$, imply the convergence of the series. Then we say that S has the Tauberian index k (7). Thus the CESÀRO and ABEL methods have the index 1. It is plain that if we are to prove Theorem 2 by Tauberian methods, we must use some method of summation whose Tauberian index is δ.

BOREL's exponential method has the index $\frac{1}{2}$. We proved this in 1916 (8), and at the same time introduced a modification of BOREL's method. We defined the limit of a divergent sequence

$$s_n = a_0 + a_1 + \dots + a_n$$

as

$$s = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi n}} \sum_{m=-\infty}^{\infty} e^{-\frac{m^2}{2n}} s_{m+n},$$

(7) Naturally we choose k as small as possible. We need the phrase only for general explanations and it is unnecessary to be precise in our definitions.

(8) HARDY and LITTLEWOOD, 1.

where s_{m+n} is to be replaced by 0 if the suffix is negative. This definition is equivalent to BOREL's for « delicately divergent » series, and in particular for series (such as FOURIER series) whose terms tend to zero. In particular, it has the same Tauberian index $\frac{1}{2}$.

A little later VALIRON [9] obtained very extensive generalisations of our results. We are concerned here only with a quite special case. If we define s by

$$s=\lim_{n\to\infty}\frac{n^{\frac{1}{2}l-1}}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}e^{-\frac{1}{2}m^2n^{l-2}}s_{m+n}, \tag{10.1}$$

where $1\leqq l<2$, then the Tauberian condition is

$$a_n=O(n^{\frac{1}{2}l-1}) \tag{10.2}$$

and the index is $1-\frac{1}{2}l$. This may be made as small as we please by taking l sufficiently near to 2, and the case in which we are interested is that in which l is a little less than 2.

When $l=2$, (10.2) becomes $a_n=O(1)$, and the method cannot sum a FOURIER series unless it is convergent.

11. - We call the method of summation defined by (10.1) the method (V, l). Its use in the theory of FOURIER series depends upon the following theorem.

THEOREM 8. - *If* $\varphi(t)$ *satisfies* (2.4), *then* $\sum a_n$ *is summable* (V, l) *for* $1\leqq l<2$, *and in particular summable* (B).

We have to show that

$$T_n=\frac{n^{\frac{1}{2}l-1}}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}e^{-\frac{1}{2}m^2n^{l-2}}t_{m+n}\to 0, \tag{11.1}$$

where

$$t_n=\int_0^c\varphi(t)\frac{\sin nt}{t}\,dt. \tag{11.2}$$

In (11.1), t_{m+n} is to be replaced by 0 if $m+n\leqq 0$. We may however drop this convention, and suppose t_{m+n} to be defined by (11.2) for all m and n. For $t_n=O(|n|)$, and the change in T_n which results from this change of convention is

$$O\Bigl(n^{\frac{1}{2}l-1}\sum_{m=-\infty}^{-n}|m+n|\,e^{-\frac{1}{2}m^2n^{l-2}}\Bigr)=O\Bigl(n^{\frac{1}{2}l-1}\sum_{\mu=n}^{\infty}\mu e^{-\frac{1}{2}\mu^2n^{l-2}}\Bigr)=$$

$$=O\Bigl(n^{\frac{1}{2}l-1}\int_n^{\infty}xe^{-\frac{1}{2}x^2n^{l-2}}dx\Bigr)=O\Bigl(n^{\frac{1}{2}l+1}\int_1^{\infty}ye^{-\frac{1}{2}y^2n^{l}}dy\Bigr),$$

which obviously tends to zero.

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[9] VALIRON, 10.

Substituting from (11.2) into (11.1), we obtain

$$T_n = \frac{n^{\frac{1}{2}l-1}}{\sqrt{2\pi}} \int_0^c \frac{\varphi(t)}{t} Q(t, n) dt,$$

where

$$Q(t, n) = Q = \sum_m e^{-\frac{1}{2}m^2 n^{l-2}} \sin (m+n)t.$$

Writing H for n^{l-2}, we have

$$Q = \mathbf{I}\left\{\sum e^{-\frac{1}{2}m^2 H} e^{i(m+n)t}\right\} = R \sin nt,$$

where

$$R = \sum e^{-\frac{1}{2}m^2 H} \cos mt.$$

By a familiar formula in the theory of elliptic functions

(11.3) $$R = \sqrt{\frac{2\pi}{H}} \sum \exp\left\{-\frac{2\pi^2}{H}\left(m - \frac{t}{2\pi}\right)^2\right\} = \sqrt{\frac{2\pi}{H}}\, S = n^{-\frac{1}{2}l+1}\sqrt{2\pi} S,$$

say; and what we have to prove is that

(11.4) $$T_n = \int_0^c \varphi(t) \frac{\sin nt}{t} S dt \to 0.$$

12. - We write

(12.1) $$S = \sum \exp\left\{-\frac{2\pi^2}{H}\left(m - \frac{t}{2\pi}\right)^2\right\} = e^{-\frac{t^2}{2H}} + S_2 = S_1 + S_2$$

(taking in S_2 all the terms of S for which $m \neq 0$). If, as we may suppose, $c < \pi$, then

$$\left(m - \frac{t}{2\pi}\right)^2 > \left(m - \frac{1}{2}\right)^2 > \frac{1}{4} m^2 \qquad (m \neq 0)$$

and

$$S_2 < 2 \sum_1^\infty e^{-\frac{m^2\pi^2}{2H}} = 2\sum_1^\infty e^{-\frac{1}{2}\pi^2 m^2 n^{2-l}} < 4e^{-\frac{1}{2}\pi^2 n^{2-l}}$$

for large n. If follows that

$$\int_0^c \varphi(t) \frac{\sin nt}{t} S_2 dt = O\left\{ne^{-\frac{1}{2}\pi^2 n^{2-l}} \int_0^c |\varphi(t)| dt\right\} \to 0.$$

We may therefore replace S by S_1 in (11.4), and the proof of (11.4) is reduced to a proof that

(12.2) $$U_n = \int_0^c \varphi(t) \frac{\sin nt}{t} e^{-\frac{1}{2}t^2 n^{2-l}} dt \to 0.$$

It will be observed that this integral reduces to DIRICHLET's integral for $l=2$.

We now write

$$U_n = \int_0^c = \int_0^{n^{-1}} + \int_{n^{-1}}^{n^{-r}} + \int_{n^{-r}}^c = V_1 + V_2 + V_3,$$

say, choosing r small enough to make

$$h=2-l-2r>0.$$

Then

$$|V_1|\leqq n\int_0^{n^{-1}}|\varphi(t)|\,dt\to 0,$$

and

$$|V_2|\leqq\int_{n^{-1}}^{n^{-r}}\frac{|\varphi(t)|}{t}\,dt\to 0$$

as in § 3; it is here only that we use (2.4). Finally

$$|V_3|\leqq e^{-\frac{1}{2}n^h}\int_{n^{-r}}^{c}\frac{|\varphi(t)|}{t}\,dt=O(n^r e^{-\frac{1}{2}n^h})\to 0.$$

13. - We can deduce Theorem 2 (and a more general theorem) by combining Theorem 8 with appropriate Tauberian theorems, which we state as lemmas.

Lemma β. - *If* $\sum a_n$ *is summable* (V, l), *and satisfies* (10.2), *then it is convergent.*

This, as we stated in § 10, was proved by VALIRON (as a special case of a much more general theorem).

Lemma γ. - *In Lemma* β *the condition* (10.2) *may be replaced by the more general condition*

(13.1) $$a_n>-An^{\frac{1}{2}l-1}.$$

This has been proved explicitly when $k=1$ by SCHMIDT, VIJAYARAGHAVAN and WIENER. The lemma as stated requires an adaptation of their methods which has been undertaken for us by Mr. J. HYSLOP.

Taking $l=2-2\delta$, and combining Theorem 8 with Lemma β, we obtain Theorem 2. If we use Lemma γ, we obtain

THEOREM 9. - *It is sufficient that* (i) $\varphi(t)$ *should satisfy* (2.4) *and* (ii) a_n *should satisfy* (13.1).

We have no direct proof of this theorem.

Criteria of Young's type.

14. - We consider next a group of criteria suggested by the well-known criterion of YOUNG. In these there is no « order » condition, but there are two conditions on $\varphi(t)$. It is characteristic of these criteria, as of YOUNG's, that $\varphi(t)$ is assumed to be of bounded variation except at $t=0$, or at any rate for $0<\delta\leqq t\leqq c$, with some c and arbitrary δ.

YOUNG's test runs: *it is sufficient that*

$$\Phi^*(t)=\int_0^t |\varphi(u)|\,du=o(t) \quad (^{10}) \tag{14.1}$$

and

$$\int_0^t |d(u\varphi)|=O(t). \tag{14.2}$$

An interesting special case is that in which

$$\varphi(t)=o(1), \tag{14.3}$$

$\varphi(t)$ is an integral except at $t=0$, and

$$\varphi'(t)=O\left(\frac{1}{t}\right) \tag{14.4}$$

or at any rate

$$\varphi'(t)>-\frac{A}{t}. \tag{14.5}$$

In this last case, if ψ is φ' when $\varphi'<0$ and 0 otherwise, then

$$|\varphi'|=\varphi'-2\psi$$

and

$$\int_0^t |d(u\varphi)|\leqq\int_0^t u|\varphi'|\,du+\int_0^t|\varphi|\,du=\int_0^t u\varphi'du-2\int_0^t u\psi du+\int_0^t|\varphi|\,du\leqq$$

$$\leqq\left|\int_0^t u\varphi'du\right|+2At+o(t)=\left|t\varphi(t)-\int_0^t\varphi du\right|+2At+o(t)<2At+o(t),$$

so that (14.2) is certainly satisfied. A rather more detailed version of a similar argument will be given in § 16.

15. - When we modify YOUNG's criterion in the manner suggested by Theorems 2 and 3, we obtain

THEOREM 10. - *It is sufficient that* (i) $\varphi(t)$ *should satisfy* (5.1) *and* (ii) *that*

$$\Psi_\Delta^*(t)=\int_0^t |d(u^\Delta\varphi)|=O(t) \tag{15.1}$$

for some Δ.

The emphasis here is on large positive Δ, whereas in Theorems 2 and 3 it was on small positive δ. We may suppose $\Delta>1$. From (15.1) it follows that

$$\Psi_\Delta(t)=\int_0^t d(u^\Delta\varphi)=O(t) \tag{15.2}$$

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(10) This condition may be replaced by

$$\Phi(t)=\int_0^t\varphi(u)du=o(t),$$

or by still more general conditions: see POLLARD (8), HARDY and LITTLEWOOD (3). We cannot prove corresponding extensions of Theorem 10.

and so that

(15.3) $$\varphi(t) = O(t^{1-\Delta}).$$

We choose r so that

(15.4) $$r > \frac{1}{\Delta}$$

and split up $S(\lambda)$ as in § 3. Then $S_1(\lambda) \to 0$ and $S_2(\lambda) \to 0$ as in § 5. As regards $S_3(\lambda)$, we have

$$S_3(\lambda) = \int_{\lambda^{-r}}^{c} \varphi \frac{\sin \lambda t}{t} dt = -\frac{1}{\lambda}\int_{\lambda^{-r}}^{c} \frac{\varphi}{t} d(\cos \lambda t) = -\left[\varphi \frac{\cos \lambda t}{\lambda t}\right]_{\lambda^{-r}}^{c} + \frac{1}{\lambda}\int_{\lambda^{-r}}^{c} \cos \lambda t d\frac{\varphi}{t} =$$

$$= -\left[\varphi \frac{\cos \lambda t}{\lambda t}\right]_{\lambda^{-r}}^{c} - \frac{\Delta+1}{\lambda}\int_{\lambda^{-r}}^{c} \frac{\varphi}{t^2} \cos \lambda t \, dt + \frac{1}{\lambda}\int_{\lambda^{-r}}^{c} \frac{\cos \lambda t}{t^{\Delta+1}} d\Psi_\Delta = S_4(\lambda) + S_5(\lambda) + S_6(\lambda),$$

say. Here

$$S_4(\lambda) = O\{\lambda^{-1}(\lambda^{-r})^{-\Delta}\} = O(\lambda^{r\Delta-1}) = o(1),$$

and

$$S_5(\lambda) = O\left(\frac{1}{\lambda}\int_{\lambda^{-r}}^{c} t^{-\Delta-1} dt\right) = O(\lambda^{r\Delta-1}) = o(1),$$

by (15.3) and (15.4). Finally

$$|S_6(\lambda)| \leqq \frac{1}{\lambda}\int_{\lambda^{-r}}^{c} t^{-\Delta-1} |d\Psi_\Delta| = \frac{1}{\lambda}\int_{\lambda^{-r}}^{c} t^{-\Delta-1} d\Psi_\Delta^* = \frac{1}{\lambda}[t^{-\Delta-1}\Psi_\Delta^*]_{\lambda^{-r}}^{c} + \frac{\Delta+1}{\lambda}\int_{\lambda^{-r}}^{c} t^{-\Delta-2}\Psi_\Delta^* dt <$$

$$< O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{\lambda}\int_{\lambda^{-r}}^{c} \frac{dt}{t^{\Delta+1}}\right) = O(\lambda^{-1}) + O(\lambda^{r\Delta-1}) = o(1).$$

16. - The special case corresponding to the special case of YOUNG's theorem quoted in § 14 is

THEOREM 11. - *It is sufficient that* (i) $\varphi(t)$ *should satisfy* (2.4), (ii) $\varphi(t)$ *should be an integral except at* $t=0$, *and* (iii) *that*

(16.1) $$\varphi'(t) > -\frac{A}{t^\Delta}.$$

It is sufficient to prove that (15.1) is satisfied. In the argument which follows $0<\varepsilon<t$, and O's are uniform in t and ε (the constants which they imply are independent of both t and ε).

We have first

(16.2) $$\int_{\varepsilon}^{t} |d(u^\Delta \varphi)| \leqq \Delta \int_{\varepsilon}^{t} u^{\Delta-1} |\varphi| du + \int_{\varepsilon}^{t} u^\Delta |d\varphi| \leqq \Delta t^{\Delta-1}\int_{0}^{t} |\varphi| du +$$

$$+ \int_{\varepsilon}^{t} u^\Delta |d\varphi| = O(t^\Delta) + \int_{\varepsilon}^{t} u^\Delta |d\varphi| = O(t) + \int_{\varepsilon}^{t} u^\Delta |d\varphi|.$$

Next, if we define ψ as in § 14, we have

$$(16.3)\quad \int_\varepsilon^t u^\Delta |d\varphi| = \int_\varepsilon^t u^\Delta |\varphi'| du = \int_\varepsilon^t u^\Delta \varphi' du - 2\int_\varepsilon^t u^\Delta \psi du < \left|\int_\varepsilon^t u^\Delta \varphi' du\right| + 2At.$$

Finally

$$(16.4)\quad \int_\varepsilon^t u^\Delta \varphi' du = t^\Delta \varphi(t) - \varepsilon^\Delta \varphi(\varepsilon) - \Delta \int_\varepsilon^t u^{\Delta-1}\varphi(u)du = O(t^\Delta) = O(t),$$

by (2.4). From (16.2), (16.3) and (16.4) it follows that

$$\int_0^t |d(u^\Delta \varphi)| = \lim_{\varepsilon\to 0}\int_\varepsilon^t |d(u^\Delta\varphi)| = O(t),$$

which is (15.2). This proves Theorem 11.

17. - The theorem which corresponds here to Theorem 7 is

THEOREM 12. - *It is sufficient that*

(i) $$\varphi(t) = O\left(\frac{1}{\log\frac{1}{t}}\right),$$

(ii) $\varphi(t)$ *is an integral except at* $t=0$, *and*

(iii) $$\varphi'(t) = O(t^{-1-\delta})$$

for any positive δ.

We leave this theorem and its obvious generalisations to the reader.

The conjugate series.

18. - There are simular theorems concerning the convergence of the series conjugate to a FOURIER series. If we suppose (making the simplifications corresponding to those of § 1) that $\psi(t)$ is odd and

$$\psi(t) \sim \sum a_n \sin nt,$$

then the problem is that of the convergence of $\sum a_n$. We state one theorem only, which corresponds to Theorem 2.

THEOREM 13. - *If*

(i) $$\psi(t) = o\left(\frac{1}{\log\frac{1}{t}}\right),$$

(ii) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(t)\cot\tfrac{1}{2}t\,dt = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_\varepsilon^\pi \psi(t)\cot\tfrac{1}{2}t\,dt = s,$$

(iii) $a_n = O(n^{-\delta})$ *for some positive* δ, *then*

$$(18.1)\quad \sum a_n = s.$$

The standard arguments show that (18.1) is equivalent to

$$\int_0^c \psi(t)\,\frac{\cos \lambda t}{t}\,dt \to 0;$$

and this may be proved by arguments similar to those of § 3.

Transforms of Theorems 2 and 11.

19. - There is a theorem about general trigonometrical series which is in a sense the « reciprocal » or « transform » of Theorem 2 ([11]).

THEOREM 14. - *If* (i) $a_n = O(n^{-\delta})$ *for some positive* δ, *and* (ii)

$$s_n - s = a_1 + a_2 + \dots + a_n - s = o\left(\frac{1}{\log n}\right), \tag{19.1}$$

then

$$\chi(t) = \sum_1^\infty a_n \frac{\sin nt}{nt} \to s \tag{19.2}$$

when $t \to 0$.

We may express (19.2) by saying that $\sum a_n$ *is summable* $(R, 1)$, *i. e.* by « RIEMANN's first mean », *to* s. It is familiar that $\sum a_n$ is summable $(R, 2)$ whenever it is convergent.

We may suppose $s=0$. We choose r so that

$$r\delta > 1 \tag{19.3}$$

and write

$$\mu = [t^{-1}], \qquad \nu = [t^{-r}], \tag{19.4}$$

$$\chi(t) = \sum_1^\nu + \sum_{\nu+1}^\infty = \chi_1(t) + \chi_2(t). \tag{19.5}$$

Here

$$\chi_2(t) = O\left(\frac{1}{t}\sum_{\nu+1}^\infty n^{-1-\delta}\right) = O(t^{r\delta-1}) = o(1), \tag{19.6}$$

so that it is enough to prove that

$$\chi_1(t) = o(1). \tag{19.7}$$

Summing partially, we have ([12])

$$\chi_1(t) = \sum_1^\nu a_n \frac{\sin nt}{nt} = \sum_1^{\nu-1} s_n \Delta \frac{\sin nt}{nt} + s_\nu \frac{\sin \nu t}{\nu t} = \chi_3(t) + s_\nu \frac{\sin \nu t}{\nu t} = \chi_3(t) + o(1), \tag{19.8}$$

([11]) In our note 2 we gave a general description of a heuristic process of « reciprocation » which often enables us to derive one theorem about trigonometrical series from another. Theorem 14 was derived from Theorem 2 in this way; but the process requires, as usual, a certain amount of adjustment of the data, and is difficult to describe precisely.

([12]) Here $\Delta u_n = u_n - u_{n+1}$: Δ has no connection with the Δ of § 15.

by (19.1). Also

(19.9) $$\chi_3(t)=\sum_1^{\nu-1} s_n \Delta \frac{\sin nt}{nt}=\sum_1^{\mu}+\sum_{\mu+1}^{\nu-1}=\chi_4(t)+\chi_5(t),$$

say. In χ_4, $nt \leqq 1$ and

$$\frac{\sin nt}{nt}=1-\frac{1}{6}n^2t^2+\dots,\qquad \Delta\frac{\sin nt}{nt}=O(nt^2);$$

so that

(19.10) $$\chi_4(t)=\sum_1^{\mu} o(nt^2)=o(\mu^2t^2)=o(1).$$

On the other hand, in χ_5, $nt \geqq 1$ and

$$\Delta\frac{\sin nt}{nt}=O\left(\frac{1}{n^2t}\right)+O\left(\frac{1}{n}\right)=O\left(\frac{1}{n}\right);$$

and so

(19.11) $$\chi_5(t)=o\left(\sum_{\mu+1}^{\nu-1}\frac{1}{n\log n}\right)=o(\log\log t^{-r}-\log\log t^{-1})=o(1).$$

Collecting our results from (19.8)-(19.11), we obtain (19.7).

If $\sum a_n \cos nt$ is the FOURIER series of $\varphi(t)$, we can state the conclusion in the form

(19.12) $$\frac{1}{t}\int_0^t \varphi(u)du \to s.$$

20. - We end by proving

THEOREM 15. - *If* (i) $\sum a_n \cos nt$ *is the Fourier series, or Cauchy-Fourier series, of* $\varphi(t)$, (ii) s_n *satisfies* (19.1), *and* (ii)

(20.1) $$\varphi(t)=O(t^{-\Delta})$$

for some Δ, *then*

(20.2) $$\chi(t)=\frac{1}{t}\int_0^t \varphi(u)du=\sum a_n\frac{\sin nt}{nt}\to s$$

when $t\to 0$.

This theorem is related to Theorem 11 much as Theorem 14 is related to Theorem 2. We have however replaced (16.1) by the more restrictive condition (20.1). There is no doubt a theorem with a « one-sided » condition, but we have not attempted this generalisation.

Some hypothesis is required to establish a connection between the series and $\varphi(t)$, and the most natural hypothesis for this purpose is (i). When we say that $\sum a_n \cos nt$ is the CAUCHY-FOURIER series of $\varphi(t)$, we mean that $\varphi(t)$ is LEBESGUE integrable except at 0 and that

$$a_n=\frac{2}{\pi}\int_0^\pi \varphi(t)\cos ntdt=\frac{2}{\pi}\lim_{\varepsilon\to 0}\int_\varepsilon^\pi \varphi(t)\cos ntdt.$$

In these circumstances $n^{-1}a_n$ is the FOURIER sine coefficient of the odd and continuous function $\chi(t)$, and

$$\chi(t)=\sum a_n \frac{\sin nt}{nt},$$

the series being summable $(C, 1)$. We shall in fact prove incidentally that the series is convergent.

We take

$$s=0, \qquad \Delta>1, \qquad r>2\Delta>\Delta+1, \qquad \nu=[t^{-r}], \tag{20.3}$$

and (assuming provisionally the convergence of the series) write

$$\chi(t)=\left(\sum_{1}^{\nu}+\sum_{\nu+1}^{\infty}\right) a_n \frac{\sin nt}{nt}=\chi_1(t)+\chi_2(t). \tag{20.4}$$

We show that $\chi_1(t)\to 0$ as in § 19, and it remains to prove that

$$\chi_2(t)=\sum_{\nu+1}^{\infty} a_n \frac{\sin nt}{nt}\to 0. \tag{20.5}$$

We write [13]

$$\chi_2(t)=\lim_{N\to\infty}\sum_{\nu+1}^{N} a_n \frac{\sin nt}{nt}=\lim_{N\to\infty}\chi_{2,N}(t). \tag{20.6}$$

Then

$$\chi_{2,N}(t)=\sum_{\nu+1}^{N}\frac{\sin nt}{nt}\,\frac{2}{\pi}\int_0^{\pi}\varphi(\theta)\cos n\theta d\theta=\frac{2}{\pi t}\int_0^{\pi}\varphi(\theta)\sum_{\nu+1}^{N}\frac{\sin nt\cos n\theta}{n}\,d\theta= \tag{20.7}$$

$$=\frac{1}{\pi t}\int_0^{\pi}\varphi(\theta)S_{\nu,N}(t-\theta)d\theta+\frac{1}{\pi t}\int_0^{\pi}\varphi(\theta)S_{\nu,N}(\theta+t)d\theta=\chi_{3,N}(t)+\chi_{4,N}(t),$$

say, where

$$S_{\nu,N}(u)=\sum_{\nu+1}^{N}\frac{\sin nu}{n}. \tag{20.8}$$

We shall prove (a) that $\chi_{3,N}(t)$ and $\chi_{4,N}(t)$ tend to limits $\chi_3(t)$ and $\chi_4(t)$ when t is positive and fixed and $N\to\infty$, and (b) that

$$\chi_{3,N}(t)\to 0, \qquad \chi_{4,N}(t)\to 0 \tag{20.9}$$

when $t\to 0$, uniformly in N. It will then follow that $\chi_{2,N}(t)$ tends to a limit $\chi_2(t)$ when $N\to\infty$ (so that the series of the theorem is convergent), and that

$$\chi_2(t)=\lim_{N\to\infty}\chi_{2,N}(t)=\lim_{N\to\infty}\chi_{3,N}(t)+\lim_{N\to\infty}\chi_{4,N}(t)=\chi_3(t)+\chi_4(t)\to 0 \tag{20.10}$$

when $t\to 0$; and this will prove the theorem.

---

[13] If φ is LEBESGUE integrable, so that $\sum a_n \cos nt$ is a FOURIER series, then the introduction of N is unnecessary. We may replace N at once by ∞, the term by term integration in the argument which follows being justified by « bounded convergence ».

21. - We require the following properties of $S_{\nu,N}(u)$. In the first place

$$|S_{\nu,N}(u)| < A \tag{21.1}$$

for all u, ν, N. Next

$$|S_{\nu,N}(u)| < \frac{A}{\nu|u|} \tag{21.2}$$

for $|u| \leqq \frac{3}{2}\pi$ and all N. Thirdly

$$|S'_{\nu,N}(u)| = \left|\sum_{\nu}^{N} \cos nu\right| < \frac{A}{|u|} \tag{21.3}$$

for $|u| \leqq \frac{3}{2}\pi$ and all ν, N. Finally

$$S_{\nu,N}(u) \to S_\nu(u) = \sum_{\nu+1}^{\infty} \frac{\sin nu}{n}, \tag{21.4}$$

for every u and ν, when $N \to \infty$, and $S_\nu(u)$ has the properties expressed by putting $N = \infty$ in (21.1) and (21.2).

22. - We may confine our attention to $\chi_{3,N}(t)$, the corresponding discussion for $\chi_{4,N}(t)$ being similar but a little simpler. We write

$$\chi_{3,N}(t) = \frac{1}{\pi t}\int_0^{\pi} \varphi(\theta) S_{\nu,N}(t-\theta)\,d\theta = \frac{1}{\pi t}\left(\int_0^{t^2} + \int_{t^2}^{t-t^r} + \int_{t-t^r}^{t+t^r} + \int_{t+t^r}^{\pi}\right) = \tag{22.1}$$

$$= \omega_{1,N}(t) + \omega_{2,N}(t) + \omega_{3,N}(t) + \omega_{4,N}(t) = \omega_1 + \omega_2 + \omega_3 + \omega_4,$$

and consider ω_3, ω_2, ω_4 and ω_1 in turn.

First

$$\omega_3 = \frac{1}{\pi t}\int_{t-t^r}^{t+t^r} \varphi(\theta) S_{\nu,N}(t-\theta)\,d\theta \to \frac{1}{\pi t}\int_{t-t^r}^{t+t^r} \varphi(\theta) S_\nu(t-\theta)\,d\theta \tag{22.2}$$

when $N \to \infty$, since φ is LEBESGUE integrable in the range and $S_{\nu,N} \to S_\nu$ boundedly. Also

$$\omega_3 = O\left(t^{-1}\int_{t-t^r}^{t+t^r} \theta^{-\Delta}\,d\theta\right) = O(t^{r-1-\Delta}) = o(1), \tag{22.3}$$

uniformly in N, by (20.1), (20.3) and (21.1).

Secondly

$$\omega_2 = \frac{1}{\pi t}\int_{t^2}^{t-t^r} \varphi(\theta) S_{\nu,N}(t-\theta)\,d\theta \to \frac{1}{\pi t}\int_{t^2}^{t-t^r} \varphi(\theta) S_\nu(t-\theta)\,d\theta, \tag{22.4}$$

for the same reasons as in (22.2). Also

$$\omega_2 = O\left(t^{-1}\int_{t^2}^{t-t^r} \frac{\theta^{-\Delta}}{\nu(t-\theta)}\,d\theta\right) = O\left(t^{r-1}\int_{t^2}^{t-t^r} \frac{\theta^{-\Delta}}{t-\theta}\,d\theta\right) = O\left(t^{r-\Delta-1}\int_{t}^{1-t^{r-1}} \frac{u^{-\Delta}}{1-u}\,du\right),$$

by (20.1), (20.3) and (21.2). The integral here is

$$\int_t^{\frac{1}{2}} + \int_{\frac{1}{2}}^{1-t^{r-1}} = O(t^{1-\Delta}) + O(\log t) = O(t^{1-\Delta}),$$

and so, by (20.3),

(22.5) $$\omega_2 = O(t^{r-2\Delta}) = o(1),$$

uniformly in N. A similar argument shows that

(22.6) $$\omega_4 \to \frac{1}{\pi t}\int_{t+t^r}^{\pi} \varphi(\theta) S_\nu(t-\theta)\, d\theta$$

and

(22.7 $$\omega_4 = o(1),$$

uniformly in N.

Finally, if

(22.8) $$\Phi(t) = \int_0^t \varphi(u)\,du = \lim_{\varepsilon\to 0}\int_\varepsilon^t \varphi(u)\,du,$$

we have

(22.9) $$\omega_1 = \frac{1}{\pi t}\int_0^{t^2} \varphi(\theta) S_{\nu,N}(t-\theta)\,d\theta = \frac{\Phi(t^2)}{\pi t} S_{\nu,N}(t-t^2) - \frac{1}{\pi t}\int_0^{t^2} \Phi(\theta) S'_{\nu,N}(t-\theta)\,d\theta.$$

The first term tends to

$$\frac{\Phi(t^2)}{\pi t} S_\nu(t-t^2).$$

The second is

$$-\frac{1}{\pi t}\int_0^{t^2} \Phi(\theta)\sum_{\nu+1}^{N} \cos n(t-\theta)\,d\theta = -\frac{1}{\pi t}\int_0^{t^2} \Phi(\theta)\,\frac{\sin\left(N+\frac{1}{2}\right)(t-\theta) - \sin\left(\nu+\frac{1}{2}\right)(t-\theta)}{2\sin\frac{1}{2}(t-\theta)}\,d\theta$$

and (by the RIEMANN-LEBESGUE theorem) tends to

$$\frac{1}{\pi t}\int_0^{t^2} \Phi(\theta)\,\frac{\sin\left(\nu+\frac{1}{2}\right)(t-\theta)}{2\sin\frac{1}{2}(t-\theta)}\,d\theta = -\frac{1}{\pi t}\int_0^{t^2} \Phi(\theta) S'_\nu(t-\theta)\,d\theta.$$

Hence

(22.10) $$\omega_1 \to \frac{\Phi(t^2)}{\pi t} S_\nu(t-t^2) - \frac{1}{\pi t}\int_0^{t^2} \Phi(\theta) S'_\nu(t-\theta)\,d\theta = \frac{1}{\pi t}\int_0^{t^2} \varphi(\theta) S_\nu(t-\theta)\,d\theta.$$

Again, the first term in (22.9) is

$$o\left\{\frac{1}{\nu t(t-t^2)}\right\} = o\left(\frac{1}{\nu t^2}\right) = o(t^{r-2}) = o(1),$$

uniformly in N, by (21.2) and (20.3); and the second is

$$O\left(\frac{1}{t}\right)\int_0^{t^2} o(1)\,O\left(\frac{1}{t}\right)d\theta=o(1),$$

by (21.3). Hence

$$\omega_1=o(1) \tag{22.11}$$

uniformly in N.

It follows from (22.2), (22.4), (22.6) and (22.10) that

$$\chi_{3,N}(t)\to\frac{1}{\pi t}\int_0^{\pi}\varphi(\theta)S_\nu(t-\theta)d\theta=\chi_3(t)$$

when $N\to\infty$, for any fixed $t>0$; and from (22.3), (22.5), (22.7) and (22.11) that

$$\chi_{3,N}(t)=o(1),$$

uniformly in N. There are similar results for $\chi_{4,N}(t)$; and the theorem follows as was explained at the end of § 20.

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CORRECTIONS

p. 52, *line* 2. Read $n^{\frac{1}{2}l-1}$.

p. 54, *line* 14. For $2\int_0^t u\phi\, du$ read $2\int_0^t u\psi\, du$.

p. 55, (15.4). Read $r < 1/\Delta$.

p. 59, *line* 2. For $\chi(t)$ read $t\chi(t)$.

COMMENTS

§§ 3, 4. The proof of Theorem 2 given here is a repetition of that given in 1932, 9.

§§ 3–7. The results of Theorems 2–4 have been generalized by G. W. Morgan, *Annali Pisa* (2), 4 (1935), 373–82.

§ 13. An alternative proof of Theorem 9, using 'rearrangements', is given by Hardy and Littlewood in 1935, 6, and O. Szász, *Bull. Amer. Math. Soc.* 48 (1942), 705–11, has proved a stronger result by similar methods (see the comments on 1935, 6). Another proof of Theorem 9 of a Tauberian character, in which Valiron's summability method is replaced by Riesz summability by exponential means, is given by F. T. Wang, *Proc. London Math. Soc.* (2), 47 (1942), 308–25. See also F. C. Hsiang, *Proc. Amer. Math. Soc.* 9 (1958), 37–44, and S. Kumari, ibid. 293–9, and references given there.

p. 54, *footnote* (10). W. C. Randels, *Annals of Math.* 36 (1935), 837–58, has given an example of a function satisfying both (15.1) and the condition

$$\int_0^t \phi(u)\, du = o\,(t/\log(1/t))$$

whose Fourier series is divergent. Another example has been given by F. T. Wang, loc. cit.

§ 15. A convergence test of 'Lebesgue' type which includes Theorem 10 has been given by G. I. Sunouchi, *Tôhoku Math. J.* (2), 4 (1952), 187–93.

§ 19. It follows from Theorem 14 that if $s_n(\theta)$ is the nth partial sum of the Fourier series of f at θ, and $\phi(t) = \frac{1}{2}\{f(\theta+t)+f(\theta-t)\}$, and if (i) $a_n \cos n\theta + b_n \sin n\theta = O(n^{-\delta})$ for some positive δ, and (ii) $s_n(\theta)-s = o\,(1/\log n)$, then

$$\phi_1(t) = \frac{1}{t}\int_0^t \phi(u)\, du \to s \quad \text{as } t \to 0+.$$

This result has been extended to Cesàro means of positive order by C.-T. Loo, *Trans. Amer. Math. Soc.* 56 (1944), 508–18, and here, rather surprisingly, the condition (i) can be omitted. Thus Loo proves that if $\sigma_n^\alpha(\theta)$ is the nth (C,α) mean of the Fourier series of f, and ϕ_β is the (C,β) mean of ϕ, then for $\alpha > 0$ the condition $\sigma_n^\alpha(\theta)-s = o\,(1/\log n)$ implies that $\phi_{1+\alpha}(t) \to s$ as $t \to 0+$.

A result in the opposite direction to this, again without the condition (i), has been proved by F. T. Wang, *J. London Math. Soc.* 22 (1947), 40–7, namely that if

$$\alpha > 0 \quad \text{and} \quad \phi_\alpha(t) = o\,(1/\log(1/t)) \quad \text{as } t \to 0+,$$

then the Fourier series of f at θ is summable (C,α).

NOTE ON LEBESGUE'S CONSTANTS IN THE THEORY OF FOURIER SERIES

G. H. HARDY†.

[Extracted from the Journal of the London Mathematical Society, Vol. 17, 1942.]

1. Lebesgue's constants are defined by

$$L_n = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \frac{|\sin Nt|}{\sin t}\, dt, \tag{1.1}$$

where

$$N = 2n+1. \tag{1.2}$$

Fejér‡ proved that

$$L_n = \frac{1}{N} + \frac{2}{\pi} \sum_{\nu=1}^{n} \frac{1}{\nu} \tan \frac{\nu\pi}{N}, \tag{1.3}$$

and Szegö§ that

$$L_n = \frac{16}{\pi^2} \sum_{\nu=1}^{\infty} \left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2N\nu-1}\right) \frac{1}{4\nu^2-1}. \tag{1.4}$$

From (1.4) Watson|| has deduced an asymptotic formula (in Poincaré's sense) for L_n, viz.

$$L_n \sim \frac{4}{\pi^2} \left\{\log N + A_0 + \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} A_\nu}{N^{2\nu}}\right\}, \tag{1.5}$$

where

$$A_0 = 2 \sum_{1}^{\infty} \frac{\log m}{4m^2-1} - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = 2{\cdot}441\ldots, \tag{1.6}$$

$$A_\nu = \frac{2^{2\nu-1}-1}{\nu} B_\nu \left(1 - \sum_{m=1}^{\nu} \frac{B_m \pi^{2m}}{2m!}\right) \quad (\nu \geqslant 1), \tag{1.7}$$

and B_m is Bernoulli's number. Gronwall¶ was the first to obtain such an expansion, but his formulae for the coefficients contain errors of calculation**, and are in any case less simple than Watson's.

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† Received 11 December, 1941; read 11 December, 1941.

‡ *Annales de l'École Normale* (3), 28 (1911), 63–103 (103).

§ *Math. Zeitschrift*, 9 (1921), 163–166.

|| *Quarterly Journal* (Oxford), 1 (1930), 310–318.

¶ *Math. Annalen*, 72 (1912), 244–261.

** Szegö corrects one of these (in the formula for A_0).

Szegö deduced from (1.4) that

(1.8) $$\Delta L_n = L_n - L_{n+1} < 0, \quad (-1)^{r-1}\Delta^r L_n > 0 \quad (r=2, 3, \ldots).$$

2. I have recently found two other formulae which are interesting formally and lead to simple proofs of the other properties of L_n. I deduce the first of them from Fejér's formulae (1.3) and, for the sake of completeness, I reproduce Fejér's elegant proof of (1.2).

It is plain that

$$L_n = \frac{1}{\pi}\int_0^{2\pi}\left|\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\right| dt = \frac{1}{\pi}\int_0^{2\pi}\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} f(t)\,dt,$$

where $$f(t) = \operatorname{sgn}\left\{\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\right\},$$

so that L_n is the $(n+1)$-th partial sum of the Fourier series of $f(t)$ for $t=0$. Hence, since $f(t)$ is even,

(2.1) $$L_n = \tfrac{1}{2}a_0 + a_1 + \ldots + a_n,$$

where $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos mt\,dt.$$

But $f(t) = (-1)^k$ for

$$\frac{2k\pi}{2n+1} < t < \frac{2(k+1)\pi}{2n+1} \quad (k=0, 1, \ldots, 2n),$$

and straightforward calculation gives

$$a_0 = \frac{2}{2n+1}, \quad a_m = \frac{2}{m\pi}\tan\frac{m\pi}{2n+1} \quad (m=1, 2, \ldots, n),$$

so that (2.1) is Fejér's formula.

3. The first of my two formulae is

(3.1) $$L_n = \int_0^\infty \frac{\tanh Ny}{\tanh y}\,\frac{dy}{y^2+\frac{1}{4}\pi^2}.$$

This formula is easily proved by calculating

$$\frac{1}{2\pi i}\int \cot Nz \tan z \frac{dz}{z}$$

round the rectangle

$$\tfrac{1}{2}\pi - iY, \quad \tfrac{1}{2}\pi + iY, \quad -\tfrac{1}{2}\pi + iY, \quad -\tfrac{1}{2}\pi - iY,$$

and making $Y \to \infty$. The sum of the residues is Fejér's sum, and the integrals along the vertical sides of the rectangle combine to give the integral in (3.1): I need hardly set out the details of the calculation.

4. I shall now deduce Watson's expansion (1.5) from (3.1): I shall prove, as he does, that the error in stopping the series at any term has the same sign as, and is numerically less than, the first term neglected. The proof is not quite so short as Watson's, since he, starting as he does from Szegö's formula (1.4), is able to use the known results about the asymptotic expansion of the logarithmic derivative of the Gamma-function. On the other hand the coefficients A_ν present themselves in a very natural way.

I need some preliminary theorems which I state as lemmas.

5. LEMMA 1. *If*

$$f(y) = \frac{1}{\tanh y\,(y^2+\frac{1}{4}\pi^2)}, \tag{5.1}$$

then

$$f(y) = \sum_0^\infty c_\nu y^{2\nu-1}, \tag{5.2}$$

where

$$c_0 = \frac{4}{\pi^2}, \quad c_\nu = (-1)^\nu (\tfrac{1}{2}\pi)^{-2\nu-2}\left(1 - \sum_{m=1}^{\nu} \frac{B_m \pi^{2m}}{2m!}\right) \quad (\nu \geqslant 1), \tag{5.3}$$

and the series is convergent for $|y| < \frac{1}{2}\pi$.

The coefficients c_ν *are alternatively positive and negative. And if*

$$f_p(y) = \sum_0^p c_\nu y^{2\nu-1}, \tag{5.4}$$

then

$$f_{2q-1}(y) < f(y) < f_{2q}(y), \tag{5.5}$$

for $q = 0, 1, \ldots$ *and all positive* y†.

First,

$$\frac{y}{\tanh y} = 1 + 2\sum_1^\infty (-1)^{\nu-1} \frac{2^{2\nu} B_\nu}{2\nu!} y^{2\nu} \tag{5.6}$$

† The left-hand inequality being omitted when $q = 0$.

for $|y|<\pi$, and

$$\frac{1}{y^2+\frac{1}{4}\pi^2}=\sum_0^\infty(-1)^\nu(\tfrac{1}{2}\pi)^{-2\nu-2}y^{2\nu}$$

for $|y|<\frac{1}{2}\pi$. The product of the two series is convergent for $|y|<\frac{1}{2}\pi$, and the formulae (5.3) follow from the ordinary rule for multiplication. Also, putting $y=\frac{1}{2}\pi i$ in (5.6),

$$\sum_{m=1}^\infty \frac{B_m\pi^{2m}}{2m!}=1,$$

so that c_ν has the sign $(-1)^\nu$.

To prove (5.5) we observe that

$$\begin{aligned}yf(y)&=\frac{1}{y^2+\frac{1}{4}\pi^2}\left(1+2y^2\sum_1^\infty\frac{1}{y^2+r^2\pi^2}\right)\\&=\frac{1}{y^2+\frac{1}{4}\pi^2}+\frac{2y^2}{(r^2-\frac{1}{4})\pi^2}\left(\frac{1}{y^2+\frac{1}{4}\pi^2}-\frac{1}{y^2+r^2\pi^2}\right),\end{aligned}$$

and that

$$\frac{1}{y^2+a^2}=\frac{1}{a^2}-\frac{y^2}{a^4}+\ldots+(-1)^p\frac{y^{2p}}{a^{2p+2}}+(-1)^{p+1}\frac{y^{2p+2}}{a^{2p+2}(y^2+a^2)}.$$

It follows that

$$\begin{aligned}&y\{f(y)-f_p(y)\}\\&\quad=(-1)^{p+1}y^{2p+2}\left[\frac{(\frac{1}{2}\pi)^{-2p-2}}{y^2+\frac{1}{4}\pi^2}-\frac{2}{\pi^2}\sum_{r=1}^\infty\frac{1}{r^2-\frac{1}{4}}\left\{\frac{(\frac{1}{2}\pi)^{-2p}}{y^2+\frac{1}{4}\pi^2}-\frac{(r\pi)^{-2p}}{y^2+r^2\pi^2}\right\}\right].\end{aligned}$$

But

$$4-2\sum_1^\infty\frac{1}{r^2-\frac{1}{4}}=4\left(1-2\sum_1^\infty\frac{1}{4r^2-1}\right)=4\left\{1-\sum_1^\infty\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)\right\}=0,$$

and so

$$y\{f(y)-f_p(y)\}=(-1)^{p+1}2\left(\frac{y}{\pi}\right)^{2p+2}\sum_{r=1}^\infty\frac{1}{r^2-\frac{1}{4}}\frac{r^{-2p}}{y^2+r^2\pi^2},$$

which has the sign $(-1)^{p+1}$.

6. This is the main lemma; but we shall also need the values of three definite integrals, and it will be convenient to use a special notation. Suppose that A and B can be so chosen that the integrals

$$\int_0\left\{\phi(x)-\frac{A}{x}\right\}dx,\qquad\int^\infty\left\{\phi(x)-\frac{B}{x}\right\}dx$$

are convergent. Then I shall write

$$\int^{*} \phi(x)\,dx = \int_0^1 \left\{\phi(x) - \frac{A}{x}\right\} dx + \int_1^\infty \left\{\phi(x) - \frac{B}{x}\right\} dx.$$

Thus

$$\int^{*} \frac{dx}{x} = 0, \qquad \int^{*} \frac{dx}{1+x} = 0, \qquad \int^{*} \frac{e^{-x}}{x}\,dx = -\gamma,$$

where γ is Euler's constant. In these three cases

(i) $A = B = 1$, (ii) $A = 0$, $B = 1$, (iii) $A = 1$, $B = 0$

LEMMA 2. *If* $\nu = 1, 2, \ldots,$ *then*

(6.1) $$I_\nu = \int_0^\infty (\tanh u - 1)\, u^{2\nu-1}\,du = -\frac{1-2^{1-2\nu}}{2\nu}\pi^{2\nu} B_\nu.$$

Also

(6.2) $$I_0 = \int^{*} \frac{\tanh u}{u}\,du = \gamma + 2\log 2 - \log \pi,$$

(6.3) $$J = \int^{*} \frac{du}{\tanh u\,(u^2 + \frac{1}{4}\pi^2)} = \frac{4\log\pi}{\pi^2} + \frac{8}{\pi^2}\sum_1^\infty \frac{\log k}{4k^2-1}.$$

(i) First,

$$I_\nu = -2\int_0^\infty \frac{e^{-2u}}{1+e^{-2u}}\,u^{2\nu-1}\,du = -2\sum_1^\infty (-1)^{k-1}\int_0^\infty e^{-2ku}\,u^{2\nu-1}\,du$$

$$= -2^{1-2\nu}(2\nu-1)!\,(1^{-2\nu} - 2^{-2\nu} + \ldots) = -\frac{1-2^{1-2\nu}}{2\nu}\pi^{2\nu} B_\nu.$$

(ii) Next, if $s > 0$, we have

$$I_0(s) = \int_0^1 \tanh u\, u^{s-1}\,du + \int_1^\infty (\tanh u - 1)\, u^{s-1}\,du$$

$$= \int_0^\infty (\tanh u - 1)\, u^{s-1}\,du + \int_0^1 u^{s-1}\,du = -2\int_0^\infty \frac{e^{-2u}}{1+e^{-2u}}\,u^{s-1}\,du + \frac{1}{s}$$

$$= -2^{1-s}\,\Gamma(s)(1^{-s} - 2^{-s} + \ldots) + \frac{1}{s} = -2^{1-s}(1-2^{1-s})\,\Gamma(s)\,\zeta(s) + \frac{1}{s}.$$

Making $s \to 0$, and remembering that $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log 2\pi$, we obtain (6.2).

(iii) Finally,

$$J_1 = \int_0^\infty \left(\frac{1}{\tanh u} - \frac{1}{u}\right) \frac{du}{u^2+\frac{1}{4}\pi^2} = 2 \sum_{k=1}^\infty \int_0^\infty \frac{u\,du}{(u^2+k^2\pi^2)(u^2+\frac{1}{4}\pi^2)}$$

$$= \frac{4}{\pi^2} \sum_1^\infty \frac{\log 4k^2}{4k^2-1} = \frac{8}{\pi^2} \sum_1^\infty \frac{\log k}{4k^2-1} + \frac{8 \log 2}{\pi^2} \sum_1^\infty \frac{1}{4k^2-1}$$

$$= \frac{8}{\pi^2} \sum_1^\infty \frac{\log k}{4k^2-1} + \frac{4 \log 2}{\pi^2}.$$

Also

$$J - J_1 = \int_0^1 \left\{\frac{1}{u(u^2+\frac{1}{4}\pi^2)} - \frac{4}{\pi^2 u}\right\} du + \int_1^\infty \frac{du}{u(u^2+\frac{1}{4}\pi^2)} = -\frac{4}{\pi^2} \log \frac{2}{\pi};$$

and (6.3) follows.

7. We define $f(y)$ and $f_p(y)$ as in (5.1) and (5.4). Then

$$(7.1) \qquad L_n = \int_0^\infty \frac{\tanh Ny}{\tanh y} \frac{dy}{y^2+\frac{1}{4}\pi^2} = \int_0^\infty \tanh Ny\, f(y)\, dy$$

$$= C + \int_0^1 \tanh Ny\, f(y)\, dy - 2 \int_1^\infty \frac{e^{-2Ny}}{1+e^{-2Ny}} f(y)\, dy,$$

where

$$(7.2) \qquad C = \int_1^\infty f(y)\, dy.$$

We begin by calculating $L_{n,p}$, the result of replacing $f(y)$ by $f_p(y)$ in the last two terms of (7.1). We obtain

$$(7.3) \qquad L_{n,p} = P_0 + \sum_1^p P_m,$$

where

$$(7.4) \qquad P_0 = C + c_0 \left(\int_0^1 \frac{\tanh Ny}{y}\, dy - 2 \int_1^\infty \frac{e^{-2Ny}}{1+e^{-2Ny}} \frac{dy}{y}\right),$$

$$(7.5) \qquad P_m = c_m \left(\int_0^1 \tanh Ny\, y^{2m-1}\, dy - 2 \int_1^\infty \frac{e^{-2Ny}}{1+e^{-2Ny}} y^{2m-1}\, dy\right).$$

Here

$$P_m = \frac{c_m}{N^{2m}} \left(\int_0^N \tanh u\, u^{2m-1}\, du - 2 \int_N^\infty \frac{e^{-2u}}{1+e^{-2u}} u^{2m-1}\, du\right)$$

$$= \frac{c_m}{N^{2m}} \left\{\int_0^N \tanh u\, u^{2m-1}\, du + \int_N^\infty (\tanh u - 1)\, u^{2m-1}\, du\right\}$$

$$= \frac{c_m}{N^{2m}} \left(I_m + \int_0^N u^{2m-1}\, du\right) = \frac{c_m I_m}{N^{2m}} + \frac{c_m}{2m},$$

if $m > 0$; and

$$P_0 = C + c_0\left(\int_0^N \frac{\tanh u}{u}\,du + \int_N^\infty \frac{\tanh u - 1}{u}\,du\right)$$

$$= C + c_0\left(I_0 + \int_1^N \frac{du}{u}\right) = C + c_0(I_0 + \log N).$$

Hence

$$L_{n,p} = \frac{4}{\pi^2}\log N + E_0 + \sum_1^p \frac{E_m}{N^{2m}}, \tag{7.6}$$

where

$$E_0 = \int_1^\infty f(y)\,dy + \frac{4}{\pi^2} I_0 + \sum_1^p \frac{c_m}{2m}, \tag{7.7}$$

and

$$E_m = c_m I_m \tag{7.8}$$

if $m > 0$.

Substituting for c_m and I_m, from (5.3) and (6.1), in (7.8), we find that

$$E_m = (-1)^{m-1}\frac{4}{\pi^2} A_m, \tag{7.9}$$

where A_m is defined by (1.7). As regards E_0, we have

$$\sum_1^\infty \frac{c_m}{2m} = \int_0^1 \left\{f(y) - \frac{4}{\pi^2 y}\right\} dy,$$

and so

$$E_0 = E_0^* - \sum_{p+1}^\infty \frac{c_m}{2m}, \tag{7.10}$$

where

$$E_0^* = J + \frac{4}{\pi^2} I_0 = \frac{8}{\pi^2}\sum_1^\infty \frac{\log k}{4k^2 - 1} + \frac{4}{\pi^2}(\gamma + 2\log 2),$$

on substituting the values of J and I_0 from (6.3) and (6.2). Since

$$\gamma + 2\log 2 = -\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})},$$

this gives

$$E_0^* = \frac{4}{\pi^2} A_0, \tag{7.11}$$

where A_0 is defined by (1.6). It now follows from (7.6), (7.9), (7.10), and (7.11) that

$$L_{n,p} + \sum_{p+1}^\infty \frac{c_m}{2m} = \frac{4}{\pi^2}\left\{\log N + A_0 + \sum_1^p \frac{(-1)^{m-1} A_m}{N^{2m}}\right\}, \tag{7.12}$$

which is the sum of the first $p+2$ terms in Watson's expansion.

On the other hand

$$L_n - L_{n,p} = \int_0^1 \tanh Ny\{f(y)-f_p(y)\}\,dy - 2\int_1^\infty \frac{e^{-2Ny}}{1+e^{-2Ny}}\{f(y)-f_p(y)\}\,dy$$

$$= \int_0^1 \{f(y)-f_p(y)\}\,dy - 2\int_0^\infty \frac{e^{-2Ny}}{1+e^{-2Ny}}\{f(y)-f_p(y)\}\,dy$$

$$= \sum_{p+1}^{\infty} \frac{c_m}{2m} - 2\int_0^\infty \frac{e^{-2Ny}}{1+e^{-2Ny}}\{f(y)-f_p(y)\}\,dy;$$

and so, after (7.12),

$$(7.13)\quad L_n = \frac{4}{\pi^2}\left\{\log N + A_0 + \sum_1^p \frac{(-1)^{m-1}A_m}{N^{2m}}\right\} - 2\int_0^\infty \frac{e^{-2Ny}}{1+e^{-2Ny}}\{f(y)-f_p(y)\}\,dy.$$

But, by (5.5), $f(y)-f_p(y)$ has the sign $(-1)^{p-1}$, and therefore the last term in (7.13) has the sign $(-1)^p$. It follows that Watson's series is asymptotic, the error in stopping at any term being of the same sign as, and less in absolute value than, the first term neglected.

8. It is plain from (3.1) that L_n is an increasing function of n. Also

$$\frac{d}{dn}\tanh Ny = 2y\,\mathrm{sech}^2\,Ny$$

decreases with n, so that $\Delta^2 L_n < 0$ and l_n is concave. But I do not see how to deduce the general inequality in (1.8) directly from (3.1); and for this purpose I use the second formula to which I referred in § 2.

This is

$$(8.1)\qquad L_n = \frac{4}{\pi^2}\int_0^\infty \frac{\sinh Ny}{\sinh y}\log\coth\tfrac{1}{2}Ny\,dy;$$

it may be proved directly by complex integration†.

We consider the imaginary part of $\int F(z)\,dz$, where

$$F(z) = F(x+iy) = \frac{\sin Nz}{\sin z}\log(i\cot\tfrac{1}{2}Nz),$$

† I found this formula first indirectly by a transformation of (3.1). The method of proof which I use here was suggested to me by Prof. Watson.

and the contour of integration is the rectangle $(0, \pi, \pi+iY, iY)$, the logarithmic singularities of $F(z)$ at the points

$$\frac{k\pi}{N} \quad (0 \leqslant k \leqslant N = 2n+1)$$

being avoided in the first instance by small semicircles (or quadrants) of radius ρ above the real axis. The integral tends to zero when $\rho \to 0$ and $Y \to \infty$.

It can easily be verified that there is a branch of the logarithm which has an imaginary part $\frac{1}{2}\pi i$ in the intervals

$$(8.2) \qquad 0 < x < \frac{\pi}{N}, \quad \frac{2\pi}{N} < x < \frac{3\pi}{N}, \quad \ldots, \quad \frac{(N-1)\pi}{N} < x < \pi,$$

and an imaginary part $-\frac{1}{2}\pi i$ in the complementary intervals of $(0, \pi)$. Since $\sin Nx$ is positive in the intervals (8.2), and negative in the complementary intervals,

$$\mathfrak{J}\{F(z)\} = \tfrac{1}{2}\pi i \frac{|\sin Nx|}{\sin x}$$

along the real part of the contour. Also

$$F(iy) = \frac{\sinh Ny}{\sinh y} \log \coth \tfrac{1}{2}Ny, \qquad F(\pi+iy) = -\frac{\sinh Ny}{\sinh y} \log \coth \tfrac{1}{2}Ny.$$

Hence

$$\int_0^{\pi} \frac{|\sin Nx|}{\sin x}\, dx = \frac{4}{\pi} \int_0^{\infty} \frac{\sinh Ny}{\sinh y} \log \coth \tfrac{1}{2}Ny\, dy,$$

which is (8.1).

Since

$$\sinh Ny \log \coth \tfrac{1}{2}Ny = 1 - 2\sum_1^{\infty} \frac{e^{-2kNy}}{4k^2-1},$$

(8.1) may be written as

$$(8.3) \qquad L_n = \frac{4}{\pi^2} \int_0^{\infty} \frac{1}{\sinh y} \left(1 - 2\sum_1^{\infty} \frac{e^{-2kNy}}{4k^2-1}\right) dy.$$

The function in brackets is positive, and its r-th derivative with respect to n has the sign $(-1)^{r-1}$, so that (8.3) makes Szegö's inequalities (1.8) intuitive.

We can also write (8.3) as

$$L_n = \frac{8}{\pi^2}\sum_1^\infty \frac{1}{4k^2-1}\int_0^\infty \frac{1-e^{-2kNy}}{\sinh y}\,dy.$$

Since
$$\int_0^\infty \frac{1-e^{-2kNy}}{\sinh y}\,dy = 2\left(1+\frac{1}{3}+\dots+\frac{1}{2Nk-1}\right),$$

this gives another proof of Szegö's formula.

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NOTES ON FOURIER SERIES (II): ON THE GIBBS PHENOMENON

G. H. HARDY *and* W. W. ROGOSINSKI*.

1. In what follows $\theta = \xi$ is a point of continuity or jump (ordinary discontinuity) of a real, periodic, and integrable function $f(\theta)$, and

(1. 1) $$C = \tfrac{1}{2}\{f(\xi-0)+f(\xi+0)\}, \quad D = f(\xi+0)-f(\xi-0).$$

We shall suppose, to fix our ideas, that $D \geqslant 0$; if not, we may consider $f(-\theta)$ instead of $f(\theta)$.

We write $s_n(\theta) = s_n(\theta, f)$ for the partial sum of the F.s. of f. Then the *Gibbs set* $G(\xi) = G(\xi, f)$ of f, for $\theta = \xi$, is the aggregate of the limits η of $s_n(\theta)$ when $n \to \infty$ and $\theta \to \xi$ in any manner. It is either a single point (then necessarily C) or a closed interval (finite or infinite).

If f is the simple function

(1. 2) $$\phi(\theta-\xi) = \frac{1}{\pi}\sum_{1}^{\infty}\frac{\sin n(\theta-\xi)}{n},$$

for which

$$\phi(-0) = -\tfrac{1}{2}, \quad \phi(0) = 0, \quad \phi(+0) = \tfrac{1}{2}, \quad C = 0, \quad D = 1,$$

then $G(\xi)$ is the closed interval

(1. 3) $$I = \langle C-HD, \; C+HD\rangle,\dagger$$

here $\langle -H, H\rangle$, where

(1. 4) $$H = \frac{1}{\pi}\int_0^{\pi}\frac{\sin u}{u}\,du = \cdot 58\ldots > \tfrac{1}{2}.$$

Thus $G(\xi)$ is a closed interval symmetrical about the origin, of length $2HD > D$. This is the "Gibbs phenomenon".

All this is true for fairly general classes of functions. Thus it is true whenever

(1. 5) $$f(\theta) = g(\theta)+D\phi(\theta-\xi) = g(\theta)+\psi(\theta),$$

where g is any function whose F.s. is uniformly convergent near $\theta = \xi$, in particular any function continuous and of bounded variation in an interval including ξ. It is natural to ask how much of it is true for a general f with a jump at ξ.

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* Received 26 May, 1943; read 17 June, 1943.

† We use (a, b), $\langle a, b\rangle$, $(a, b\rangle$, $\langle a, b)$ for open, closed, and half-closed intervals.

There are two theorems of Rogosinski which give a partial answer to the question. Rogosinski proved that

(I)* *G always includes I* (so that its length is always at least $2HD$);

(II)† *if f is continuous at ξ, then G is symmetrical about C;*

and it might seem at first sight that the condition of continuity in (II) should be superfluous. For we may suppose without loss of generality that $f(\xi) = C$, and then, if we decompose f as in (1.5), we have

$$g(\xi-0) = g(\xi) = g(\xi+0) = C,$$

so that g is continuous at ξ. Thus $G(\xi, g)$ is symmetrical about C, and $G(\xi, \psi)$ is symmetrical about 0; and this might lead us to expect that $G(\xi, f)$ should be symmetrical about C.

Our object here is to show that this is not so, by proving

(III) *there is a function $f(\theta)$, continuous except for a jump at ξ, whose Gibbs interval $G(\xi)$ is finite but not symmetrical about C.*

The explanation will be, to put it roughly, that G is formed by the combination of two symmetrical intervals in an entirely unsymmetrical way.

We interpolate here one further remark, which will be important in the sequel, about the special function ϕ. We take $\xi = 0$. Then

$$s_n(h, \phi) \to \frac{1}{\pi}\int_0^a \frac{\sin u}{u}\, du$$

if $n \to \infty$, $h \to 0$ and $nh \to a$. In particular $s_n(h, \phi) \to \frac{1}{2}$ when $nh \to \infty$.

2. We take $\xi = 0$, $C = 0$, $D = 1$, and decompose f as in (1.5). We begin by constructing a continuous g with the properties:

(i) *$s_n(h, g) \to 0$ if $n \to \infty$, $h \to 0$, and either $h \leqslant 0$ or else $h > 0$ and $h = O(n^{-1})$;*

(ii) *$G(g)$ includes $\eta = 1$;*

(iii) *$G(g)$ is finite.*

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* *Schriften d. Königsberger gelehrten Gesellschaft*, 3 (1926), 57–98.

† *Math. Annalen*, 95 (1925), 110–134 (128). Both of these theorems are proved in our Cambridge Tract *Fourier series* (now in course of printing). Rogosinski also proved that G is exactly I whenever the F.c. of f are $O(n^{-1})$, but this theorem is not relevant here.

We assume at first only (α) that g is continuous, (β) that $g = 0$ for $-\pi < \theta \leqslant 0$, and ($\gamma$) that $\theta^{-1}g(\theta)$ is integrable; and prove that (i) is true of any such g. For this, it is sufficient to prove that

$$|J| = \left|\frac{1}{\pi}\int_0^\delta g(\theta)\frac{\sin n(\theta-h)}{\theta-h}\,d\theta\right| < \epsilon \tag{2.1}$$

for a $\delta = \delta(\epsilon)$, large enough n, and small enough h satisfying the conditions of (i).

If $h < 0$, then

$$|J| \leqslant \frac{1}{\pi}\int_0^\delta \frac{|g(\theta)|}{\theta}\,d\theta < \epsilon$$

for an appropriate δ. We may therefore suppose $h \geqslant 0$. Then

$$\left|\frac{1}{\pi}\int_0^{\frac{1}{2}h} g(\theta)\frac{\sin n(\theta-h)}{\theta-h}\,d\theta\right| \leqslant \frac{1}{\pi}\int_0^{\frac{1}{2}h}\frac{|g(\theta)|}{\theta}\,d\theta \to 0,$$

and

$$\left|\frac{1}{\pi}\int_{\frac{3}{2}h}^{\delta} g(\theta)\frac{\sin n(\theta-h)}{\theta-h}\,d\theta\right| \leqslant \frac{3}{\pi}\int_{\frac{3}{2}h}^{\delta}\frac{|g(\theta)|}{\theta}\,d\theta < \tfrac{1}{3}\epsilon$$

for an appropriate δ. Hence

$$|J-K| \leqslant \tfrac{2}{3}\epsilon, \tag{2.2}$$

where

$$K = \frac{1}{\pi}\int_{\frac{1}{2}h}^{\frac{3}{2}h} g(\theta)\frac{\sin n(\theta-h)}{\theta-h}\,d\theta, \tag{2.3}$$

for small h, and it is sufficient to prove that $K \to 0$. But

$$|K| \leqslant \frac{n}{\pi}\int_{\frac{1}{2}h}^{\frac{3}{2}h}|g(\theta)|\,d\theta \leqslant \frac{3nh}{2\pi}\int_{\frac{1}{2}h}^{\frac{3}{2}h}\frac{|g(\theta)|}{\theta}\,d\theta \to 0$$

because $nh = O(1)$; and this, with (2. 2), proves (2. 1). It is to be observed that we have not assumed nh bounded in proving (2. 2).

3. We have now to prove (ii) and (iii), and for this we must define $g(\theta)$ more precisely. We take

$$h_k = 4^{-2k}\pi, \quad n_k = 4^{4^{k+1}} \quad (k = 1, 2, \ldots), \tag{3.1}$$

and denote the interval $(\frac{1}{2}h_k, \frac{3}{2}h_k)$ by i_k. Since $9h_{k+1} < h_k$, these intervals are separated. We take

$$g(\theta) = h_k^{\frac{1}{2}}\sin n_k(\theta-h_k)\,\mathrm{sgn}(\theta-h_k) \tag{3.2}$$

in i_k, and $g(\theta) = 0$ elsewhere. Since $n_k h_k$ is an even multiple of π, $g(\theta)$ is continuous; and since $g(\theta) = O(\theta^{\frac{1}{2}})$, $\theta^{-1}g$ is integrable. Thus $g(\theta)$ satisfies conditions (α)–(γ) of § 2, and therefore satisfies (i). In order to prove that it satisfies (ii) and (iii), it is sufficient, after (2. 2), to prove (1) that $K > 1$ for arbitrarily large n and corresponding small h, and (2) that K is bounded.

For (1), we take $n = n_k$, $h = h_k$. Then

$$K = \frac{h_k^{\frac{1}{2}}}{\pi}\int_{\frac{1}{2}h_k}^{\frac{3}{2}h_k}\frac{\sin^2 n_k(\theta-h_k)}{|\theta-h_k|}\,d\theta = \frac{h_k^{\frac{1}{2}}}{\pi}\int_0^{\frac{1}{2}n_k h_k}\frac{1-\cos 2u}{u}\,du$$
$$\sim \frac{h_k^{\frac{1}{2}}}{\pi}\log\tfrac{1}{2}n_k h_k \sim \frac{h_k^{\frac{1}{2}}}{\pi}\log n_k = \frac{4\log 4}{\sqrt{\pi}} > 1.$$

4. It remains to prove (2): for this, we need a (trivial) lemma.

Lemma. *Suppose that $p > 0$, $q > 0$, $c > 0$, $M = \max(p,q)$,*

$$P = \int_\rho^\sigma \frac{\sin pu\cos qu}{u}\,du, \quad Q = \int_\rho^\sigma \frac{\sin pu\sin qu}{u}\,du.$$

Then there is a constant A such that (a) $|P| < A$ for all ρ and σ, (b) $|Q| < A$ for $p \leqslant \frac{1}{2}q$ or $p \geqslant 2q$ and all ρ and σ, and (c)

$$|Q| < A\log(2+2Mc)$$

for $\frac{1}{2}q \leqslant p \leqslant 2q$, $|\rho| < c$, $|\sigma| \leqslant c$.

We may plainly suppose $\rho = 0$, $\sigma > 0$. Then (a) is familiar. For (b),

$$2Q = \int_0^\sigma \frac{\cos(p-q)u-\cos(p+q)u}{u}\,du = \int_{|p-q|\sigma}^{(p+q)\sigma}\frac{1-\cos w}{w}\,dw,$$

which is positive and less than $2\log\dfrac{p+q}{|p-q|} \leqslant 2\log 3$. Finally, for (c),

$$2Q = \int_0^{(p+q)\sigma}\frac{1-\cos w}{w}\,dw - \int_0^{|p-q|\sigma}\frac{1-\cos w}{w}\,dw$$
$$\leqslant \int_0^1 \frac{1-\cos w}{w}\,dw + \int_1^{(p+q)\sigma}\frac{1-\cos w}{w}\,dw < \tfrac{1}{4}+2\log(2+2Mc)$$
$$< A\log(2+2Mc)$$

for an appropriate A.*

* The second integral in the last line is absent if $(p+q)\sigma \leqslant 1$.

Since $9h_{k+1} < h_k$, there is at most one interval $i_k = (\frac{1}{2}h_k, \frac{3}{2}h_k)$ which has points in common with $i = (\frac{1}{2}h, \frac{3}{2}h)$; and, if there is one, then $\frac{1}{3}h_k \leqslant h \leqslant 3h_k$. If there is none, $K = 0$. If there is one, then K is the sum of at most two* integrals of the type

$$\pm \frac{h_k^{\frac{1}{4}}}{\pi} \int_R^S \frac{\sin n(\theta-h)}{\theta-h} \sin n_k(\theta-h_k)\, d\theta,$$

where R and S lie in i. But

$$\int_R^S \frac{\sin n(\theta-h)}{\theta-h} \sin n_k(\theta-h_k)\, d\theta = \int_\rho^\sigma \frac{\sin nu}{u} \sin n_k(u+h-h_k)\, du$$

$$= \sin n_k(h-h_k) \int_\rho^\sigma \frac{\sin nu \cos n_k u}{u}\, du + \cos n_k(h-h_k) \int_\rho^\sigma \frac{\sin nu \sin n_k u}{u}\, du,$$

where $|\rho| \leqslant \frac{1}{2}h$ and $|\sigma| \leqslant \frac{1}{2}h$. The first integral is bounded; and the second is bounded unless $\frac{1}{2}n_k \leqslant n \leqslant 2n_k$, in which case its modulus is less than

$$A \log\{2+h\max(n, n_k)\}.$$

It follows that

$$K = O(h_k^{\frac{1}{4}} \log n_k h_k) = O(4^{-k}.4^k) = O(1),$$

and that g has all the properties demanded in § 2.

5. It is now easy to prove (III). We define f by (1. 5), with $\xi = 0$, $D = 1$ and g the g of § 3. The points of $G(f)$ are contributed by (a) pairs (n, h) for which $h \leqslant 0$ or $h > 0$ and $nh = O(1)$, and (b) pairs (n, h) for which $h > 0$ and $nh \to \infty$.

The interval $G(g)$ is $\langle -\eta_0, \eta_0 \rangle$, where η_0 is finite and at least 1; and all of this interval, except the origin, is contributed by the pairs (b) only. For such pairs $s_n(h, \psi) \to \frac{1}{2}$; so that their contribution to $G(f)$ is the interval $\langle -\eta_0+\frac{1}{2}, \eta_0+\frac{1}{2} \rangle$. On the other hand the pairs (a) contribute nothing to $G(g)$ except the origin, and their contribution to $G(\psi)$ is the interval $\langle -H, H \rangle$. Since $H < 1 < \eta_0+\frac{1}{2}$, $G(f)$ includes $\eta_0+\frac{1}{2}$ but not $-\eta_0-\frac{1}{2}$, and is therefore unsymmetrical about the origin.

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* Two if h_k is in i, since $\text{sgn}(\theta-h_k)$ changes sign at $\theta = h_k$; otherwise one only.

(*b*) Summability of a Fourier Series or its Conjugate

INTRODUCTION TO PAPERS ON THE SUMMABILITY OF A FOURIER SERIES OR ITS CONJUGATE

The early papers of Hardy, and of Hardy and Littlewood, on the summability of Fourier series were concerned with Cesàro summability. In 1913, 4, Hardy completed a theorem of Lebesgue by showing that if the series

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)$$

is the Fourier series of a function $f \in L(-\pi,\pi)$, then the series is summable (C,δ) for all $\delta>0$ almost everywhere. In this paper he made also an improvement on results of W. H. Young by showing that the series

$$\sum_{n=2}^{\infty}\frac{a_n\cos n\theta+b_n\sin n\theta}{\log n}$$

is convergent almost everywhere.

Hardy's joint papers with Littlewood on Cesàro summability of Fourier series centre round the problem of summability (C), that is to say the problem of determining necessary and sufficient conditions for the summability (C,δ) of the Fourier series of f at a given point θ for *some* $\delta \geqslant -1$. No solution is known for the corresponding problem where δ is given, nor is one likely to exist, but for this more general problem Hardy and Littlewood obtained (1924, 1) a complete solution, namely that some Cesàro mean of the Fourier series of f converges if and only if some Cesàro mean of f converges. More precisely, they prove that if $f \in L(-\pi,\pi)$,

$$\phi(t)=\phi_0(t)=\tfrac{1}{2}\{f(\theta+t)+f(\theta-t)\},$$

and for any positive integer α we define

$$\phi_\alpha(t)=\frac{1}{t}\int_0^t\frac{du_1}{u_1}\int_0^{u_1}\cdots\int_0^{u_{\alpha-2}}\frac{du_{\alpha-1}}{u_{\alpha-1}}\int_0^{u_{\alpha-1}}\phi(u_\alpha)\,du_\alpha, \tag{1}$$

then in order that the Fourier series of f is summable (C,δ) to the sum s for some $\delta \geqslant -1$, it is necessary and sufficient that $\phi_\alpha(t)\to s$ as $t\to 0$ for some $\alpha \geqslant 0$.

The function ϕ_α here is the (H,α) mean of ϕ, and, as observed by Hardy and Littlewood in 1924, 1 and 1924, 4, this mean has equivalent properties to the (C,α) mean

defined by the relation

$$\phi_\alpha(t) = \frac{\alpha}{t^\alpha}\int_0^t (t-u)^{\alpha-1}\phi(u)\,du \quad (t > 0,\ \alpha > 0), \tag{2}$$

a definition which is applicable to non-integral α. Using this latter definition, Hardy and Littlewood gave in 1927, 2 an improvement on their solution of the sufficiency part of the summability (C) problem.

A number of papers related to 1924, 1 were concerned with conditions under which a Fourier series is either summable (C, δ) for all $\delta > 0$ or not summable (C). The main results here are that this is true if either of the following conditions is satisfied:

(i) there exists $A > 0$ such that $f(x) \geqslant -A$ for all x of some neighbourhood of θ (1926, 10),

(ii) $\int_0^t |\phi(u)|^p\,du = O(t)$ for some $p \geqslant 1$ (1931, 5).

Under the condition (i), a necessary and sufficient condition for summability (C) is that $\phi_1(t)$ tends to a limit as $t \to 0+$, and under (ii) a necessary and sufficient condition is that $\phi_\alpha(t)$ tends to a limit for some $\alpha > 1/p$. Another related result (1928, 3) is that if $\int_0^t |d(u\phi)| = O(t)$ as $t \to 0+$, then the Fourier series of f at θ is either convergent or non-summable (C).

Hardy and Littlewood considered also the more difficult problem of the summability (C) of the conjugate series

$$\sum (b_n \cos n\theta - a_n \sin n\theta),$$

and again obtained a complete solution, namely that some (C, δ) mean of the conjugate series converges to the sum s if and only if some (C, α) mean of

$$\chi(t) = \frac{1}{2\pi}\int_t^\pi \{f(\theta+t)-f(\theta-t)\}\cot\tfrac{1}{2}t\,dt$$

converges to s. In particular, they showed that the conjugate series of an integrable f is almost everywhere summable (C, δ) for all $\delta > 0$, thus improving a result of Plessner, which gave only summability $(C, 1)$.

In addition to their work on ordinary Cesàro summability, Hardy and Littlewood considered also strong Cesàro summability of order 1. The concept of strong Cesàro summability was first introduced in 1913, 11, and a local condition for strong summability was obtained there for a function of the class $L^2(-\pi, \pi)$. This result was generalized by various authors, and in 1927, 3 Hardy and Littlewood gave a further extension, obtaining a test for strong summability in the case where $f \in L^p(-\pi, \pi)$, with $p > 1$. The case $p = 1$ was left open, and in 1935, 5 they showed, by a counter-example of extreme sophistication, that their local conditions applicable for $p > 1$ fail when $p = 1$.

In 1936, 2 Hardy and Littlewood obtained an inequality ostensibly related to results concerning strong summability, but also related to the coefficient inequalities studied by W. H. Young, Hausdorff, and Hardy and Littlewood themselves. This paper led to further generalizations of the coefficient inequalities by H. R. Pitt and others, culminating in the recent work of E. M. Stein and G. Weiss. The paper also contains a useful inequality concerning conjugate functions.

The two papers 1947, 1 and 1949, 1, written by Hardy in collaboration with Rogosinski, are concerned with Riemann summability. Of the two general types of Riemann summability, (R,p) and R_p (where p is a positive integer), the methods (R,p) are familiar; they are regular for $p \geqslant 2$, and are Fourier-effective for all p (i.e. they sum any Fourier series almost everywhere). In 1947, 1 the regular method R_2 is discussed, and 1949, 1 deals with the more difficult non-regular method R_1. These methods are not comparable with $(R,2)$ and $(R,1)$ respectively. The papers give necessary and sufficient conditions for the summability of a Fourier series by these methods, and show that both methods are Fourier-effective. The first paper contains also a useful general theorem on 'kernel methods' (cf. H–R, pp. 56–62).

To the papers of this section should be added certain other results contained in books by Hardy and his collaborators, e.g. the 'kernel methods' discussed on pp. 56–62 of H–R.

NOTE ON DIVERGENT FOURIER SERIES.

By *G. H. Hardy*, Trinity College, Cambridge.

§ 1. IF $f(x)$ is continuous for all values of x such that

$$a \leqq x \leqq b,$$

and has at most a finite number of maxima and minima between a and b, then for $a < x < b$

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{2}{b-a}\sum_{n=1}^{\infty}\int_a^b \cos\frac{2\pi m(x-t)}{b-a} f(t)\,dt$$

$$= a_0 + \sum_1^{\infty}\left\{a_m \cos\frac{2\pi mx}{b-a} + b_m \sin\frac{2\pi mx}{b-a}\right\}$$

where

$$a_0 = \frac{1}{b-a}\int_a^b f(t)\,dt,$$

$$a_m = \frac{2}{b-a}\int_a^b \cos\frac{2\pi mt}{b-a} f(t)\,dt,$$

$$b_m = \frac{2}{b-a}\int_a^b \sin\frac{2\pi mt}{b-a} f(t)\,dt.$$

This is in fact the ordinary form of Fourier's theorem.*

* For $z = a$ or b the series represents $\frac{1}{2}\{f(a) + f(b)\}$.

1904, 11 *Messenger of Mathematics*, 33, 137–44.

Now suppose that $f(x)$ satisfies these conditions except that it has a finite number of singularities α_k

$$(k=1, 2, \ldots, r;\ a<\alpha_1<\ldots<\alpha_r<b),$$

such that

$$f(x)-\sum_{i=1}^{i=m_k}\frac{A_{k,i}}{(x-\alpha_k)^i}$$

satisfies the conditions in the neighbourhood of α_k. Then $f(x)$ can be expanded in a Fourier series which is divergent but summable for all values of z in (a, b) except the points α_k.

For

$$\cot\frac{\pi(x-\alpha_k)}{b-a}=2\sin\frac{2\pi(x-\alpha_k)}{b-a}+2\sin\frac{4\pi(x-\alpha_k)}{b-a}+\ldots$$

$$=2\overset{\infty}{\underset{1}{S}}\sin\frac{2m\pi(x-\alpha_k)}{b-a}$$

for all values of x in (a, b) except α_k. Moreover, this equation may be differentiated any number of times.* Hence

$$\left(\frac{d}{dx}\right)^{i-1}\cot\frac{\pi(x-\alpha_k)}{b-a}$$

$$=2\left(\frac{2\pi}{b-a}\right)^{i-1}\overset{\infty}{\underset{1}{S}}m^{i-1}\sin\left\{\frac{2m\pi(x-\alpha_k)}{b-a}+\tfrac{1}{2}(i-1)\pi\right\}$$

for all such values of x. The part of

$$\left(\frac{d}{dx}\right)^{i-1}\cot\frac{\pi(x-\alpha_k)}{b-a}$$

which becomes infinite for $x=\alpha_k$ is

$$\frac{b-a}{\pi}\,\frac{(-)^{i-1}(i-1)!}{(x-\alpha_k)^i}.$$

Hence

$$f(x)-\frac{\pi}{b-a}\sum_{i=1}^{i=m_k}\frac{(-)^{i-1}A_{k,i}}{(i-1)!}\left(\frac{d}{dx}\right)^{i-1}\cot\frac{\pi(x-\alpha_k)}{b-a}$$

* *Cambridge Philosophical Transactions*, Vol. XIX, pp. 303–305.

remains finite for $x = \alpha_k$, and

$$f(x) - \frac{\pi}{b-a} \sum_{k=1}^{r} \sum_{i=1}^{i=m_k} \frac{(-)^{i-1} A_{k,i}}{(i-1)!} \left(\frac{d}{dx}\right)^{i-1} \cot \frac{\pi(x-\alpha_k)}{b-a}$$

remains finite throughout (a, b) and clearly satisfies the Fourier conditions. This function may therefore be expanded in a convergent series

$$a_0' + \sum_1^\infty \left\{a_m' \cos\frac{2\pi m x}{b-a} + b_m' \sin\frac{2\pi m x}{b-a}\right\},$$

valid throughout (a, b). Thus the divergent series

$$a_0' + \sum_1^\infty \left\{a_m' \cos\frac{2\pi m x}{b-a} + b_m' \sin\frac{2\pi m x}{b-a}\right\}$$

$$+ \sum_{k=1}^{r} \sum_{i=1}^{i=m_k} \frac{(-)^{i-1} A_{k,i}}{(i-1)!} \left(\frac{2\pi}{b-a}\right)^{i} \overset{\infty}{\underset{1}{S}} m^{i-1} \sin\left\{\frac{2m\pi(x-\alpha_k)}{b-a} + \tfrac{1}{2}(i-1)\pi\right\}$$

represents $f(x)$ for $a < x < b$ except at the points α_k.

§ 2. Suppose, for instance, that

$$a = 0, \quad b = 2\pi,$$

$$m_k = 1 \quad (k = 1, 2, \ldots, r).$$

Then

$$f(x) = a_0' + \sum_1^\infty \{a_m' \cos mx + b_m' \sin mx\} + \sum_{k=1}^{r} A_k \overset{\infty}{\underset{1}{S}} \sin m(x-\alpha_k)$$

$$= a_0 + \overset{\infty}{\underset{1}{S}} \{a_m \cos mx + b_m \sin mx\},$$

where

$$a_0 = a_0',$$

$$a_m = a_m' - \sum_{k=1}^{r} A_k \sin m\alpha_k,$$

$$b_m = b_m' + \sum_{k=1}^{r} A_k \cos m\alpha_k.$$

Now

$$a_0' = \frac{1}{2\pi} \int_0^{2\pi} \{f(x) - \sum_{k=1}^{r} \tfrac{1}{2} A_k \cot \tfrac{1}{2}(x-\alpha_k)\}\, dx$$

$$= \frac{1}{2\pi} P \int_0^{2\pi} f(x)\, dx.$$

Since $$P\int_0^{2\pi} \cot\tfrac{1}{2}(x-\alpha_k)\,dx = 0.$$

And

$$a_m = \frac{1}{\pi}\int_0^{2\pi} \{f(x) - \sum_{k=1}^{r} \tfrac{1}{2}A_k \cot\tfrac{1}{2}(x-\alpha_k)\} \cos mx\,dx$$

$$- \sum_{k=1}^{r} A_k \sin m\alpha_k$$

$$= \frac{1}{\pi} P\int_0^{2\pi} f(x) \cos mx\,dx,$$

since

$$P\int_0^{2\pi} \tfrac{1}{2}\cot\tfrac{1}{2}(x-\alpha_k) \cos mx\,dx = -\pi \sin m\alpha_k.$$

Similarly $$b_m = \frac{1}{\pi} P\int_0^{2\pi} f(x) \sin mx\,dx.$$

Thus in the case in which $f(x)$ has a finite number of simple poles between 0 and 2π (or between a and b), but otherwise satisfies the conditions stated at the beginning, Fourier's theorem retains its ordinary form, if the series is interpreted as a summable divergent series, and the integrals as principal values.

The preceding analysis generally fails when a or b is a pole of $f(x)$, since then the function which we subtract from $f(x)$, in order to make the difference continuous for $x=a$, becomes infinite also for $x=b$.

§ 3. It is possible to find divergent Fourier series to represent functions which have singularities of types other than that discussed in § 1. I shall illustrate this by considering a special case. I suppose that $a=-\pi$, $b=\pi$, and that $f(x)$ becomes infinite for $x=0$ in such a way that

$$f(x) - \frac{A}{|x|^\mu}$$

remains continuous for $x=0$, μ being any positive quantity. Let

$$f(x) = \left(\frac{1+x}{1-x}\right)^\mu = c_0 + c_1 x + c_2 x^2 + \ldots,$$

where $c_0 = 1$. The series, inside the unit circle, represents the branch of $f(x)$ which $= 1$ for $x = 0$; and, by a theorem of M. Borel's, it is summable within the strip bounded by the lines

$$R(x) = -1, \quad R(x) = 1.$$

Hence the sum of the series

$$c_0 + c_1 e^{i\theta} + c_2 e^{2i\theta} + \ldots$$

is the value of the branch in question at the point $x = e^{i\theta}$; and it is easy to see that if $0 < \theta < \pi$ this is

$$e^{\frac{1}{2}\mu\pi i} \, | \cot \tfrac{1}{2}\theta \, |^{\mu},$$

and if $-\pi < \theta < 0$ it is

$$e^{-\frac{1}{2}\mu\pi i} \, | \cot \tfrac{1}{2}\theta \, |^{\mu}.$$

Hence

$$c_0 + c_1 \cos\theta + c_2 \cos 2\theta + \ldots$$
$$= \cos \tfrac{1}{2}\mu\pi \, | \cot \tfrac{1}{2}\theta \, |^{\mu}, \quad (-\pi < \theta < \pi).$$

If $0 < \mu < 2$,

$$\lim_{\theta=0} \left\{ | \tfrac{1}{2} \cot \tfrac{1}{2}\theta \, |^{\mu} - \frac{1}{|\theta|^{\mu}} \right\} = 0,$$

and
$$f(\theta) - \frac{A}{2^{\mu}} \, | \cot \tfrac{1}{2}\theta \, |^{\mu}$$

can be expanded in a Fourier series

$$a_0' + \sum_1^{\infty} (a_m' \cos m\theta + b_m' \sin m\theta),$$

so that
$$f(\theta) = a_0 + \mathop{S}_1^{\infty} (a_m \cos mx + b_m \sin mx),$$

where
$$a_m = a_m' + \frac{A c_m}{2^{\mu} \cos \frac{1}{2}\mu\pi},$$
$$b_m = b_m'$$

for all values of θ, $-\pi < \theta < \pi$, save 0.

If $2p < \mu < 2(p+1)$ we can find constants $k_1, \ldots, k_p$, such that

$$\lim_{\theta=0} \left\{ | \tfrac{1}{2} \cot \tfrac{1}{2}\theta \, |^{\mu} - \frac{1}{|\theta|^{\mu}} - \frac{k_1}{|\theta|^{\mu-2}} - \ldots - \frac{k_p}{|\theta|^{\mu-2p}} \right\} = 0,$$

and consequently constants $l_1, \ldots, l_p$, such that

$$\lim_{\theta=0} \left\{ |\tfrac{1}{2}\cot\tfrac{1}{2}\theta|^{\mu} + l_1 |\tfrac{1}{2}\cot\tfrac{1}{2}\theta|^{\mu-2} + \ldots + l_p |\tfrac{1}{2}\cot\tfrac{1}{2}\theta|^{\mu-2p} - \frac{1}{|\theta|^{\mu}} \right\} = 0,$$

and so a divergent Fourier expansion for $f(\theta)$.

If we observe that

$$\left(\frac{1+x}{1-x}\right)^{\mu} = \left(\frac{2}{1-x}\right)^{\mu} \{1 - \tfrac{1}{2}(1-x)\}^{\mu}$$

$$= 2^{\mu}(1-x)^{-\mu} - \mu 2^{\mu-1}(1-x)^{-\mu+1} + \frac{\mu.\mu-1}{1.2} 2^{\mu-2}(1-x)^{-\mu+2} - \ldots,$$

we see that

$$c_m = 2^{\mu} \frac{\mu.\mu+1\ldots\mu+m-1}{1.2\ldots m} \left(1 - \frac{1}{2}\frac{\mu.\mu-1}{1.\mu+m-1} + \frac{1}{2^2}\frac{\mu.\mu-1.\mu-1.\mu-2}{1.2.\mu+m-1.\mu+m-2} - \frac{1}{2^3}\frac{\mu.\mu-1.\mu-2.\mu-1.\mu-2.\mu-3}{1.2.3.\mu+m-1.\mu+m-2.\mu+m-3} + \ldots\right),$$

so that the dominant term in c_m is comparable with $\dfrac{\Gamma(\mu+m)}{\Gamma(m+1)}$ or with $m^{\mu-1}$.

Hence the dominant term in coefficient of $\cos m\theta$ in the expansion of $f(\theta)$ is comparable with $m^{\mu-1}$.

If $0<\mu<1$ the series for $f(\theta)$ is convergent. If $s<\mu<s+1$, it is, in the language of M. Césaro,* s times indeterminate. A similar remark of course applies to the series considered

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* Césaro, *Bull. des Sc. Math.*, Vol. XIV., 1890. See Borel's *Leçons sur les séries divergentes*, pp. 87–93.

If $u_1 + u_2 + \ldots$ is any series,

$$s_n = u_1 + \ldots + u_n,$$

$$s_n' = \frac{s_1 + \ldots + s_n}{n},$$

$$s_n^2 = \frac{s_1^1 + \ldots + s_n^1}{n}, \text{ etc.,}$$

the series is *convergent, once indeterminate, twice indeterminate*, etc., according as $s_n, s_n^1, s_n^2, \ldots$ is the first of the quantities s_n^m which has a limit for $n=\infty$.

in § 1. If M is the greatest of the integers m_k, the series is M times indeterminate. If, for instance, s_m is the sum of the first m terms of a series of the special type considered in § 2,

$$\lim_{m=\infty} \frac{s_1 + \ldots + s_m}{m} = f(x).$$

§ 4. The preceding part of this note was written before I had seen M. L. Fejér's important paper, "Untersuchungen über Fourierschen Reihen," recently printed in the *Mathematische Annalen.** In this paper he proves that if $f(x)$ is continuous throughout $(0, 2\pi)$ and

$$s_n = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt + \frac{1}{\pi} \sum_{\nu=1}^{n} \int_0^{2\pi} f(t) \cos \nu (t - x)\, dt,$$

then
$$\frac{s_0 + s_1 + \ldots + s_{n-1}}{n}$$

tends uniformly to the limit $f(x)$ for $n = \infty$. The importance of this extension lies on the one hand in the fact that the *only* condition imposed on $f(x)$ is simple continuity, and on the other hand in the uniformity of the convergence.

This result enables us at once to extend the results of §§ 1 and 2. For, since the divergent series there employed for

$$\left(\frac{d}{dx}\right)^{i-1} \cot \frac{\pi (x - \alpha_k)}{b - a}$$

is i times indeterminate, it follows at once that a function $f(x)$ continuous throughout (a, b) except at a finite number of points α_k distinct from a and b, and such that

$$f(x) - \sum_{i=1}^{i=m_k} \frac{A_{k,i}}{(x - \alpha_k)^i}$$

remains continuous near $x = \alpha_k$ may be represented for all values in (a, b) other than the values α_k by a divergent series which is at most only M times indeterminate, M being the greatest of the numbers m_k.

It does not, however, so far as I can see, follow that the series is necessarily *summable*. In a recent paper† I showed

* Vol. 58.

† *Quarterly Journal*, Vol. XXXV., p. 41.

that a series even *simply* indeterminate is not necessarily summable, unless a certain further condition is satisfied; and it does not seem to me possible to modify M. Fejér's reasoning so as to show that this condition is satisfied by his Fourier series without subjecting $f(x)$ to new conditions. Nor does it seem possible to prove directly that if $f(x)$ is continuous the series is summable, by the method which I adopted in the paper in the *Camb. Phil. Trans.* already referred to. In fact the proposition seems to me in all probability untrue, though I have not actually constructed any example to the contrary.*

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* In a note attached to his paper, M. Fejér gives a proof that the limit of

$$\vartheta_4(\nu, q) = 1 + 2\sum_{n=1}^{\infty} (-)^n q^{n^2} \cos 2n\pi\nu,$$

for $q = 1$ is 0 (except for certain exceptional values of ν), based on an extension which he gives to a known theorem of Frobenius and Hölder. As a matter of fact it follows from the theorem itself. For, for $q = 1$ the series for $\vartheta_4(\nu, q)$ takes the form

$$1 - 2\cos 2\nu\pi + 0 + 0 + 2\cos 4\nu\pi + 0 + 0 + 0 + 0 - 2\cos 6\nu\pi + 0 + \dots,$$

in which $u_n = 0$ if n is not a square, and $u_n = (-)^p\, 2\cos 2p\nu\pi$, if $n = p^2$.

Now consider the series

$$1 - e^{2\nu\pi i} + 0 + 0 + e^{4\nu\pi i} + 0 + 0 + \dots.$$

Here
$$s_0 = 1,$$

$$s_1 = s_2 = s_3 = \frac{1 - e^{4\nu\pi i}}{1 + e^{2\nu\pi i}},$$

$$s_4 = \dots = s_8 = \frac{1 + e^{6\nu\pi i}}{1 + e^{2\nu\pi i}},$$

and so on. Thus if

$$n = p^2 + q \quad (0 \leq q < 2p + 1),$$

$$s_0 + s_1 + \dots + s_n = \frac{1}{1 + e^{2\nu\pi i}} \left[\sum_1^p (2r-1)\{1 - (-)^r e^{2r\nu\pi i}\} + (q+1)\{1 + (-)^r e^{2(r+1)\nu\pi i}\} \right],$$

and it follows at once that

$$\lim. \frac{s_0 + s_1 + \dots + s_n}{n+1} = \frac{1}{1 + e^{2\nu\pi i}} = \tfrac{1}{2}(1 - i\tan\nu\pi),$$

and from this the result stated follows.

In this case it is easy to see that the further condition which ensures that the series is summable is also satisfied. The same method may be applied to other ϑ-series.

ON THE SUMMABILITY OF FOURIER'S SERIES

By G. H. HARDY.

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1. The investigations contained in this paper were suggested to me by a very interesting theorem established by Prof. W. H. Young,* viz., that *if*

$$\tfrac{1}{2}a_0+\Sigma\,(a_n \cos nx + b_n \sin nx) \equiv \tfrac{1}{2}a_0+\Sigma A_n$$

is the Fourier's series of a summable function $f(x)$, then the series

$$\Sigma n^{-\delta} A_n,$$

where δ is any positive number, converges almost everywhere.†

It has been proved by Dr. Marcel Riesz that, *if a series ΣA_n is summable $(C\delta)$, then $\Sigma n^{-\delta} A_n$ is convergent.* The proof of this theorem (in a considerably more general form) will appear in the *Cambridge Tract* "The General Theory of Dirichlet's Series" which Dr. Riesz and I are preparing in collaboration. This theorem of Riesz shows that Young's result above referred to could be deduced as a corollary from the theorem which follows.

THEOREM 1. — *The Fourier's series of any summable function is summable $(C\delta)$, for any positive value of δ, almost everywhere.*

That this theorem should be true appears *a priori* as highly probable. Fejér‡ proved that the series is summable $(C1)$ whenever

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

* *Comptes Rendus*, December 23rd, 1912. For the moment I state a part only of Young's result. If we insert the additional condition that $f(x)$ has its *square* summable, we obtain a theorem equivalent to one given by Prof. Hobson (*Proc. London Math. Soc.*, Ser. 2, Vol. 12, pp. 297–308). In so far as Fourier's series are concerned, Prof. Hobson's theorem is a special case of Prof. Young's; but Prof. Hobson's is much more general in another respect, viz. in that it is applicable to all series of normal functions, and not merely to Fourier's series.

† *I.e.*, with the possible exception of a set of values of x of measure zero.

‡ *Math. Ann.*, Vol. 58, pp. 51–69.

1913, 4 *Proceedings of the London Mathematical Society* (2), 12, 365–72.

exists, and Lebesgue* that it is summable almost everywhere; and as Riesz† and Chapman‡ have shown that, in Fejér's theorem, the index 1 may be replaced by any positive δ, it is natural to expect Lebesgue's result to be capable of a similar generalisation.

2. I proceed now to the proof of Theorem 1. In proving it I shall use, not the definition of summability $(C\delta)$ employed by Chapman§ and Knopp,‖ but a different definition, due to Riesz,¶ and shown by him** to be equivalent to the former. I shall say that a series ΣA_n is summable $(C\delta)$, to sum s, if

$$\sum_{n<\omega}\left(1-\frac{n}{\omega}\right)^{\delta} A_n \to s,$$

as the continuous variable ω tends to infinity.

Now suppose that $f(x)$ is a summable function with period 2π, and that, in the customary notation of the theory of Fourier's series††

$$f(x) \sim \tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx) \equiv \tfrac{1}{2}a_0 + \Sigma A_n.$$

Further, let

$$C_q(t) = \frac{t^q}{\Gamma(q+1)}\left\{1 - \frac{t^2}{(q+1)(q+2)} + \frac{t^4}{(q+1)(q+2)(q+3)(q+4)} - \dots\right\},$$

so that $$C_0(t) = \cos t, \quad C_1(t) = \sin t.$$

Then Prof. Young‡‡ has established the formula

$$(1) \quad \tfrac{1}{2}a_0 + \sum_{n<\omega}\left(1-\frac{n}{\omega}\right)^{\delta} A_n$$
$$= \frac{\Gamma(1+\delta)}{\pi}\int_0^{\infty} t^{-1-\delta} C_{1+\delta}(t)\left\{f\left(x+\frac{t}{\omega}\right) + f\left(x-\frac{t}{\omega}\right)\right\} dt.$$

Here δ is any positive number. The integral is absolutely convergent at

* *Math. Ann.*, Vol. 61, pp. 251–280.
† *Comptes Rendus*, Nov. 22, 1909.
‡ *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 369–409.
§ *L.c. supra.*
‖ *Dissertation*, Berlin, 1907.
¶ *Comptes Rendus*, Nov. 22, 1909.
** *Ibid.*, June 12, 1911.
†† Hobson, *Theory of Functions of a Real Variable*, p. 715.
‡‡ *Quarterly Journal*, Vol. 43, pp. 161–177. See also Young, *Leipziger Berichte*, Vol. 63, pp. 369–387, where methods of summation are considered of which Cesàro's method (as modified by Riesz) is a special case. Young considers only integral values of ω, but his proof is perfectly general.

infinity; for Prof. Young has proved that

$$t^{-q}C_q(t) = O(t^{-q}) \quad or \quad O(t^{-2}),$$

according as $0 \leqslant q \leqslant 2$ or $2 \leqslant q$. It is, in fact, easy, by the use of methods similar to those which have been used by Dr. Barnes and myself in investigations concerning the asymptotic expansions of functions defined by Taylor's series,* to obtain the much more precise equation

$$C_q(t) = \frac{t^{q-2}}{\Gamma(q-1)}\{1+o(1)\}+\cos(t-\tfrac{1}{2}q\pi)+o(1).$$

But this equation is not required for our present purpose.

3. It is a simple deduction from the formula (1) that the Fourier's series is summable $(C\delta)$, and has the sum

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\},$$

wherever this limit exists. This deduction is made by Prof. Young, and constitutes a new and simpler proof of the theorem of Riesz-Chapman referred to in § 1. In order to prove Theorem 1, we have to generalise this result.

For any value of x for which $f(x)$ is defined (and therefore almost everywhere) we have

$$(2) \quad \tfrac{1}{2}a_0+\sum_{n<\omega}\left(1-\frac{n}{\omega}\right)^{\delta}A_n-f(x) = \frac{\Gamma(1+\delta)}{\pi}\int_0^\infty t^{-1-\delta}C_{1+\delta}(t)\,\phi\left(\frac{t}{\omega}\right)dt,$$

where $$\phi(\lambda) = f(x+\lambda)+f(x-\lambda)-2f(x).$$

We write $$\Phi(\lambda) = \int_0^\lambda |\phi(\mu)|\,d\mu,$$

and we suppose that, as $\lambda \to 0$,

$$(3) \quad \Phi(\lambda) = o(\lambda).$$

This is a condition of which Lebesgue has made great use, and which he has shown to be satisfied almost everywhere.†

We shall prove that, when the condition (3) is satisfied,

$$(4) \quad J = \int_0^\infty t^{-1-\delta}C_{1+\delta}(t)\,\phi\left(\frac{t}{\omega}\right)dt \to 0,$$

as $\omega \to \infty$. From this Theorem 1 will follow at once.

* Barnes, *Phil. Trans. Roy. Soc.*, (A), Vol. 206, pp. 249–297; Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 401–431.

† See Lebesgue, *Leçons sur les séries trigonométriques*, pp. 15–16; de la Vallée-Poussin, *Cours d'Analyse Infinitésimale*, 2nd edition, Vol. 2, pp. 115–162, 163.

It is evident that we may, without loss of generality, suppose $\delta < 1$, When $\delta \geqslant 1$ the theorem is a mere corollary of Lebesgue's.

We write $$J = \int_0^1 + \int_1^\omega + \int_\omega^\infty = J_1 + J_2 + J_3.$$

In the first place, since $t^{-1-\delta} C_{1+\delta}(t)$ is bounded for $0 < t < 1$, we have

(5) $$J_1 = O \int_0^1 \left| \phi\left(\frac{t}{\omega}\right) \right| dt = O\left\{ \omega \Phi\left(\frac{1}{\omega}\right) \right\} = o(1).$$

Secondly, since $C_{1+\delta}(t)$ is bounded* for $1 \leqslant t$, we have

(6) $$\begin{aligned} J_2 &= O \int_1^\omega t^{-1-\delta} \left| \phi\left(\frac{t}{\omega}\right) \right| dt \\ &= O\left\{ \omega^{-\delta} \int_{1/\omega}^1 \xi^{-1-\delta} \,|\, \phi(\xi) \,|\, d\xi \right\} \\ &= O\left\{ \omega^{-\delta}\Phi(1) - \omega \Phi\left(\frac{1}{\omega}\right) - (1+\delta)\, \omega^{-\delta} \int_{1/\omega}^1 \xi^{-2-\delta} \Phi(\xi)\, d\xi \right\} \\ &= o(1) + o(1) + O\left\{ \omega^{-\delta} \int_{1/\omega}^1 o(\xi^{-1-\delta})\, d\xi \right\} \\ &= o(1) + o(1) + o(1) = o(1). \end{aligned}$$

Finally,

(7) $$\begin{aligned} J_3 &= O \int_\omega^\infty t^{-1-\delta} \left| \phi\left(\frac{t}{\omega}\right) \right| dt \\ &= O\left\{ \omega^{-\delta} \int_1^\infty \xi^{-1-\delta} \,|\, \phi(\xi) \,|\, d\xi \right\} \\ &= O(\omega^{-\delta}) = o(1). \end{aligned}$$

From (5), (6), and (7) it follows that $J = o(1)$; and so the proof is completed.

4. Prof. Young's result is deducible from Theorem 1, whereas the converse is not the case.† But Prof. Young has gone further, and shown that the series

$$\Sigma \frac{A_n}{(\log n)^{2+\delta}}, \quad \Sigma \frac{A_n}{\log n \,(\log \log n)^{2+\delta}}, \quad \ldots$$

are convergent almost everywhere. These results cannot be deduced from

* Young, *Quarterly Journal*, *l.c.*, pp. 163–164.

† The series $\Sigma n^{-1-\delta-ti}$ is convergent (absolutely), but Σn^{-1-ti} is not summable by any of Cesàro's means (Riesz, *Comptes Rendus*, June 21, 1909; Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301–320).

Theorem 1, for $n^{-\delta}$ is the *least* convergence factor which will always convert a series summable $(C\delta)$ into a convergent series.*

It is naturally suggested that summability $(C\delta)$ is not the *most* that can (almost always) be asserted about a Fourier's series, and that such a series must be almost always summable by methods less powerful than any method $(C\delta)$. For this purpose the methods of "infinitely small" or "functional" order, considered by Mr. Chapman and myself in a recent paper,† appear to be appropriate. I have obtained certain results in this direction: thus, for example, the Fourier's series of any periodic and continuous function is uniformly summable $(C\kappa)$ for certain functional forms of κ. But my results are not as yet sufficiently definite or final to justify publication.

5. I have, however, found, in another direction, a theorem from which it is possible, in an extremely simple manner, to deduce all Prof. Young's results and even go a little further.

THEOREM 2.—*If s_n is the sum of the first n terms A_n‡ of the Fourier's series of a summable function $f(x)$, then*

$$s_n = o(\log n)$$

almost everywhere.

The theorem will evidently be proved if we can show that, for some positive η,

$$\int_0^\eta \phi(t)\frac{\sin nt}{t}\,dt = o(\log n), \tag{8}$$

whenever the condition (3) of § 3 is satisfied. Here n is to be regarded as a continuous variable which tends to infinity.

But
$$\begin{aligned}\int_0^\eta = \int_0^{1/n} + \int_{1/n}^\eta &= O\left\{n\int_0^{1/n}|\phi(t)|\,dt + \int_{1/n}^\eta \frac{|\phi(t)|}{t}\,dt\right\} \\ &= o(1) + O\left\{\frac{\Phi(\eta)}{\eta} - n\Phi\left(\frac{1}{n}\right) + \int_{1/n}^\eta \frac{\Phi(t)}{t^2}\,dt\right\} \\ &= o(1) + O(1) + o(1) + O\int_{1/n}^\eta o\left(\frac{1}{t}\right)dt \\ &= O(1) + o(\log n) = o(\log n);\end{aligned}$$

and the theorem is therefore proved.

* This may be shown by means of the series $1^\delta - 2^\delta + 3^\delta - \dots$.

† *Quarterly Journal*, Vol. 42, pp. 181–215.

‡ It is convenient to ignore the term $\frac{1}{2}a_0$.

Now let a_n be a sequence of positive numbers which tend steadily to zero as $n \to \infty$, and satisfy the conditions that $a_n = O(1/\log n)$ and that the series

$$\Sigma (a_n - a_{n+1}) \log n$$

is convergent. Then

$$\sum_1^n a_\nu A_\nu = \sum_1^{n-1} s_\nu \Delta a_\nu + s_n a_n.$$

The last term is of the form

$$o(\log n)\, O(1/\log n) = o(1),$$

and the series $\Sigma s_\nu \Delta a_\nu$ is absolutely convergent. Hence $\Sigma a_\nu A_\nu$ is convergent. The conditions imposed on a_ν are satisfied, if, *e.g.*,

$$a_n = \frac{1}{(\log n)^{1+\delta}},\quad \frac{1}{\log n\,(\log\log n)^{1+\delta}},\quad \ldots .$$

Hence we obtain, from Theorem 2, a slightly generalised form of Prof. Young's result, viz.,

The series $\quad \Sigma \dfrac{A_n}{(\log n)^{1+\delta}},\quad \Sigma \dfrac{A_n}{\log n\,(\log\log n)^{1+\delta}},\quad \ldots$

converge almost everywhere.

It is, however, possible to assert rather more than this.

THEOREM 3.—*The series* $\quad \Sigma \dfrac{A_n}{\log n}$

*converges almost everywhere.**

For ΣA_n is, by Lebesgue's theorem, summable $(C1)$ almost everywhere, and therefore

$$\Sigma \frac{A_n}{\log n}$$

is so also.†

Now the necessary and sufficient condition that a series ΣB_n, known to

* Prof. Young had conjectured the truth of this theorem, and, by a curious coincidence, a letter of his suggesting it to me reached me just as I had completed my own proof. The theorem was also discovered independently by Dr. Marcel Riesz.

† The first and second differences of $a_n = 1/\log n$ are ultimately positive. The summability of the second series therefore follows from a theorem proved by Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 4, pp. 247–265. Much more general forms of this theorem were given later by Hardy, *Math. Ann.*, Vol. 64, pp. 77–94; and still more general forms, which may be regarded as final, by Bromwich, *Math. Ann.*, Vol. 65, pp. 350–369; Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, pp. 255–264, and Vol. 8, pp. 277–294; Bohr, *Bidrag til de Dirichlet'ske Raekkers Theorie*, Copenhagen, 1910. That $\Sigma A_n/(\log n)$ is summable $(C1)$ almost everywhere may also be deduced from the fact that it is a Fourier series, since $\Sigma (\cos nx)/(\log n)$ is one: see Young, *Proc. London Math. Soc.*, Ser. 2, Vol. 12, pp. 41–71.

be summable ($C1$), should be convergent, is that

$$B_1+2B_2+\ldots+nB_n = o(n).*$$

Taking $B_n = A_n/(\log n)$, we find

$$\sum_1^n \nu B_\nu = \sum_1^n \frac{\nu A_\nu}{\log \nu} = \sum_1^{n-1} s_\nu \Delta \frac{\nu}{\log \nu} + \frac{n s_n}{\log n}$$

$$= \sum_1^{n-1} o(\log n)\, O\left(\frac{1}{\log n}\right) + o(n)$$

$$= \sum_1^{n-1} o(1) + o(n) = o(n).$$

Hence ΣB_n, being almost everywhere summable, and also almost everywhere convergent if summable, must be almost everywhere convergent. It should be observed that the series

$$\Sigma \log n \,\Delta \frac{1}{\log n}$$

is divergent, so that our previous reasoning would not apply.

6. The question naturally arises as to whether the equation

$$s_n = o(\log n)$$

of Theorem 2 is the best possible of its kind. I have not proved rigorously that this is so, but it seems to me very probable.

Fejér† has given an exceedingly simple and ingenious method for the construction of trigonometrical series which are the Fourier's series of continuous functions, but which cease to converge for isolated values of x, or for an enumerable, everywhere dense, set of values of x. Thus, for example, he has constructed a pure cosine series

$$\Sigma a_n \cos nx,$$

such that the $(g_1+g_2+\ldots+g_{\nu-1}+\frac{1}{2}g_\nu)$-th partial sum of the series Σa_ν is greater than

$$\frac{1}{\nu^2} \log \tfrac{1}{2} g_\nu.‡$$

Fejér takes $g_\nu = 2 . 2^{\nu^3}$, and in his example s_n is sometimes of order

* See, *e.g.*, Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301–320, where the analogous condition, which enables us to infer summability (Ck) from summability (C, $k+1$), is given.

† *Crelle's Journal*, Vol. 138, pp. 22–53; *Annales de l'École Normale*, Vol. 28, pp. 63–103.

‡ *L.c. supra*, p. 85.

$\sqrt[3]{(\log n)}$. It is easy, by choosing still more rapidly increasing forms of g_ν, to replace $\sqrt[3]{(\log n)}$ by $(\log n)^{1-\delta}$, or indeed by any function of the form $o(\log n)$: and the same remarks apply to his examples which cease to converge for values of x which are everywhere dense. Inasmuch as these values form a set of measure zero, Fejér's examples do not prove that the equation of Theorem 2 is the best possible; but, at any rate, they suggest it very forcibly.

Another class of non-convergent Fourier's series of continuous functions was defined by Schwarz and du Bois-Reymond, and has been further discussed by Prof. Hobson.* The oscillation of these series is of order $\log\log n$, and it does not seem possible, at any rate without a much more delicate discussion, to modify the functions so as to obtain oscillation of any notably more pronounced type.

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* *Theory of Functions of a Real Variable*, pp. 701-707; see also *Proc. London Math. Soc.*, Ser. 2, Vol. 3, pp. 48-61.

COMMENTS

The main results of this paper were communicated to the London Mathematical Society at its meeting on 13 February 1913. See *Proceedings* (2), 12 (1913), xxviii–xxix.

§ 4. The results mentioned at the end of this section do not appear to have been published.

§ 5. The result of Theorem 2 had been proved earlier by Lebesgue.

§ 6. It has been shown that the result $s_n = o(\log n)$ is best possible (see e.g. Z I, p. 298).

ANALYSE MATHÉMATIQUE. — *Sur la série de Fourier d'une fonction à carré sommable.* Note de M. **G.-H. Hardy** et **J.-E. Littlewood**.

1. Soit $f(u)$ une fonction de u, sommable et possédant la période 2π, et envisageons une valeur ordinaire x de u, c'est-à-dire une valeur de u pour laquelle l'expression

$$s = \frac{1}{2}[f(x+0)+f(x-0)]$$

a une valeur déterminée. On sait, d'après un théorème connu de M. Féjer, que la série de Fourier

$$\frac{1}{2}A_0 + \Sigma A_m = \frac{1}{2}a_0 + \Sigma(a_m \cos mu + b_m \sin mu),$$

de $f(u)$ est sommable pour $u = x$ par les moyennes de Cesarò du premier ordre, et a la somme s. C'est ce que nous pouvons exprimer en écrivant

$$\lim \frac{(s_0 - s) + (s_1 - s) + \ldots + (s_n - s)}{n+1} = 0$$

où

$$s_m = \frac{1}{2}A_0 + A_1 + \ldots + A_m.$$

2. On doit à MM. Lebesgue, Marcel Riesz et Chapman des généralisations très intéressantes du théorème de M. Féjer. Nous avons trouvé, pour les fonctions à carré sommable, un résultat d'un caractère un peu différent.

I. *Le carré de $f(u)$ étant sommable, on a, pour $u = x$,*

$$\lim \frac{(s_0 - s)^2 + (s_1 - s)^2 + \ldots + (s_n - s)^2}{n+1} = 0,$$

et

$$\lim \frac{|s_0 - s| + |s_1 - s| + \ldots + |s_n - s|}{n+1} = 0.$$

La seconde égalité est un corollaire de la première, comme on le voit immédiatement en faisant application de l'inégalité de Schwarz.

Il est convenable, pour la démonstration de notre théorème, d'envisager,

1913, 11 (with J. E. Littlewood) *Comptes Rendus*, 156, 1307–9.

au lieu de s_n, l'expression

$$S_n = \frac{1}{2}A_0 + A_1 + \ldots + A_{m-1} + \frac{1}{2}A_m.$$

Il suffit évidemment de démontrer notre résultat pour S_n; et l'on peut supposer, sans restreindre la généralité, que $s = 0$. Cela étant, on trouve, par l'application des formules de Fourier, les relations

$$S_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\frac{1}{2}\cot\frac{1}{2}u \sin nu\, du,$$

$$\sum_1^\infty S_n r^n \sin n\theta = \frac{r(1-r^2)\sin n\theta}{\pi} \times \int_{-\pi}^{\pi} f(u+x)\frac{\cos^2\frac{1}{2}u\, du}{[1-2r\cos(\theta-u)+r^2][1-2r\cos(\theta+u)+r^2]},$$

où $0 < r < 1$. De la dernière relation on peut déduire qu'à tout nombre $\varepsilon > 0$ correspond un nombre $\delta > 0$, tel que

$$\left|\sum_1 S_n r^n \sin n\theta\right| < \varepsilon\left[\sqrt{\left(\frac{1}{1-r}\right)} + \frac{|\sin\theta|}{1-2r^2\cos 2\theta + r^4}\right],$$

pour $1-\delta < r < 1$. Nous intégrons le carré de cette inégalité de $\theta = 0$ à $\theta = 2\pi$ et nous trouvons

$$\lim(1-r)\sum_1 S_n^2 r^{2n} = 0,$$

et *a fortiori*

$$\lim(1-r)\sum_1^\nu S_n^2 r^{2n} = 0,$$

d'où découle notre théorème, en supposant $r = 1 - \frac{1}{\nu}$.

3. M. Lebesgue a démontré que le procédé de M. Féjer donne pour somme $f(x)$ en tous les points où la fonction

$$|\varphi(t)| = |f(x+t) + f(x-t) - 2f(x)|$$

est pour $t = 0$ la dérivée de son intégrale indéfinie, ce qui a lieu *presque partout*. Nous avons démontré également le théorème suivant :

II. *Les résultats du théorème I subsistent, avec $s = f(x)$, presque partout.*

4. Il est intéressant de vérifier l'exactitude de nos résultats sur les

exemples qu'a donnés M. Féjer, de fonctions continues qui possèdent des séries de Fourier divergentes.

On pourrait supposer que les relations du théorème I sont des conséquences immédiates de la sommabilité de la série ΣA_m et de la convergence de la série ΣA_m^2. Mais M. Fekete a démontré, par des exemples, que cela n'a nullement lieu. En effet la série

$$\Sigma \sin(\log n)^2 \frac{\log n}{n}$$

satisfait à ces deux conditions sans vérifier les relations de notre théorème. M. Fekete nous a obligeamment communiqué son analyse intéressante sur ce point.

COMMENTS

The property $\sum_{m=0}^{n} |s_m - s|^q = o(n)$, which is first introduced in this paper, was later known as strong Cesàro summability of order 1, with index q. For further references see the comments on 1927, 3.

On the Fourier Series of a Bounded Function

Mr. G. H. Hardy and Mr. J. E. Littlewood.

Our paper "Abel's Theorem and its converse" contains a statement, but no proof, of the following theorem:—

The Fourier series of a summable function $f(x)$, bounded in the neighbourhood of the particular value of x considered, is either summable (C, k) for ALL *positive values of k (however small) or summable for* NO *value of k (however large). The necessary and sufficient condition for summability is that*

$$\frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt \to S \tag{1}$$

when $h \to 0$; and S is then the sum of the series.

As this theorem contains the complete solution of the problem of Cesàro summability for the Fourier series of a bounded function, it seems worth while to indicate the general lines of a proof.

We use unpublished generalisations of three known theorems. In the first place there is a well known theorem (due to M. Riesz and Chapman) that if $f(x)$ is *continuous* then the series is *summable* (C, k) for every posi-

1918, 4 (with J. E. Littlewood) *Proceedings of the London Mathematical Society* (2), 17, xiii–xv.

tive k. Substantially the same argument shows that if $f(x)$ is *bounded* then the series is *bounded* (C, k) for every positive k.

In the second place, we have proved* that if a series is bounded (C, r) and summable (C, s), where r and s are integers and $r < s$, then it is summable (C, k) for every integral value of k between r and s. Dr. Riesz afterwards proved that this result remains true when the restriction that r, s and k are integers is removed. This theorem, and others of a similar character, have since been generalised widely by Mr. K. A. Rau, in a dissertation which obtained the Smith's Prize for 1918 in the University of Cambridge; but neither Dr. Riesz's nor Mr. Rau's work has yet been published.

It is evident that the first part of our theorem follows at once from a combination of these results.

We have next to show that (1) is necessary and sufficient for summability. That it is *sufficient* follows from a known theorem of Lebesgue,† which asserts that (1) implies summability with $k = 2$. That it is *necessary* may be proved as follows. Fejér has shown‡ that, if the series is summable $(C, 1)$ to sum S, then

$$\frac{\phi(h)}{16h^4} = \frac{\Delta_{2h}^4 F_4(x)}{16h^4}$$

$$= \frac{F_4(x+4h)-4F_4(x+2h)+6F_4(x)-4F_4(x-2h)+F_4(x-4h)}{16h^4} \to S$$

when $h \to 0$, $F_4(x)$ being the fourth iterated integral of $f(x)$. Taking, as we may do without loss of generality, $S = 0$, we have

$$\phi(h) = o\,(h^4).$$

Also

$$\phi'''(h) = 64\{\chi(h) - \tfrac{1}{2}\chi(\tfrac{1}{2}h)\},$$

where

$$\chi(h) = F_1(x+4h) - F_1(x-4h)$$

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* "Contributions to the arithmetic theory of series", *Proc. London Math. Soc.*, Ser. 2, Vol. 11, pp. 411–478. The particular result referred to is not stated as an explicit theorem anywhere in our paper; but it is an immediate deduction from Theorem 3. It is also virtually included in Theorem 19. But the latter theorem involves a parameter β which we assume to be positive; and here β is zero, so that some restatement of the argument is required.

The work of Dr. Riesz to which we refer is contained in a letter addressed to one of us on November 28th, 1912. In this letter Dr. Riesz generalises our Theorem 3; and the theorem required here is an immediate deduction from his generalisation.

† "Recherches sur la convergence des séries de Fourier", *Math. Annalen*, Vol. 61, pp. 251–280 (pp. 278–279).

‡ "Untersuchungen über Fourierschen Reihen", *Math. Annalen*, Vol. 58, pp. 51–69 (p. 69).

is the integral of a bounded function. From this it is easily deduced, by arguments similar to those of Section II of our paper already referred to, that $\phi'''(h) = o(h)$,* and hence that $\chi(h) = o(h)$; which completes the proof of our theorem.

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* This is the third "unpublished generalisation" to which we referred above. For generalisations of some of the older theorems in this direction see Landau, "Einige Ungleichungen für zweimal differentiierbare Funktionen", *Proc. London Math. Soc.*, Ser. 2, Vol. 13, pp. 43–49.

COMMENTS

A better proof of the result of this paper is given in 1924, 1 (Theorem C1). The result is included in Theorem A of 1926, 10.

Solution of the Cesàro summability problem for power-series and Fourier series.

By

G. H. Hardy in Oxford and J. E. Littlewood in Cambridge.

1. Introduction.

1.1. In this memoir we solve a problem which has apparently not been stated in quite the same form before. The nature of the problem is most easily explained in terms of Fourier series.

The "convergence problem" for Fourier series is still unsolved. If $f(t)$ is integrable (L)[1] in $(-\pi, \pi)$, and $t = x$ is a point of the interval, there is no simple property of $f(t)$ which is known to be both necessary and sufficient for the convergence of its Fourier series when $t = x$. There are simple sufficient conditions, but these are known not to be necessary; and the necessary conditions are known not to be sufficient.

The same gap appears when we consider the summability of the series by an assigned Cesàro mean, for example its summability $(C, 1)$. This problem is, as Fejér has shown, in some ways more fundamental than that of convergence, and it also is unsolved. Thus continuity of the function is sufficient for summability $(C, 1)$ of the series, but it is by no means necessary.

The question which we raise here is different. We ask, not when the series is summable by some particular mean, but when it is summable *by some mean or another, i. e.* when it is *summable* (C). And the justification for insistence on this particular question is that it happens to admit a quite simple and complete solution, not only for Fourier series but for power-series as well. That this should be so is not altogether surprising. The reader who is familiar with the work of H. Bohr and

[1]) Integrable in the sense of Lebesgue, i. e. "summable". The latter word would be very embarrassing in this memoir. We might say "integrable" simply; but we shall have occasion later to use the word in a slightly extended sense.

1924, 1 (with J. E. Littlewood) *Mathematische Zeitschrift*, 19, 67–96.

M. Riesz will remember that there is a similar phenomenon in the theory of Dirichlet's series. The "convergence problem" for a Dirichlet's series $\Sigma a_n n^{-s}$, the problem of determining the abscissa of convergence of the series in terms of the properties of the analytic function $f(s)$ associated with the series, is unsolved. So is the problem of determining the various abscissae of summability; but the limit of these abscissae, the number Λ such that the series is summable when $\sigma > \Lambda$,[2] and not when $\sigma < \Lambda$, can be defined in the manner required: it is the least number λ for which $f(s)$ is regular, and of finite order in t, when $\sigma \geqq \lambda + \varepsilon > \lambda$.[3]

1.2. In § 2 we prove a general arithmetic theorem (a theorem in the pure theory of infinite series) on which all our subsequent reasoning depends. The statement of this theorem (Theorem A) requires a few words of preface. We take our series in the form

$$A = \Sigma a_n = a_0 + a_1 + a_2 + \dots,$$

and use A either for the series or for its sum (when it is summable). We write

$$A_n = A_n^0 = a_0 + a_1 + \dots + a_n,\ A_n^1 = A_0^0 + \dots + A_n^0,\ A_n^2 = A_0^1 + \dots + A_n^1, \dots$$

with a corresponding notation in other letters. If $C_n = 1$, so that

$$C_n^r = \frac{(n+r)(n+r-1)\dots(n+1)}{r!} = \frac{\Gamma(n+r+1)}{\Gamma(n+1)\,\Gamma(r+1)},$$

and

$$A_n^r \sim C_n^r A,$$

the series is said to be summable (C, r) to sum A; and in these circumstances we write

$$\Sigma a_n = A \quad (C, r).$$

Summability $(C, 0)$ is the same thing as convergence; and A is said to be summable $(C, -1)$ if it is convergent and $n a_n \to 0$.[4] We are not, except in two subsidiary theorems, concerned with non-integral orders of summability.

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[2]) $s = \sigma + it$.

[3]) See, for example, G. H. Hardy and M. Riesz: The general theory of Dirichlet's series, Camb. Math. Tracts 18 (1915), p. 56.

[4]) This definition of summability $(C, -1)$ has been used by W. H. Young: On the convergence of the derived series of Fourier series, Proc. London Math. Soc. (2), 17 (1916), pp. 195–236 (209–210). There is another memoir of Young [On the convergence of a Fourier series and its allied series, Proc. London Math. Soc. (2), 10 (1911), pp. 254–272] which contains a number of ideas which we have found of great value in this investigation, in particular the idea of a connection be-

We say that

$$u_n \to u \quad (C, r)$$

if the series whose n-th partial sum is u_n is summable (C, r) to sum u. Thus A is summable (C, r) if, and only if, $A_n \to A \ (C, r)$.

We assume certain elementary properties of summable series which are recapitulated in § 2. 4; all are direct consequences of the definitions. An excellent account of the general theory, for both integral and non-integral orders of summability, may be found in a dissertation by Andersen[5]).

Theorem A. *The necessary and sufficient condition that $A = \Sigma a_n$ should be summable (C, r) is that there should be a system of numbers*

$$a_{n,s} \quad (s = 0, 1, 2, \ldots, r+1;\ n = 0, 1, 2, \ldots) \tag{1.21}$$

such that

$$a_{n,0} = a_n, \quad a_{n,s-1} = (n+1)(a_{n,s} - a_{n+1,s}) \qquad (s > 0), \tag{1.22}$$

and $A_{r+1} = \Sigma a_{n,r+1}$ is summable $(C, -1)$. In these circumstances

$$a_{n,s} = \sum_n^\infty \frac{a_{\nu,s-1}}{\nu+1} \quad (C, r-s) \tag{1.23}$$

for $s = 1, 2, \ldots, r+1$; the series A_s is summable $(C, r-s)$; and the sums of all the series are the same.

There are a number of variants of this theorem, sometimes more useful than the theorem itself. Before passing on to our main problems, we indicate some of these variants and of their applications.

In § 3 we apply Theorem A to power-series, and prove

Theorem B. *The necessary and sufficient conditions that A should be summable (C) are that (1) the function*

$$f(x) = \Sigma a_n x^n \tag{1.24}$$

should be regular and of finite order in the unit circle, i. e. that

$$|f(x)| < K(1 - |x|)^{-K} \qquad (|x| < 1), \tag{1.241}$$

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tween the summability (C, r) of the Fourier series of a function $\varphi(t)$ and the summability $(C, r-1)$ of that of

$$\varphi_1(t) = \frac{1}{t}\int_0^t \varphi(u)\,du.$$

Young does not, so far as we know, prove any of the theorems which we prove here, but it was from an examination of his work that our researches originated.

[5]) A. F. Andersen, Studier over Cesàros Summabilitetsmetode (Copenhagen 1921).

where the K's are independent of x; (2) there should be a number k such that if

$$(1.25)\quad f_1(x)=\frac{1}{1-x}\int_x^1 f(x_1)\,dx_1,\quad f_2(x)=\frac{1}{1-x}\int_x^1 f_1(x_1)\,dx_1,\quad \ldots,$$

then

$$(1.26)\qquad f_k(x)\to A$$

when $x\to 1$ in any manner from within the circle.

The integrals which appear in the formulae (1.25) may be taken indifferently along the straight line $(x, 1)$ or the broken line $(x, 0)$, $(0, 1)$; and are to be regarded as elementary generalised integrals of the type

$$\lim_{\xi\to 1}\int_x^{\xi},$$

where $\xi\to 1$ along whichever path of integration is chosen.

Theorem B 1. *If $f(x)$ is bounded in the circle, then A is either summable by every Cesàro mean of positive order, or summable by no Cesàro mean of any order. The necessary and sufficient condition that it should be summable is that $f_1(x)\to A$ when $x\to 1$ in any manner from within the circle.*

In § 4 we consider Fourier series. Using one of the trivial variants of Theorem A, we prove

Theorem C. *The necessary and sufficient condition that the Fourier series of an integrable function $f(t)$ should be summable (C), to sum A, for $t=x$, is that there should be a number k such that, if*

$$(1.271)\qquad \varphi(t)=f(x+t)+f(x-t)-2A,$$

$$(1.272)\quad \varphi_1(t)=\frac{1}{t}\int_0^t \varphi(t_1)\,dt_1,\quad \varphi_2(t)=\frac{1}{t}\int_0^t \varphi_1(t_1)\,dt,\quad \ldots$$

then

$$(1.28)\qquad \varphi_k(t)\to 0$$

when $t\to 0$.

It will appear later that this theorem has a dual interpretation; that the word "integrable" in the enunciation may be interpreted either as meaning "integrable (L)" or in a slightly more general sense; and that there is a similar ambiguity in regard to the integrals (1.272).

We may express our condition more concisely by saying that it is necessary and sufficient that $\varphi(t)$ should be *continuous in mean* for $t=0$. We prove Theorem C first on the assumption that the square of $f(t)$ is

summable, when the proof is simple. The general proof depends on more delicate considerations, and is decidedly more difficult than anything else in the memoir.

Finally we prove

Theorem C1. *If $f(t)$ is bounded in an interval including $t=x$, then the Fourier series of $f(t)$, for $t=x$, is either summable by every Cesàro mean of positive order, or summable by no Cesàro mean of any order. The necessary and sufficient condition that it should be summable is that $\varphi_1(t)\to 0$ when $t\to 0$.*

This last theorem we enunciated without proof in a former memoir[6]).

We must express our thanks to Mr. A. E. Ingham of Trinity College, Cambridge, for a number of valuable suggestions. We owe to him in particular the proof that it is indifferent, in Theorem B, which form of the definitions of the functions $f_s(x)$ is adopted, and the precise form of the functions $\psi_s(t)$ used in the proof of Theorem C.

We must also add a few words concerning the relation of our arithmetic theorem A to certain theorems of Knopp: for a fuller statement we may refer to Herr Knopp's note which follows this memoir. Knopp's theorem of 1917[6a]), with $\lambda=1$, is equivalent to Theorem A when $r=1$, and is, we believe, the first theorem of this particular character. The generalisations which he has made of it are different in form, but are naturally equivalent in substance to our general theorem. The applications of this theorem which we make here show very clearly that theorems of this type are more important than might have been anticipated.

2. Proof of Theorem A.

2.1. Lemma 1. *If*

$$a_n=(n+1)(b_n-b_{n+1}) \tag{2.11}$$

then

$$A_n^r=(r+1)B_n^r-(n+1)B_{n+1}^{r-1} \qquad (r\geqq 0). \tag{2.12}$$

If $r=0$, B_{n+1}^{-1} is to be interpreted as meaning b_{n+1}.

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[6]) G. H. Hardy and J. E. Littlewood, Abel's Theorem and its converse, Proc. London Math. Soc. (2), **18** (1918), pp. 205–235 (p. 235, Theorem Z). We had already published a sketch of a proof, inferior to our present one: see "On the Fourier series of a bounded function", Proc. London Math. Soc. (2), **17** (Records of proceedings at meetings, 6. Dec. 1917).

[6a]) K. Knopp, Über die Oszillationen einfach unbestimmter Reihen, Sitzungsberichte d. Berliner Math. Gesellschaft **16** (1917), S. 45–50. Later Hardy [A theorem concerning summable series, Proc. Camb. Phil. Soc. **20** (1920), pp. 304–307] proved a theorem, concerning the case $r=1$, $\lambda=1$, identical with Knopp's.

We can verify the result at once when $r = 0, 1$. Assuming it true for $r = s$, we have

$$A_n^{s+1} = (s+1)(B_0^s + B_1^s + \ldots + B_n^s) - B_1^{s-1} - 2B_2^{s-1} - \ldots - (n+1)B_{n+1}^{s-1}$$
$$= (s+2)B_n^{s+1} - (n+1)B_{n+1}^s,$$

and the conclusion is established by induction.

Lemma 2. *If B is summable $(C, r-1)$ then A is summable (C, r), and $A = B$.*

If $r = 0$, the result follows at once from (2.12), which may then be written

$$A_n = B_n - (n+1)b_{n+1}.$$

If $r > 0$, we have

$$B_{n+1}^{r-1} = C_{n+1}^{r-1} B + o(n^{r-1}),$$

and therefore, by summation

$$B_n^r = C_n^r B + o(n^r).$$

Hence, by (2.12),

$$A_n^r = ((r+1)C_n^r - (n+1)C_{n+1}^{r-1})B + o(n^r) = C_n^r B + o(n^r).$$

2.2. Lemma 3. *If A is summable (C, r), then*

$$B_n^{r-1} = h\,C_n^r + C_n^{r-1} A + o(n^{r-1}), \tag{2.21}$$

where h is a constant. When $r = 0$, the right hand side is to be interpreted as meaning

$$h + o\left(\frac{1}{n}\right).$$

We may suppose without loss of generality that $A = 0$.[7] If $r = 0$,

$$(n+2)B_n - (n+1)B_{n+1} = B_n - (n+1)b_{n+1} = A_n = o(1),$$

$$\frac{B_n}{n+1} - \frac{B_{n+1}}{n+2} = \frac{A_n}{(n+2)(n+1)} = o\left(\frac{1}{n^2}\right).$$

Hence there is a constant h such that

$$\frac{B_n}{n+1} = h + o\left(\frac{1}{n}\right),$$

$$b_{n+1} = \frac{B_n}{n+1} + o\left(\frac{1}{n}\right) = h + o\left(\frac{1}{n}\right),$$

which is (2.21). If $r > 0$,

$$(n+r+2)B_n^r - (n+1)B_{n+1}^r \tag{2.22}$$
$$= (r+1)B_n^r - (n+1)B_{n+1}^{r-1} = A_n^r = o(n^r).$$

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[7]) If we decrease b_0 by A we do the same to a_0 and leave the other a's unaltered, and A_n^{r-1}, B_n^{r-1} are each decreased by $C_n^{r-1} A$.

If we write

(2.23) $$B_n^r = (n+r+1)(n+r)\dots(n+1)\varphi_n,$$

substitute in (2.22), and reduce, we obtain

(2.24) $$(n+r+2)(n+r+1)\dots(n+1)(\varphi_n - \varphi_{n+1}) = A_n^r = o(n^r).$$

(2.25) $$\varphi_n - \varphi_{n+1} = o\left(\frac{1}{n^2}\right).$$

Hence φ_n tends to a limit φ and

(2.26) $$\varphi_n = \varphi + \sum_n^\infty \frac{A_\nu^r}{(\nu+r+2)\dots(\nu+1)} = \varphi + \sum_n^\infty o\left(\frac{1}{\nu^2}\right) = \varphi + o\left(\frac{1}{n}\right),$$

(2.27) $$B_n^r = (n+r+1)(n+r)\dots(n+1)\varphi + o(n^r).$$

Substituting in (2.22), and dividing by $n+1$, we obtain

(2.28) $$B_{n+1}^{r-1} = h\,C_{n+1}^r + o(n^{r-1}),$$

where

(2.281) $$h = (r+1)!\,\varphi;$$

and this is (2.21) with $A = 0$ and $n+1$ in place of n.

2.3. Lemma 4. *The necessary and sufficient condition that A should be summable (C, r) is that (2.11) should have a solution b_n such that B is summable $(C, r-1)$.*

The condition is *sufficient*, by Lemma 2. To prove it *necessary*, take $A = 0$, and suppose that b_n is any solution of (2.11). All other solutions are of the form $\mathfrak{b}_n = b_n - \mathfrak{h}$, where $\mathfrak{h}$ is a constant. If we take $\mathfrak{h} = h$, we have

$$\mathfrak{B}_n^{r-1} = B_n^{r-1} - h\,C_n^r = o(n^{r-1}),$$

by (2.21), so that $\mathfrak{B}$ is summable $(C, r-1)$ to sum 0.

It is evident that only one b_n can satisfy the conditions.

Lemma 5. *If A is summable (C, r) then*

(2.31) $$A' = \sum \frac{a_n}{n+1}$$

is summable $(C, r-1)$, and

(2.32) $$A' = (r+1)! \sum_0^\infty \frac{A_n^r}{(n+r+2)\dots(n+1)},$$

the latter series being absolutely convergent. And if

(2.33) $$b_n = \sum_n^\infty \frac{a_\nu}{\nu+1} \quad (C, r-1),$$

then B is summable $(C, r-1)$, and $B = A$.

It follows from Lemma 4 that there is a solution of (2.11) such that $\mathfrak{B}$ is summable $(C, r-1)$. But then $\sum(\mathfrak{b}_n - \mathfrak{b}_{n+1})$ is also summable $(C, r-1)$, to sum $\mathfrak{b}_0$, *i. e.* A' is summable $(C, r-1)$ to sum $\mathfrak{b}_0$.

Hence b_n exists, and it is plainly a solution of (2.11), so that $b_n = \mathfrak{b}_n + \mathfrak{h}$, where $\mathfrak{h}$ is a constant. But $b_n \to 0\,(C, r-1)$, from its definition; and $\mathfrak{b}_n \to 0\ (C, r-1)$, because $\mathfrak{B}$ is summable $(C, r-1)$. Hence $\mathfrak{h} = 0$, $b_n = \mathfrak{b}_n$, and B is summable $(C, r-1)$.

Finally, the h and φ of Lemma 3 are now zero, and

$$A' = b_0 = B_0^r = (r+1)!\,\varphi_0 = (r+1)!\sum_0^\infty \frac{A_n^r}{(n+r+2)\ldots(n+1)},$$

by (2.26). This last result is of course well known.[8])

Lemma 5 shows that the particular solution of (2.11), referred to in Lemma 4, is given by (2.33). This is what is required for our immediate purpose; but we insert two additional lemmas for the sake of completeness.

Lemma 6. *If A' is summable $(C, r-1)$, b_n is defined by* (2.33), *and B is summable $(C, r-1)$, then A is summable (C, r).*

Lemma 7. *The necessary and sufficient condition that A should be summable (C, r) is that, if b_n is defined by* (2.33), *B should be summable $(C, r-1)$,*

Lemma 6 is a corollary of Lemma 2, since b_n satisfies (2.11), and Lemma 7 results from the combination of Lemmas 5 and 6.

2.4. The preceding lemmas contain the proof of Theorem A. Suppose first that A_{r+1} is summable $(C, -1)$. Applying Lemma 2 repeatedly, we see that A_s is summable $(C, r-s)$ for $s = r, r-1, \ldots, 0$. Similarly, if A is summable (C, r), we see, by applying Lemma 4 repeatedly, that numbers $a_{n,s}$ exist such that A_s is summable $(C, r-s)$ for $s = 1, 2, \ldots, r+1$. Finally it follows, from Lemma 5, that these numbers are connected by (1.23).

It is instructive to recapitulate the properties of summable series which we have assumed in the preceding argument. We have assumed only that, if A is summable (C, r), then

(1) $$a_n \to 0 \quad (C, r),$$

(2) $$a_n + a_{n+1} + \ldots = A - A_{n-1} \to 0 \quad (C, r),$$

(3) $$\sum(a_n - a_{n+1}) = a_0 \quad (C, r),$$

[8]) H. Bohr, Sur la série de Dirichlet, Comptes Rendus, 11 Jan. 1909; M. Riesz, Sur les séries de Dirichlet, *ibid.*, 15 July 1909. See also Bohr's dissertation "Bidrag til de Dirichlet'ske Raekkers Theori" (Copenhagen, 1910).

(4) a change in one term of A will affect its sum in the obvious manner,

(5) the addition of a constant (other than 0) to each of its terms will destroy its summability.

All these properties are quite trivial deductions from the definitions.

2.5. We have proved Theorem A in the form most convenient for applications to *power-series*. For *Fourier series*, a slightly different form is preferable.

Theorem A1. *Theorem A remains true if* (1.22) *is replaced by*

$$a_{0,s} = a_0 = 0, \qquad a_{n,0} = a_n \ (n > 0), \tag{2.51}$$
$$a_{n,s-1} = n(a_{n,s} - a_{n+1,s}) \quad (n > 0,\ s > 0),$$

and (1.23) *by*

$$a_{n,s} = \sum_n^\infty \frac{a_{\nu,s-1}}{\nu} \quad (C, r-s). \tag{2.52}$$

This amounts to little more than a restatement of Theorem A in a slightly different notation. If, in (2.51), we write $\alpha_{n-1,s}$ for $a_{n,s}$, then A_s is replaced by

$$\mathsf{A}_s = 0 + \alpha_{0,s} + \alpha_{1,s} + \ldots,$$

and the relations between the α's are those of Theorem A. Hence Theorem A1 is a corollary of Theorem A; but we must add a sixth to the propositions (1)—(5) of § 2.4, viz.

(6) the summability and sum of the series are not affected by inserting or deleting a zero term at the beginning.

2.6. Theorem A provides a very powerful weapon in the general theory of summable series, and the following examples of its application to the proof of known theorems are perhaps worthy of notice.

(1) *If A is summable (C, r), and $n a_n \to 0$, then A is convergent.* For

$$a_{n,1} = \sum_n^\infty \frac{a_\nu}{\nu+1} = \sum_n^\infty o\left(\frac{1}{\nu^2}\right) = o\left(\frac{1}{n}\right);$$

and similarly $a_{n,s} = o\left(\frac{1}{n}\right)$ for every s. Hence A_r is summable $(C, -1)$, and therefore A is summable $(C, r-1)$. Repeating the argument r times, A is convergent (and so, naturally, summable $(C, -1)$).

(2) *If A is summable (C, r), and*

$$t_n = a_1 + 2a_2 + \ldots + n a_n = o(n),$$

then A is convergent.

For

$$\sum_n^N \frac{a_\nu}{\nu+1} = \sum_n^N \frac{t_\nu - t_{\nu-1}}{(\nu+1)\nu} = 2\sum_n^{N-1} \frac{t_\nu}{(\nu+2)(\nu+1)\nu} - \frac{t_{n-1}}{(n+1)n} + \frac{t_N}{(N+1)N}.$$

Making $N\to\infty$, we obtain

$$a_{n,1} = 2\sum_n^\infty \frac{t_\nu}{(\nu+2)(\nu+1)\nu} - \frac{t_{n-1}}{(n+1)n} = \sum_n^\infty o\left(\frac{1}{\nu^2}\right) + o\left(\frac{1}{n}\right) = o\left(\frac{1}{n}\right).$$

Hence A_1 is summable $(C, -1)$, by (1); and therefore A is convergent.

(3) *If A is summable (C, r), and*

$$\sum n^p |a_n|^{p+1} \qquad (p \geqq 0)$$

is convergent, then A is convergent.

The conclusion is trivial if $p=0$. If $p>0$, we have

$$a_{n,1} = \sum_n^\infty \frac{a_\nu}{\nu+1} = \sum_n^\infty \left((\nu+1)^{-\frac{2p+1}{p+1}} \cdot (\nu+1)^{\frac{p}{p+1}} a_n\right)$$

$$= O\left(\left(\sum_n^\infty \nu^{-\frac{2p+1}{p}}\right)^{\frac{p}{p+1}} \left(\sum_n^\infty \nu^p |a_\nu|^{p+1}\right)^{\frac{1}{p+1}}\right) = O\left(\left(n^{-\frac{p+1}{p}}\right)^{\frac{p}{p+1}}\right) o(1) = o\left(\frac{1}{n}\right).$$

Hence A_1 is summable $(C, -1)$, and therefore A is convergent.

(4) If A is summable (C, r), and $a_n > -\frac{K}{n}$, where K is a constant, we can prove, as in (1) above, that $a_{n,1}, a_{n,2}, \ldots$ satisfy the same condition as a_n. We thus reduce the proof of the theorem "*if A is summable (C, r), and $a_n > -\frac{K}{n}$, then A is convergent*" to its proof in the special case when $r=1$. Of this a considerable number of proofs have been given: it does not seem to be possible to simplify them materially by use of Theorem A.[8a])

(5) There are theorems similar to Theorem A (and its attendant lemmas) but concerned with summation by Hölder's means. These theorems are of interest because, when combined with Theorem A, they lead to a new proof of the equivalence of the two methods of summation.

Write

$$\mathfrak{A}_n^0 = A_n, \qquad \mathfrak{A}_n^1 = \frac{\mathfrak{A}_0^0 + \ldots + \mathfrak{A}_n^0}{n+1}, \qquad \mathfrak{A}_n^2 = \frac{\mathfrak{A}_0^1 + \ldots + \mathfrak{A}_n^1}{n+1}, \ \ldots.$$

Then A is *summable* (H, r) if $\mathfrak{A}_n^r \to A$. If we suppose that

(2.61) $$a_0 = 0, \quad b_0 = 0, \quad a_{n+1} = n(b_n - b_{n+1}) \qquad (n \geqq 0),$$

[8a]) See however the following note of Herr Knopp.

we find without difficulty that[9])

$$\mathfrak{A}_n^0 = \mathfrak{B}_n^0 - n b_n, \quad \mathfrak{A}_n^r = 2\mathfrak{B}_n^r - \mathfrak{B}_n^{r-1} \qquad (r \geqq 1). \tag{2.62}$$

Following the lines of §§ 2.1 — 2.3, we prove that *the necessary and sufficient condition that A should be summable (H, r) is that there should be a solution b_n of* (2.61) *such that B is summable $(H, r-1)$, and that this solution is given by*

$$b_0 = 0, \quad b_n = \frac{a_{n+1}}{n} + \frac{a_{n+2}}{n+1} + \dots \quad ((H, r-1),\ n > 0). \tag{2.63}$$

The equivalence theorem follows from a combination of this proposition with Lemma 7. Let us suppose that the theorem is true for means of order r. Then

$$a_0 + a_1 + a_2 + a_3 + \dots = A \qquad (H, r+1) \tag{2.64}$$

is equivalent to

$$0 + 0 + a_2 + a_3 + \dots = A - a_0 - a_1 \quad (H, r+1),$$

and therefore to

$$0 + b_1 + b_2 + b_3 + \dots = A - a_0 - a_1 \qquad (H, r),$$

where

$$b_n = \frac{a_{n+1}}{n} + \frac{a_{n+2}}{n+1} + \dots \qquad (H, r)$$

for $n > 0$. In the last two equations, H may be replaced by C. If now we write

$$a_n = a'_{n-2}\ (n > 1), \qquad b_n = b'_{n-1} \qquad (n > 0),$$

we have

$$b'_n = \frac{a'_n}{n+1} + \frac{a'_{n+1}}{n+2} + \dots \qquad (C, r).$$

Thus (2.64) is equivalent to

$$b'_0 + b'_1 + b'_2 + b'_3 + \dots = A - a_0 - a_1 \qquad (C, r),$$

and therefore to

$$a'_0 + a'_1 + a'_2 + a'_3 + \dots = A - a_0 - a_1 \qquad (C, r+1),$$

or to

$$a_0 + a_1 + a_2 + a_3 + \dots = A \qquad (C, r+1). \tag{2.65}$$

The equivalence theorem is thus established by induction.

3. Proof of Theorem B.

3.1. We consider now the power-series (1.24). We may obviously suppose that the unit circle is the circle of convergence, the theorem being otherwise trivial.

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[9]) Equations equivalent to these are given by Young in the second memoir quoted in [4]).

The condition (1.241) is equivalent to

(3.11) $$a_n = O(n^K)$$

for some K. For if $a_n = O(n^H)$ then

$$f(x) = O(\sum n^H |x|^n) = O(1-|x|)^{-H-1};$$

and if $f(x) = O(1-|x|)^{-H}$ then

$$a_n = \frac{1}{2\pi i}\int \frac{f(x)}{x^{n+1}}\,dx = O(n^H),$$

on integrating round the circle $|x| = 1 - \frac{1}{n}$.

3.2. (1) We prove first that the conditions are *necessary*; and this is obvious so far as (3.11), or (1.241), is concerned.

Suppose that A is summable (C, r), and that $A_1, \ldots$ are defined as in Theorem A. If $|x| < 1$, we have

$$\sum \frac{a_n}{n+1} x^{n+1} = \sum (a_{n,1} - a_{n+1,1}) x^{n+1} = a_{0,1} - (1-x)\sum a_{n,1} x^n,$$

and therefore

(3.21) $$\varphi_1(x) = \sum a_{n,1} x^n = \frac{1}{1-x}\left(a_{0,1} - \sum \frac{a_n}{n+1} x^{n+1}\right)$$
$$= \frac{1}{1-x}\sum \frac{a_n}{n+1}(1-x^{n+1}) \quad (C, r-1),$$

since $\sum \frac{a_n}{n+1}$ is summable $(C, r-1)$. But

(3.22) $$\sum \frac{a_n}{n+1}(1-x^{n+1}) = \lim_{\xi\to 1} \sum \frac{a_n}{n+1}(\xi^{n+1} - x^{n+1})$$
$$= \lim_{\xi \to 1} \int_x^{\xi} f(x_1)\,dx_1 = \int_x^1 f(x_1)\,dx_1,$$

the path of integration being either of those contemplated in the enunciation of Theorem B, and ξ tending to 1 along it[10]).

From (3.21) and (3.22) we deduce

(3.23) $$\varphi_1(x) = \sum a_{n,1} x^n = \frac{1}{1-x}\int_x^1 f(x_1)\,dx_1 = f_1(x).$$

Similarly

(3.24) $$\sum a_{n,s} x^n = f_s(x).$$

[10]) When A is summable, $f(x)$ is continuous in any domain interior to the circle and bounded near 1 by two chords of the circle, and Cauchy's Theorem is valid for the triangle $(x, 0, 1)$. It is therefore indifferent in which way $f_1(x)$, or any of the succeeding functions, is defined. In the second half of the proof, where we have to prove our condition *sufficient*, this point will demand more careful consideration.

Take now $s=r+1$. Then A_{r+1} is summable $(C, -1)$, to sum A, and therefore, by a known theorem,

$$f_{r+2}(x) \to A$$

when $x \to 1$ in any manner from within the circle.

The theorem which we use here is this: *if $A=\Sigma a_n$ is convergent, and $a_n = o\left(\frac{1}{n}\right)$, then*

$$(3.25) \qquad F(x) = \frac{1}{1-x}\sum \frac{a_n}{n+1}(1-x^{n+1}) = \frac{1}{1-x}\int_x^1 \sum a_n x_1^n \, dx_1$$

$$= \frac{1}{1-x}\int_x^1 f(x_1)\, dx_1 \to A,$$

when $x \to 1$. We insert a proof of this theorem, since it has not been published, in its simplest form, before[11]).

In the first place, since $\sum \frac{a_n}{n+1}$ is absolutely convergent, the two forms of $F(x)$ in (3.25) are equivalent. Next

$$(3.26) \qquad F(x) = \frac{1}{1-x}\sum_{n \leqq N} + \frac{1}{1-x}\sum_{n>N} = F_1 + F_2,$$

where

$$(3.261) \qquad N = \left[\frac{1}{|1-x|}\right].$$

But

$$1-x^{n+1} = (n+1)(1-x) + O(n^2|1-x|^2),$$

and therefore

$$(3.27) \qquad F_1 = \sum_{n \leqq N} a_n + O\Big(|1-x| \sum_{n \leqq N} o(1)\Big)$$

$$= A + o(1) + o(N|1-x|) = A + o(1);$$

while

$$(3.28) \qquad F_2 = O\left(\frac{1}{|1-x|}\sum_{n>N} o\left(\frac{1}{n^2}\right)\right) = o\left(\frac{1}{N|1-x|}\right) = o(1).$$

From (3.26)—(3.28) we deduce (3.25). The conditions of Theorem B are therefore *necessary.*

3.3. (2) It remains to prove that the conditions are sufficient.

[11]) The theorem was first stated, but without proof, in our memoir "Contributions to the arithmetic theory of series", Proc. London Math. Soc. (2), **11** (1912), pp. 411—478 (478, Theorem 50). The proof of the correlative "Tauberian" theorem is given there, and also in Landau's *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie* (1916), pp. 43—45. In our later memoir referred to in [6]), the theorem appears as a special case of a more difficult theorem.

We must consider first the question of the equivalence of the two definitions of the functions $f_s(x)$. If $f_1(x)$ is defined by the broken line, then plainly

$$(1-x)f_1(x) = C - \sum \frac{a_n x^{n+1}}{n+1} = \int_0^1 f(x)\,dx - \mathfrak{f}(x),$$

say, the existence of the integral, as $\lim\limits_{\xi \to 1} \int_0^\xi$, being implied in our definition; and it is evident that $f_1(x)$ is regular for $|x| < 1$. But if $f_1(x)$ is defined by the line $(x, 1)$, it is not evident that $f_1(x)$ is an analytic function, and in fact this is not true without some restriction.

The *existence* of $f_1(x)$, for every x, means that $\mathfrak{f}(\xi)$ tends to a limit L when $\xi \to 1$ along any chord of the circle, in which case

$$(1-x)f_1(x) = L - \mathfrak{f}(x).$$

Here L is *prima facie* a function (not generally analytic) of x, so that f_1 is not necessarily analytic. It is sufficient, for f_1 to be analytic, that L *should be independent of* x; and this is true, but *only in virtue of the condition that* $a_n = O(n^K)$. Consider a region D, interior to the circle, and bounded near 1 by two chords of the circle; let L_1 and L_2 be the values of L corresponding to the two chords bounding D; and let D' be a slightly smaller region of the same type, interior to D. Then $\mathfrak{f}(x) = O(|1-x|^{-K})$ in D, and $\mathfrak{f} = L_1 + o(1)$, $\mathfrak{f} = L_2 + o(1)$ on the two sides of D; from which it follows, by well known theorems of Phragmén, Lindelöf, and Montel[12]), that $L_2 = L_1$ and that $\mathfrak{f} \to L_1$ uniformly in D'. Hence L is independent of x, and f_1 is analytic. We can now apply Cauchy's Theorem to the triangle $(x, 0, 1)$, and it appears that the two definitions of f_1 are equivalent. It is plain, moreover, that the same argument can be applied to the succeding functions $f_2, f_3, \ldots$.

Returning to our main argument, we observe first that it is enough to prove that *if* (3.11) *is satisfied, and* $f(x) \to A$, *then* A *is summable.* For suppose that this is established, and that what we are actually given is that

$$f_r(x) \to A.$$

Then the functions $f_s(x)$ are regular for $|x| < 1$, so that

$$f_s(x) = \Sigma\, a_{n,s} x^n,$$

[12]) E. Phragmén and E. Lindelöf, Sur une extension d'un principe classique de l'analyse, Acta Mathematica **31** (1908), pp. 381–406; E. Lindelöf, Sur un principe général de l'analyse, Acta Societatis Fennicae **46**, no. 4 (1915), pp. 1–35 (7); P. Montel, Sur les familles de fonctions analytiques qui admettent des valeurs exceptionnelles dans un domaine, Annales scientifiques de l'École Normale Supérieure (3), **29** (1912), pp. 487–535 (519). See also p. 218 of our memoir referred to in [6]).

say, for $|x|<1$. Also

$$f_{s-1}(x)=-\frac{d}{dx}((1-x)f_s(x)),$$

from which it follows that the numbers $a_{n,s}$ are related in the manner prescribed in Theorem A. But A_r is summable, and therefore A is summable, by Theorem A.

We may suppose then, without loss of generality, that $f(x)\to A$; and we may also suppose that $A=0$, so that $f(x)=o(1)$. This being so, we write

$$(3.31)\qquad F_1(x)=\int_x^1 f(x_1)\,dx_1,\qquad F_2(x)=\int_x^1 F_1(x_1)\,dx_1,\ \ldots,$$

the integrals being now rectilinear; and plainly we have

$$(3.32)\qquad F_h(x)=o(|1-x|^h)$$

for every h. Further, if $|x|<1$, we have

$$F_1(x)=C-\sum\frac{a_n}{n+1}x^{n+1},\qquad F_2(x)=-Cx+D+\sum\frac{a_n}{(n+2)(n+1)}x^{n+2},$$

and generally

$$(3.33)\qquad F_h(x)=P_{h-1}(x)+(-1)^h\sum\frac{a_n}{(n+h)\ldots(n+1)}x^{n+h},$$

where $P_{h-1}(x)$ is a polynomial of degree $h-1$. If h is large enough (*e.g.* if $h=K+2$) then the series is absolutely and uniformly convergent for $|x|\leqq 1$, and

$$(3.34)\qquad |F_h(x)|=O(1)\qquad (|x|<1).$$

Suppose that

$$(3.35)\qquad r=h+1=K+3.$$

Then

$$(3.36)\qquad \frac{A_n^r}{C_n^r}=\frac{1}{2\pi i\,C_n^r}\int\frac{f(x)}{(1-x)^{r+1}}\frac{dx}{x^{n+1}},$$

the contour of integration being the circle $|x|=1-\frac{1}{n}$. Integrating h times by parts, we find

$$(3.361)\qquad \frac{A_n^r}{C_n^r}=\frac{1}{2\pi i\,C_n^r}\int F_h(x)\left(\frac{d}{dx}\right)^h\left(\frac{1}{(1-x)^{r+1}x^{n+1}}\right)dx.$$

If we develop the right hand side by Leibniz' Theorem, we obtain

$$(3.362)\qquad \frac{A_n^r}{C_n^r}=\sum_{\lambda+\mu=h} I_{\lambda,\mu},$$

where

(3.363) $$I_{\lambda,\mu} = O(n^{\mu-r}) \int F_h(x)(1-x)^{-r-1-\lambda} x^{-n-1-\mu}\, dx,$$

the constants of the O's depending only on K.

We divide the contour of integration into two parts, C_1 (for which $|\arg x| \leqq \delta$), and C_2, and $I_{\lambda,\mu}$ into two corresponding parts $I_{\lambda,\mu,1}$ and $I_{\lambda,\mu,2}$, so that

(3.364) $$I_{\lambda,\mu} = I_{\lambda,\mu,1} + I_{\lambda,\mu,2}.$$

Given $\varepsilon > 0$ we can, by (3.32), choose δ so that

$$|F_h(x)| < \varepsilon\, |1-x|^h$$

on C_1. Also $|x|^{-n-1-\mu} = O(1)$ on C_1. We have therefore

$$|I_{\lambda,\mu,1}| < H\varepsilon n^{\mu-r} \int_0^{\delta} \frac{d\theta}{(1-2\varrho\cos\theta+\varrho^2)^{\frac{1}{2}(r+1+\lambda-h)}},$$

where $\varrho = 1 - \frac{1}{n}$ and H depends only on K, so that

(3.37) $$|I_{\lambda,\mu,1}| < H\varepsilon n^{\mu-r} \int_0^{\infty} \frac{d\theta}{\left(\frac{1}{n^2}+\theta^2\right)^{\frac{1}{2}(r+1+\lambda-h)}}$$
$$< H\varepsilon n^{\mu-r} \cdot n^{r+\lambda-h} < H\varepsilon n^{\lambda+\mu-h} = H\varepsilon.$$

When δ has been fixed we have, on C_2,

$$F_h(x) = O(1), \qquad |1-x|^{-1} = O(1), \qquad |x|^{-n-1-\mu} = O(1),$$

and therefore

(3.38) $$I_{\lambda,\mu,2} = O(n^{\mu-r}) = o(1),$$

since $\mu \leqq h < r$. Finally, from (3.362), (3.364), (3.37) and (3.38) we derive

$$\left|\frac{A_n^r}{C_n^r}\right| = \left|\sum_{\lambda+\mu=h} (H\varepsilon + o(1))\right| < H\varepsilon + o(1) < H\varepsilon,$$

if n is sufficiently large. Hence A is summable (C, r) to sum 0. This completes the proof of Theorem B.

Proof of Theorem B1.

3.4. An interesting particular case is that in which $f(x)$ is *bounded* in the unit circle. In this case it is essential to take account of non-integral orders of summability, and we shall assume a good deal more of the general theory than has been necessary hitherto.

Theorem B1 is the analogue for power-series of a theorem for Fourier series stated without proof[12a] in our memoir referred to in [6]. The first part of it is not new, being contained in Andersen's dissertation[13]. Andersen, however, gives a different necessary and sufficient condition for summability, viz. that $f(x) \to A$ when $x \to 1$ by real values. In this case the necessity of the condition is trivial, though not its sufficiency.

3.5. We require two additional lemmas.

Lemma 8. *If $f(x)$ is bounded and $f_r(x) \to A$ when $x \to 1$ by real values, then $f_s(x) \to A$ for $s = 1, 2, \ldots, r.$*[14]

We may suppose, without, loss of generality, that $f(x)$ is real and $A = 0$. If we write $x = 1 - y$, $f(x) = g(y)$, so that $y \to 0$, we have

$$g_1(y) = \frac{1}{y}\int_0^y g(y_1)\,dy_1, \quad g_2(y) = \frac{1}{y}\int_0^y g_1(y_1)\,dy_1, \quad \ldots, \quad g_r(y) = o(1).$$

It is enough to prove the lemma when $r = 2$. For suppose it proved in this case. Then all of the functions $g_s(y)$ are bounded, so that $g_{r-2} = O(1)$, $g_r = o(1)$, and therefore $g_{r-1} = o(1)$. Repeating the argument, we obtain the lemma in its general form.

Supposing then $r = 2$, we have

$$\int_0^y g_1(y_1)\,dy_1 = y g_2(y) = o(y);$$

and so, if δ is fixed and $0 < \delta < 1$,

$$\int_y^{y+\delta y} g_1(y_1)\,dy_1 = o(y),$$

$$\delta y g_1(y + \theta\delta y) = o(y), \tag{3.51}$$

where $0 < \theta < 1$. But

$$g_1(y + \theta\delta y) - g_1(y) = \theta\delta y\, g_1'(y + \theta\theta_1\delta y),$$

where $0 < \theta_1 < 1$, and

$$g_1'(y) = \frac{g(y)}{y} - \frac{1}{y^2}\int_0^y g(y_1)\,dy_1 = O\left(\frac{1}{y}\right);$$

so that

$$g_1(y + \theta\delta y) = g_1(y) + \delta\, O(1).$$

[12a]) Theorem Z, p. 235.

[13]) A. F. Andersen, *l. c.*, p. 93 (Saetning X). The same theorem was stated to us by O. Szász in a letter dated 28 Oct. 1919.

[14]) There is of course nothing essentially new in this lemma, which belongs to a class which we have used repeatedly before; but it is not actually included in any previous theorem that we can quote.

Hence, by (3.51),

$$g_1(y) = \delta O(1) + \frac{1}{\delta} o(1). \tag{3.52}$$

From (3.52) it follows (by choice first of δ and then of y) that $g_1(y) \to 0$, which proves the lemma.

Lemma 9. *If $f(x)$ is bounded for $|x| < 1$, and $f_r(x) \to A$ when $x \to 1$ in any manner from within the circle, then $f_s(x) \to A$ for $s = 1, 2, \ldots, r$.*

Write

$$x = 1 - ye^{i\varphi} \quad \left(0 < y < 1, -\frac{1}{2}\pi < \varphi < \frac{1}{2}\pi\right),$$

$$f(x) = f(1 - ye^{i\varphi}) = g(y, \varphi) + ih(y, \varphi),$$

$$f_1(x) \qquad\qquad = g_1(y, \varphi) + ih_1(y, \varphi),$$

and so on, $g_1, g_2, \ldots$, for example, being formed from g as in the proof of Lemma 8. All the g's are bounded in the two variables y, φ; and $g_r = o(1)$, uniformly in φ. Repeating with these glosses the arguments used before, we find that $g_s = o(1)$ uniformly in φ. Similarly $h_s = o(1)$, which proves the lemma.

3.6. It is now easy to prove Theorem B1. In the first place, every Cesàro mean of A, of positive order, is bounded[15]). Hence[16]), if A is summable by any mean, it is summable by every mean of positive order. This proves the first part of the theorem.

If $f_1(x) \to A$, A is summable, by Theorem B, and therefore summable (C, δ) for every positive δ. The condition is therefore *sufficient*. If, on the other hand, A is summable, $f_r(x) \to A$, for some r, by Theorem B; and therefore $f_1(x) \to A$, by Lemma 9. The condition is therefore *necessary*.

The conditions of Theorem B1 are unnecessarily narrow. It should be enough to suppose (i) that $f(x)$ is bounded in an angle $|\arg x| \leq \delta$ and (ii) that

$$\int_0^{2\pi} |f(|x| e^{i\theta})| \, d\theta$$

is bounded[17]); but we have not worked out a detailed proof.

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[15]) E. Landau, Abschätzung der Koeffizientensumme einer Potenzreihe (III), Archiv der Math. u. Physik (3), **21** (1916), pp. 250–260, and *Darstellung u. s. w.*, pp. 17–19; A. F. Andersen, l. c., pp. 91–93 (Saetning IX).

[16]) A. F. Andersen, l. c., pp. 56–59 (Saetning VII). A less complete theorem was proved by Doetsch (G. Doetsch, Über die Cesàrosche Summabilität bei Reihen und eine Erweiterung des Grenzwertbegriffs bei integrablen Funktionen, Math. Zeitschrift **11** (1921), 161–179.)

[17]) Compare Theorem C 1 (§ 4).

4. Proof of Theorem C.

4.1. We proceed, taking now Theorem A 1 as our basis, to discuss the summability of the Fourier series, for $t = x$, of a periodic and integrable function $f(t)$. It is convenient to make certain preliminary simplifications.

The Fourier series of $f(t)$ for $t = x$ is, save for the constant term and a factor $\frac{1}{2}$, that of $\varphi(t)$ for $t = 0$. We may therefore confine our attention to even functions, and to the special value $t = 0$; and we may suppose that $A = 0$. We may also suppose that $a_0 = 0$, so that

$$\varphi(t) \sim a_1 \cos t + a_2 \cos 2t + \dots . \tag{4.11}$$

We shall find it convenient to introduce the auxiliary functions

$$\psi_0 = \varphi_0 = \varphi, \qquad \psi_1 = \frac{1}{2} \cot \frac{1}{2} t \int_0^t \psi_0(t_1)\, dt_1 + \gamma_1 \sin^2 \frac{1}{2} t, \tag{4.12}$$

$$\psi_2 = \frac{1}{2} \cot \frac{1}{2} t \int_0^t \psi_1(t_1)\, dt_1 + \gamma_2 \sin^2 \frac{1}{2} t, \dots,$$

where the γ's are constants whose values will be fixed in a moment. It may be verified at once that, if any one of these functions ψ_r exists, then so does the corresponding φ_r, and conversely, and that the difference between φ_r and ψ_r is in fact $o(t)$. To prove this we need only observe (i) that it is obvious when $r = 0$, and (ii) that, if it is true when $r = m$, then

$$\psi_{m+1} = \left(\frac{1}{t} + O(t)\right) \int_0^t (\varphi_m(t_1) + o(t_1))\, dt_1 + o(t_1) = \varphi_{m+1} + o(t).\text{[18]}$$

In proving Theorem C, then, we may replace every φ by the corresponding ψ.

4.2. We suppose first that φ^2 is integrable, in which case the proof is greatly simplified. Substantially the same proof is valid if we suppose only that $|\varphi|^{1+\delta}$, where $\delta > 0$, is integrable.

Lemma 10. *If φ^2 is integrable, then φ_r^2 and ψ_r^2 are integrable.*

[18]) It is important to observe that this proof holds even when some or all of the functions are not integrable (L). It is enough that they should be integrable in any sense which makes the integral a continuous function of its upper limit, and, in particular, enough that they should possess integrals of the type

$$\int_0^a F(t)\, dt = \lim_{\varepsilon \to 0} \int_\varepsilon^a F(t)\, dt$$

(elementary non-absolutely convergent integrals).

It is plainly sufficient to prove that φ_1^2 is integrable, *i. e.* that $\int\limits_{\varepsilon}^{\delta} \varphi_1^2\, dt$ is bounded when $\varepsilon \to 0$. But, if

$$\Phi_1(t) = \int\limits_0^t \varphi(t_1)\, dt_1,$$

we have

$$\Phi_1^2 \leqq t \int\limits_0^t \varphi^2\, dt_1 = o(t)$$

and so

$$\int\limits_{\varepsilon}^{\delta} \varphi_1^2\, dt = \int\limits_{\varepsilon}^{\delta} \frac{\Phi_1^2}{t^2}\, dt = \frac{(\Phi_1(\varepsilon))^2}{\varepsilon} - \frac{(\Phi_1(\delta))^2}{\delta} + 2\int\limits_{\varepsilon}^{\delta} \varphi_1 \varphi\, dt$$

$$= o(1) + O(1) + O\left(\sqrt{\int\limits_{\varepsilon}^{\delta} \varphi_1^2\, dt}\right),$$

which is impossible unless the left hand side is bounded[19]).

All that we shall use in the argument which follows is that φ_r and ψ_r are integrable (L).

4. 31. The function ψ_1 is even and periodic, since

$$\int\limits_{-\pi}^{\pi} \psi_0(t)\, dt = 0.$$

Since $\int\limits_{-\pi}^{\pi} \sin^2 \frac{1}{2} t\, dt = \pi$, we can choose γ_1 so that

$$\int\limits_{-\pi}^{\pi} \psi_1(t)\, dt = 0,$$

and ψ_1 then fulfils all the relevant conditions fulfilled by ψ_0. Proceeding in this manner, we can determine the constants $\gamma_1, \gamma_2, \ldots$ successively so that

$$\int\limits_{-\pi}^{\pi} \psi_s(t)\, dt = 0 \qquad (s = 0, 1, 2, \ldots),$$

and we may write

$$\psi_s(t) \sim a_1^s \cos t + a_2^s \cos 2t + \ldots. \tag{4.311}$$

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[19]) The proof of Lemma 10 is an adaptation of M. Riesz's proof of the corresponding theorem for infinite series. See G. H. Hardy, Note on a theorem of Hilbert, Math. Zeitschrift 6 (1920), pp. 314—317.

We have now

$$\sum_n^N \frac{a_\nu}{\nu} = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t) \sum_n^N \frac{\cos \nu t}{\nu}\, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \Phi_1(t) \sum_n^N \sin \nu t\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos\left(n-\frac{1}{2}\right)t - \cos\left(N+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\, \Phi_1(t)\, dt$$

$$\to \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_1(t) \cot\frac{1}{2}t \cos nt\, dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_1(t) \sin nt\, dt.$$

when $N \to \infty$, since $\Phi_1(t)\cot\frac{1}{2}t$ is integrable. Hence[20]

(4.312) $$a_{n,1} = a_n^1 + b_{n,1} = a_n^1 + b'_{n,1} + b''_{n,1},$$

where

(4.313) $$b'_{n,1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_1(t) \sin nt\, dt, \quad b''_{n,1} = -\frac{\gamma_1}{2\pi}\int_{-\pi}^{\pi} \sin^2\frac{1}{2}t \cos nt\, dt.$$

so that $b''_{n,1}$ is $\frac{1}{2}\gamma_1$ if $n = 1$ and zero if $n > 1$.

4.32. The series $\Sigma b''_{n,1}$ is plainly convergent to sum $\frac{1}{2}\gamma_1$. Further,

(4.321) $$\sum_1^N b'_{n,1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos\frac{1}{2}t - \cos\left(N+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}\, \Phi_1(t)\, dt \to \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\cot\frac{1}{2}t\, \Phi_1(t)\, dt$$

when $N \to \infty$. Hence $\Sigma b_{n,1}$ is convergent, to sum

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\cot\frac{1}{2}t\, \Phi_1(t)\, dt + \frac{1}{2}\gamma_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_1(t)\, dt = 0.$$

Also $b'_{n,1}$, being the Fourier sine coefficient of an integral, is $o\left(\frac{1}{n}\right)$, and therefore $b_{n,1}$ is of the same form. In other words, $\Sigma b_{n,1}$ is summable $(C, -1)$ to sum zero; and we have proved that *$a_{n,1}$ differs from a_n^1 by the general term of a series B_1, summable $(C, -1)$ to sum 0.*

As ψ_1 possesses all the relevant properties of ψ_0, we may repeat the argument. The series $B_2, B_3, \ldots$, formed from B_1 as $A_2, A_3, \ldots$ are formed from A_1, are all summable $(C, -1)$, to sum 0. and so are all

20) In the notation of Theorem A 1.

other series of similar type introduced in the argument[21]). We thus obtain

Lemma 11. *If φ^2 is integrable then $a_{n,s}$ differs from the Fourier coefficient of ψ_s by the general term of a series summable $(C, -1)$ to sum 0.*

4.4. We next recall a known theorem.

Lemma 12. *If $a_n = o\left(\frac{1}{n}\right)$, then the necessary and sufficient condition that A should be convergent is that $\varphi_1 \to A$.*

This result is due to Fatou[22]). We have generalised it, replacing the o by O, but this generalisation is not needed here.

There is now no difficulty in proving Theorem C (when φ^2 is integrable). We suppose $A = 0$. The condition is *necessary*; for, if A is summable (C, r) to sum 0, A_{r+1} is summable $(C, -1)$ to sum 0, by Theorem A 1; A^{r+1} is summable $(C, -1)$ to sum 0, by Lemma 11; and therefore ψ_{r+1}, or φ_{r+1}, tends to 0, by Lemma 12. The condition is *sufficient*; for, if φ_r, or ψ_r, tends to 0, A^r is summable $(C, 1)$ to sum 0, by Fejér's theorem; A_r is summable $(C, 1)$ to sum 0, by Lemma 11; and therefore A is summable $(C, r+1)$ to sum 0, by Theorem A 1.

4.5. We have still to prove the theorem in the general case. The proof follows the same lines, but more delicate considerations are required. The difficulty is an obvious one. The integrability (L) of φ does not necessarily imply that of φ_1, as may be seen from the trivial example in which

$$\varphi = \frac{1}{t\,(\log t)^2}:$$

here φ_1 exists, but is not integrable down to 0, and $\varphi_2, \varphi_3, \ldots$ have no meaning.

We begin by proving the *sufficiency* of our condition, which is comparatively easy. A word of preliminary explanation is required. Our condition asserts that some one of the functions $\varphi, \varphi_1, \ldots$, say φ_r, is continuous for $t = 0$, and this obviously implies the existence, in some sense, of all of $\varphi_1, \varphi_2, \ldots, \varphi_r$. If the integrals which define these functions are Lebesgue integrals, *i. e.* if all the φ's are integrable (L), then there is nothing more to prove, the proof of §§ 4.3—4.4 being valid as it

[21]) Thus $a_{n,2}$ differs from a_n^2 by (1) a term $b_{n,2}$ derived from $b_{n,1}$ as $a_{n,2}$ is from $a_{n,1}$ and (2) a term $c_{n,2}$ arising in the same manner as $b_{n,1}$. At the next stage the difference arises from three terms $b_{n,3}, c_{n,3}, d_{n,3}$; and so on. The series thus arising have all the same property.

[22]) P. Fatou, Séries trigonométriques et séries de Taylor, Acta Mathematica **30** (1906), pp. 335—400. See also pp. 228 *et seq.* of our memoir quoted in [6]).

stands. It will appear, however, that we are in any case compelled to consider integrals of a more general type, viz. "elementary generalised integrals"

$$\int_0^a F(t)\,dt = \lim_{\varepsilon\to 0}\int_\varepsilon^a F(t)\,dt,$$

where the integral on the right is a Lebesgue integral but that on the left is not. If the integrals which define the φ's are of this type, our argument requires reconsideration.

We call an integral of this type a "Cauchy" integral, and say that its integrand is integrable (C). It is to be understood that our integrals are "generalised" only in respect to the point $t=0$.

4.6. Lemma 13. *If φ is integrable (L), $\varphi_1, \varphi_2, \ldots, \varphi_{r-1}$ are integrable* (C), *and φ_r is continuous for $t=0$, then $\varphi_1, \varphi_2, \ldots, \varphi_r$ are integrable (L).*

It is convenient to make the transformations

$$t = e^{-x}, \quad F(x) = e^{-x}\varphi(t), \quad F_1(x) = e^{-x}\varphi_1(t), \quad \ldots .$$

Our data are then (i) that the integrals

$$\int_x^\infty |F(y)|\,dy, \quad F_1(x) = \int_x^\infty F(y)\,dy, \quad F_2(x) = \int_x^\infty F_1(y)\,dy, \quad \ldots$$

exist (as Cauchy integrals up to ∞), and (ii) that

$$F_r(x) = \int_x^\infty F_{r-1}(y)\,dy = o(e^{-x})$$

and our conclusion is to be that $\int_x^\infty |F_s(y)|\,dy$ exists for $s = 1, 2, \ldots, r$. We shall in fact assume only that $F_r(x) = O(e^{-ax})$, a being a positive constant.

The functions $F_s(x)$ are continuous, and tend to zero when $x\to\infty$, in virtue of their definition, and there is no loss of generality in supposing that $|F_s(x)| < 1$.

We consider the curve

$$y = F_{r-1}(x) \qquad (\xi < x < 2\xi),$$

ξ being large and positive. The set of points x for which

$$|y| > \eta = e^{-\frac{1}{2}a\xi}$$

consists of a set of (open) intervals XX'; and

$$\int_X^{X'} \eta\,dx \leqq \left|\int_X^{X'} F_{r-1}(x)\,dx\right| = O(e^{-aX}) = O(e^{-a\xi})$$

so that

$$X' - X < \frac{A e^{-a\xi}}{\eta} < A\eta, \tag{4.61}$$

where the A's are constants. Suppose now that $X \leqq x \leqq X'$. Then

$$F_{r-1}(x) = F_{r-1}(X) - (x - X) F_{r-2}(X) + \ldots + (-1)^{r-2} \frac{(x-X)^{r-2}}{(r-2)!} F_1(X)$$
$$+ (-1)^{r-1} \int_X^x dx_1 \int_X^{x_1} dx_2 \ldots \int_X^{x_{r-2}} F(x_{r-1})\, dx_{r-1},$$

$$|F_{r-1}(x)| \leqq \eta + (X' - X) + \ldots + \frac{(X'-X)^{r-2}}{(r-2)!} + \left(\int_X^{X'} dx\right)^{r-1} |F|.\text{[23]} \tag{4.62}$$

Suppose first that $r = 2$. Then (4.62) is

$$|F_1(x)| \leqq \eta + \int_X^{X'} |F|\, dx; \tag{4.63}$$

whence

$$\int_\xi^{2\xi} |F_1|\, dx \leqq \xi\eta + \Sigma \int_X^{X'} |F_1|\, dx \leqq 2\xi\eta + \Sigma (X' - X) \int_X^{X'} |F|\, dx \tag{4.64}$$
$$\leqq 2\xi\eta + \Sigma A\eta \int_X^{X'} |F|\, dx < 2\xi\eta + A\xi\eta < A\xi\eta = A\xi e^{-\frac{1}{2}a\xi},$$

by (4.63) and (4.61). From (4.64) it follows at once that $\int^\infty |F_1|\, dx$ is convergent.

If $r > 2$, (4.62) gives

$$|F_{r-1}(x)| < \eta + A(X' - X) < A\eta < Ae^{-A\xi} < Ae^{-Ax}; \tag{4.65}$$

so that $\int^\infty |F_{r-1}|\, dx$ is convergent. It is plain that, if we start afresh from (4.65) and repeat the argument, we complete the proof of the lemma.

It follows from Lemma 13 that, if the condition of Theorem C is satisfied, all of $\varphi_1, \varphi_2, \ldots$ are integrable (L). The argument of §§ 4.3 — 4.4 is therefore valid, and the *sufficiency* of the condition is established. We have now to prove that it is also *necessary*, and this requires a series of further lemmas.

4.71. Lemma 14.[24]) *The Fourier coefficients of a function, integrable* (C) *over* $(-\pi, \pi)$, *are* $o(n)$.

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[23]) Since $|F_{r-1}(X)| \leqq \eta$. This inequality fails if X is ξ. But evidently the equation preceding (4.62) holds also with X' in place of X. If X is ξ, $|F_{r-1}(X')| \leqq \eta$, and (4.62) is still true.

[24]) This lemma is of course well-known, but we are unable to give a precise reference.

For if, *e. g.*,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nt\, f(t)\, dt,$$

where $f(t)$ is integrable (C), and $\int_0^t f(t_1)\,dt_1 = F(t)$, we have

$$a_n = O(1) + \frac{n}{\pi}\int_{-\pi}^{\pi} \sin nt\, F(t)\, dt = o(n),$$

since $F(t)$ is continuous.

Lemma 15. *If* $f(t)$ *and* $\frac{F(t)}{t}$ *are integrable* (C), *then*

$$\int_0^a \frac{\cos}{\sin} nt \frac{F(t)}{t}\, dt \to 0 \quad (C, 1).$$

The lemma is trivial if $\frac{F(t)}{t}$ is integrable (L), the integral then tending to zero in the ordinary manner, by the theorem of Riemann-Lebesgue. It is in any case trivial if the *sine* is chosen, since $F(t)$ is continuous and we can apply Fejér's theorem. We need therefore only consider the cosine integral.

Denoting the integral, in this case, by v_n, we have

$$\begin{aligned} v_1 + v_2 + \ldots + v_n &= \frac{1}{2}\int_0^a \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,\frac{F(t)}{t}\, dt + O(1) \\ &= \frac{1}{2}\int_0^a \sin nt \cot\frac{1}{2}t\,\frac{F(t)}{t}\, dt + \frac{1}{2}\int_0^a \cos nt\,\frac{F(t)}{t}\, dt + O(1) \\ &= \int_0^a \sin nt\,\frac{F(t)}{t^2}\, dt + o(n), \end{aligned}$$

by Lemma 14; so that what we have to prove is that

$$i_n = \int_0^a \sin nt \frac{F(t)}{t^2}\, dt = o(n). \tag{4.711}$$

Write

$$i_n = \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{a} = k_n + l_n + m_n. \tag{4.712}$$

In the first place

$$(4.713)\qquad k_n = n\int_0^{\frac{1}{n}} \frac{\sin nt}{nt}\frac{F(t)}{t}\,dt = n\int_0^{\eta}\frac{F(t)}{t}\,dt \qquad \left(0<\eta<\frac{1}{n}\right),$$

since $\frac{\sin nt}{nt}$ decreases steadily from 1 throughout the interval of integration. Next

$$(4.714)\qquad |l_n| = \left|\int_{\frac{1}{n}}^{\delta}\sin nt\,\frac{F(t)}{t^2}\,dt\right| \leqq \mu\int_{\frac{1}{n}}^{\delta}\frac{dt}{t^2} < \mu n,$$

where μ is the maximum of $|F(t)|$ in $(0, \delta)$. Finally, when δ is fixed,

$$(4.715)\qquad m_n = o(1) = o(n)$$

by the theorem of Riemann-Lebesgue. It is plain that (4.711) follows from (4.712)—(4.715), by choice first of δ and then of n.

Lemma 16. *If* $\sum\frac{a_n}{n}$ *is convergent, then*

$$\chi(\varepsilon) = \sum\frac{a_n}{n}\int_{n\varepsilon}^{nt}\left(\frac{\sin\frac{1}{2}t_1}{\frac{1}{2}t}\right)^2 dt_1$$

is continuous for $\varepsilon = 0$.

We may replace the upper limit in the integral by ∞, since $\int_{nt}^{\infty}$ is a positive decreasing function of n. Also $\int_{n\varepsilon}^{\infty}$ is a positive decreasing function of n for every fixed $\varepsilon \geqq 0$, and is a bounded function of (n, ε). Hence

$$\sum\frac{a_n}{n}\int_{n\varepsilon}^{\infty}\left(\frac{\sin\frac{1}{2}t}{\frac{1}{2}t}\right)^2 dt$$

is uniformly convergent, and therefore continuous.

4.72. We suppose now that $\varphi(t)$ satisfies the conditions of 4.1, except that the integrals

$$\int_{-\pi}^{\pi}\varphi(t)\,dt,\qquad \int_{-\pi}^{\pi}\varphi(t)\cos nt\,dt$$

may exist only as Cauchy integrals[25]); and that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(t)\cos nt\,dt = o(1).$$

[25]) The existence of the second integral is (by the second mean-value theorem) a corollary of that of the first.

In these circumstances we shall continue to call a_n the "Fourier coefficient" of $\varphi(t)$.

Lemma 17. *If* (i) $\varphi(t)$ *is integrable* (C), (ii) $a_n = o(1)$, *and* (iii) A *is summable, then* $\varphi(t)$ (*or* $\psi(t)$) *is integrable* (C).

If

$$\Phi_1(t) = \int_0^t \varphi(t_1)\,dt_1, \qquad \Phi_2(t) = \int_0^t \varphi_1(t_1)\,dt_1,$$

we have $\Phi_1 = o(1)$, $\Phi_2 = o(t)$ and

$$\int_\varepsilon^a \varphi_1\,dt = \int_\varepsilon^a \frac{\Phi_1}{t}\,dt = \frac{\Phi_2(a)}{a} - \frac{\Phi_2(\varepsilon)}{\varepsilon} + \int_\varepsilon^a \frac{\Phi_2(t)}{t^2}\,dt.$$

What we have to prove, then, is that

$$\lim_{\varepsilon\to 0} \int_\varepsilon^a \frac{\Phi_2(t)}{t^2}\,dt \tag{4.721}$$

exists.

Now $\Phi_1(t)$, being continuous, possesses a Fourier series which is uniformly summable $(C, 1)$; and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \Phi_1(t)\sin nt\,dt = \frac{1}{n\pi}\int_{-\pi}^{\pi} \varphi(t)\cos nt\,dt = \frac{a_n}{n}$$

(since Φ_1 is periodic); so that

$$\Phi_1(t) = \sum \frac{a_n}{n}\sin nt.$$

The series is uniformly summable, and therefore uniformly convergent, since its general term is $o\left(\frac{1}{n}\right)$. Integrating, dividing by t^2, and integrating again, we obtain

$$\int_\varepsilon^a \frac{\Phi_2(t)}{t^2}\,dt = \sum \frac{a_n}{n^2}\int_\varepsilon^a \frac{1-\cos nt}{t^2}\,dt = \frac{1}{2}\sum \frac{a_n}{n}\int_{n\varepsilon}^{na} \left(\frac{\sin\frac{1}{2}t}{\frac{1}{2}t}\right)^2 dt. \tag{4.722}$$

But $\sum \frac{a_n}{n}$ is summable (since A is summable), and its general term is $o\left(\frac{1}{n}\right)$, so that it is convergent. Hence, by Lemma 16, (4.722) is continuous for $\varepsilon = 0$. That is to say, the limit (4.721) exists, which proves the lemma.

4.73. We return now to the analysis of 4.3, assuming only that $\varphi(t)$ satisfies the conditions of Lemma 17, and that A is summable.

Then ψ_1 (or φ_1) is integrable (C), by Lemma 17. And we can determine γ_1, as in § 4.31, so that

$$\int_{-\pi}^{\pi} \psi_1(t)\,dt = 0,$$

the integral being now a Cauchy integral.

We now perform the transformations of § 4.31, appealing, however, to Lemma 15 instead of the theorem of Riemann-Lebesgue. We have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos\left(N-\frac{1}{2}\right)t}{t}\,\Phi_1(t)\,dt \to 0 \quad (C, 1),$$

by Lemma 15, and therefore

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos\left(N-\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,\Phi_1(t)\,dt \to 0 \quad (C, 1).$$

Hence

$$\sum_{n}^{N} \frac{a_\nu}{\nu} \to \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos\left(n-\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,\Phi_1(t)\,dt \quad (C, 1)$$

when $N \to \infty$. But $\frac{a_\nu}{\nu} = o\left(\frac{1}{\nu}\right)$, and we may therefore omit the $(C, 1)$. That is to say, (4.312) *is still valid under our more general hypotheses.*

Next, (4.321) holds, in the $(C, 1)$ sense, by Lemma 15, and the proof that $b_{n,1} = o\left(\frac{1}{n}\right)$ is unchanged. Hence (4.321) still holds in the ordinary sense, and our discussion of the series B_1 remains valid.

We have therefore proved

Lemma 18. *If* (i) φ *is integrable* (C), (ii) $a_n = o(1)$, *and* (iii) A *is summable, then* ψ_1 *is integrable* (C). *The Fourier coefficient of* ψ, *is also* $o(1)$, *and differs from* $a_{n,1}$ *by the general term of a series summable* $(C, -1)$ *to sum* 0.

We note in passing an interesting corollary of our analysis.

Lemma 19. *If* φ *is integrable* (L), *or, more generally, satisfies the conditions of Lemma* 17, *then the necessary and sufficient condition that*

the series $\sum \frac{a_n}{n}$ *should be convergent is that* ψ_1 (*or* φ_1) *should be integrable* (C). *If this condition is satisfied, and* $a_0 = 0$, *then*

$$\sum \frac{a_n}{n} = \frac{1}{\pi} \int_{-\pi}^{\pi} \psi_1(t)\, dt. \quad (4.733)$$ [26])

4, 8. We can now complete the proof of Theorem C. We prove in fact a little more, since we need not suppose φ integrable (L). It is only necessary to assume that it satisfies the conditions of Lemma 17. For, if A is summable (C), ψ_1, by Lemma 18, satisfies the same conditions as φ, and therefore $\psi_2, \psi_3, \ldots$ also satisfy them; and the Fourier constants of these functions are related as in §§ 4. 3 and 4. 4. The proof of 4. 4 is therefore still valid.

4. 9. It remains only to prove Theorem C 1, which follows from Theorem C in the same way that Theorem B 1 followed from Theorem B. The first part of the proof is exactly the same, while the second part is a little simpler, since we use Lemma 8 instead of Lemma 9.

Postscript (July 1923).

We add two further remarks.

(1) The condition (1. 28) of Theorem C, viz.

$$\text{(a)} \qquad \varphi_k(t) = \frac{1}{t} \int_0^t \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \cdots \int_0^{t_{k-2}} \frac{dt_{r-1}}{t_{r-1}} \int_0^{t_{k-1}} \varphi(t_k)\, dt_k \to 0,$$

may be replaced by

$$\text{(b)} \qquad \varphi_{(k)}(t) = \frac{k}{t^k} \int_0^t (t-u)^{k-1} \varphi(u)\, du \to 0.$$

[26]) The formula

$$\sum \frac{a_n}{n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t) \log\left(\frac{1}{4} \operatorname{cosec}^2 \frac{1}{2} t\right) dt$$

(equivalent to (4. 733)) has been proved by Young (On a certain series of Fourier, Proc. London Math. Soc. (2) 11 (1912), pp. 357–366) on the assumption that $\varphi \log |\varphi|$ is integrable (L): a more natural condition would be the integrability of $\varphi \log |t|$. In an earlier memoir (On the Fourier constants of a function, Proc. Roy. Soc. (A), 85 (1910), pp. 14–25) he proved it on the assumption that φ^2 is integrable. In an intermediate paper (On the convergence of certain series involving the Fourier constants of a function, *ibid.* 87 (1912), pp. 217–224) he showed that is is true if $|\varphi|^{1+\delta}$ $(\delta > 0)$ is integrable, and in fact that $\Sigma\, n^{-q} a_n$ is then convergent if $(1+\delta) q > 1$.

The two forms of mean value are in fact equivalent: either of (a) or (b) implies the other. This equivalence theorem is naturally a theorem of the same character as the Knopp-Schnee equivalence theorem, or Landau's analogue for integrals[27]; but it is not identical with, nor immediately deducible from, the latter.

(2) There is a theorem, of the same character as Theorem C, concerning the trigonometrical series "allied" or conjugate to the Fourier series of $f(t)$.

We say that the integral

$$\int_0^a h(t)\,dt,$$

where $h(t)$ is a function integrable (L) over (ε, a) for every positive ε, "exists in the Cesàro sense", and has the value A, if

$$H(t) = \int_t^a h(t)\,dt, \quad H_1(t) = \int_0^t H(t_1)\,dt, \quad H_2(t) = \int_0^t H_1(t_1)\,dt, \quad \ldots$$

and

$$H_k(\varepsilon) \sim \frac{A\varepsilon^k}{k!}$$

for some value of k. This being so, our theorem is as follows:

Theorem D. *The necessary and sufficient condition that the series conjugate to the Fourier series of $f(t)$ should be summable (C), for $t = x$, is that the integral*

$$\int_0^a \frac{f(x+t) - f(x-t)}{t}\,dt$$

should exist in the Cesàro sense.

The proof does not differ in principle from that of Theorem C.

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[27]) E. Landau, Die Identität des Cesàroschen und Hölderschen Grenzwertes für Integrale. Leipziger Berichte 65 (1913), S. 131–138.

(Eingegangen am 6. April 1923.)

CORRECTIONS

p. 85. In the formula for ψ_{m+1} the second $o(t_1)$ should $o(t)$.

p. 87, (4.313). The factor 2 should be deleted from the denominator in front of the second integral.

p. 88, *line* 15. For ψ_{r+1}, ϕ_{r+1} read ψ_{r+2}, ϕ_{r+2}.

p. 92, *statement of Lemma* 16. The t in the denominator of the integrand should be t_1.

p. 93, *statement of Lemma* 17. The conclusion should read 'then $\phi_1(t)$ (or $\psi_1(t)$) is integrable (C)'.

p. 94, *line* 10 *from below*. The proof at this point is incomplete; a corrected version is given in 1926, 4, p. 234 n.

p. 94, *line* 6 *from below*. For ψ read ψ_1.

COMMENTS

The main results of this paper were communicated to the London Mathematical Society at its meeting on 8 June 1922. See *Proceedings* (2), 21 (1923), xxxi–xxxii.

§ 1. In the comments below we suppose $f \in L(-\pi, \pi)$, we write

$$\phi(t) = \phi_0(t) = \tfrac{1}{2}\{f(\theta+t)+f(\theta-t)\},$$

and for $\alpha > 0$ we denote by ϕ_α the (C, α) mean of ϕ, defined as in (2) of the Introduction to the present group of papers. When α is integral, this definition is equivalent to the definition (1) in the same Introduction.

The statement of Theorem C (p. 70) says only that some Cesàro mean of the Fourier series of f at θ converges to the sum s if and only if some Cesàro mean of ϕ has the limit s. What is actually proved in the text (§ 4; note the correction above to p. 88) is as follows:

(i) *If* $\phi_\alpha(t) \to s$ *as* $t \to 0+$, *where* α *is a non-negative integer, then the Fourier series of* f *at* θ *is summable* (C, δ) *to the sum* s *for* $\delta = \alpha+1$.

(ii) *If the Fourier series of* f *at* θ *is summable* (C, δ) *to the sum* s, *where* δ *is a non-negative integer, then* $\phi_\alpha(t) \to s$ *as* $t \to 0+$ *for* $\alpha = \delta+2$.

The cases $\alpha = 0$, 1 of (i) were already known, and indeed in the stronger form:

(iii) *If* $\alpha = 0$ *or* 1, *and* $\phi_\alpha(t) \to s$ *as* $t \to 0+$, *then the Fourier series of* f *at* θ *is summable* (C, δ) *to the sum* s *for all* $\delta > \alpha$.

Here the case $\alpha = 0$ is due to M. Riesz, and the case $\alpha = 1$ to W. H. Young. It was known too that the result of (ii) holds also for $\delta = -1$, this being the theorem of Fatou quoted as Lemma 12 (p. 88).

In 1927, 2 Hardy and Littlewood considered results of type (i) and (iii) for general α, and stated the following theorem:

(iv) *If* $\alpha \geqslant 0$, *and* $\phi_\alpha(t) \to s$ *as* $t \to 0+$, *then the Fourier series of* f *at* θ *is summable* (C, δ) *to the sum* s *for all* $\delta > \alpha$.

They proved this result only for $0 < \alpha < 1$, adding the comment that 'the proof for $\alpha > 1$ will differ in complication but not in principle'.

In the direction of (ii), they proved in 1928, 3:

(v) *If* $-1 < \delta < 0$ *and the Fourier series of f at* θ *is summable* (C, δ) *to the sum* s, *then* $\varphi_1(t) \to s$ *as* $t \to 0+$.

All these results were completed by L. S. Bosanquet, *Proc. London Math. Soc.* (2), 31 (1930), 144–64 and ibid. 33 (1932), 561, and R. E. A. C. Paley, *Proc. Camb. Phil. Soc.* 26 (1930), 173–203. Bosanquet and Paley gave proofs of the general case of (iv), and proved also:

(vi) *If* $\delta \geqslant -1$, *and the Fourier series of f at* θ *is summable* (C, δ) *to the sum* s, *then* $\varphi_\alpha(t) \to s$ *as* $t \to 0+$ *for all* $\alpha > \delta+1$.

Paley proved also that the bound for δ in the case $\alpha = 0$ of (iv) is best possible. Hardy and Littlewood had already proved in 1920, 7 and 1924, 7 that the bound for α in the case $\delta = 0$ of (vi) is best possible.

An alternative proof of (iv) and (vi), using a general Tauberian theorem, was subsequently given by N. Wiener, *Annals of Math.* 33 (1932), 1–100 (see Chapter 7). In an earlier paper (*Proc. London Math. Soc.* (2), 30 (1929), 1–8), Wiener had given a similar proof of (i) and (ii). Another alternative proof of (vi) is given by S. Verblunsky, *Proc. Camb. Phil. Soc.* 26 (1930), 152–7.

Another alternative solution of the problem of summability (C) is given in Z II, pp. 65–71.

The result of Theorem C1 is included in Theorem A of 1926, 10.

In connection with Theorem B, see also L. S. Bosanquet and M. L. Cartwright, *Math. Zeitschrift*, 37 (1933), 170–92, 416–23.

§ 4. The generalization of Lemma 12 mentioned on p. 88 is given in 1917, 10, 1920, 7, and 1924, 3.

A simpler proof of Lemma 19 was given by Hardy and Rogosinski in 1947, 1.

p. 96. The results concerning the conjugate series mentioned here are proved in full in 1926, 4.

Note on a Theorem concerning Fourier Series

G. H. HARDY *and* J. E. LITTLEWOOD.

Extracted from Records of Proceedings at the Meeting of the London Mathematical Society, December, 1922.

Dr. Marcel Riesz has indicated to us a material simplification in the proof of Theorem X of our paper "Abel's Theorem and its Converse" in Vol. 18 (1918) of the *Proceedings* (see in particular pp. 229–231).

It is a question of proving that, *if*

$$A_n = O\left(\frac{1}{n}\right),$$

and ΣA_n *converges to* A, *then*

$$\Phi(a) = \Sigma A_n \frac{\sin na}{na} \to A,$$

when $a \to 0$.

We may suppose $A = 0$ and $|nA_n| < 1$ for all values of n. Given δ, then, we can choose μ so that

$$\left|\sum_1^{\mu} A_n\right| < \delta \tag{1}$$

and

$$\left|\sum_{\nu}^{\nu'} A_n\right| < \delta, \tag{2}$$

if $\mu \leqslant \nu < \nu'$. We write

$$\Phi = \sum_1^n + \sum_{\mu+1}^m + \sum_{m+1}^{\infty} = \Phi_1 + \Phi_2 + \Phi_3,$$

where

$$m = \left[\frac{K}{a}\right].$$

We have first

$$|\Phi_3| < \frac{1}{a} \sum_{m+1}^{\infty} \frac{1}{n^2} < \frac{1}{ma} < \frac{2}{K} < \epsilon, \tag{3}$$

by choice of K. When K has been fixed, the function $(\sin na)/na$ has, in the range of summation covered by Φ_2, a finite number $k(K)$ of maxima and minima, all numerically less than unity. Hence $|\Phi_2|$ is less than the sum of $k(K)$ terms of the type (2); *i.e.*

$$|\Phi_2| < \delta k(K) < \epsilon, \tag{4}$$

by choice of δ (*i.e.* of μ). We may obviously suppose $\delta < \epsilon$. Finally, when μ is fixed, we can choose a_0 so that

$$|\Phi_1| \leqslant \left|\sum_1^{\mu} A_n\right| + \epsilon < \delta + \epsilon < 2\epsilon, \tag{5}$$

for $0 < a \leqslant a_0$, by (1). From (3), (4), and (5) our conclusion follows.

1924, 3 (with J. E. Littlewood) *Proceedings of the London Mathematical Society* (2), 22, xviii–xix.

COMMENT

The result of this paper is included in Theorem 1 of 1926, 8.

The Allied Series of a Fourier Series

G. H. Hardy *and* J. E. Littlewood.

Extracted from Records of Proceedings at the Meeting of the London Mathematical Society, June, 1923.

1. The methods which we have applied to the solution of the "summability problem" for a Fourier's series $\frac{1}{2}a_0+\Sigma(a_n \cos nx+b_n \sin nx)$ may be applied also to the corresponding problem for the allied or conjugate series $\Sigma(b_n \cos nx - a_n \sin nx)$.

Suppose that $f(t)$ is integrable (in the sense of Lebesgue) in (ϵ, a) for every positive ϵ, that

$$\chi(\epsilon) = \int_\epsilon^a f(t)\,dt, \quad \chi_1(\epsilon) = \int_0^\epsilon \chi(\epsilon_1)\,d\epsilon_1, \quad \chi_2(\epsilon) = \int_0^\epsilon \chi_1(\epsilon_1)\,d\epsilon_1, \ \ldots,$$

and that
$$\chi_k(\epsilon) \sim A\epsilon^k$$
for some value of k. Then we say that

$$\int_0^a f(t)\,dt = A \ (C),$$

or that the integral exists in the Cesàro sense. This being so, *the necessary and sufficient condition that the allied series of $f(x)$ should be summable (C) is that*

$$\int_0^a \frac{f(x+t)-f(x-t)}{t}\,dt \tag{1}$$

should exist in the Cesàro sense.

2. It is known that any Fourier series is summable (C, δ) for every positive δ and almost all values of x. The same is true of the allied series; but the theorem lies a little deeper and has not, so far as we know, been stated generally, though the corresponding theorem for summability $(C, 1)$ is proved in a recent dissertation of Plessner.*

It was first proved by Young† that the allied series is summable $(C, 1)$ whenever (i) the integral (1) exists (as an "elementary generalized integral") and (ii) the integral

$$\int_0^u |f(x+t)-f(x-t)|\,dt$$

has, for $u=0$, a differential coefficient which is zero; and there is no

* A. Plessner, "Zur Theorie der konjugierten trigonometrischen Reihen", Giessen, 1922.

† *Proc. London Math. Soc.* (2), Vol. 10 (1911), pp. 271–272.

particular difficulty in replacing 1 by δ in Young's enunciation. The second condition is satisfied almost everywhere, and, in particular, in the set, whose complementary has measure zero, in which $|f(x)-q|$ is the derivative of its integral whatever be q.

We may call this set the Lebesgue set. In the Lebesgue set, the Fourier series of $f(x)$ is summable (C, δ); but this is not generally true of the allied series, since (i) is not necessarily true in the Lebesgue set or even at a point of continuity of the function. There is, however, an interesting property of the allied series which is true in the Lebesgue set, whether (i) be true or not, and which is the strict analogue of the property of the Fourier series. This property is that *if the series is summable by* any *Cesàro mean, or, more generally, if it is summable by Abel's limit, then it is summable* (C, δ) *for every positive* δ.

The question remains whether (i) is true almost everywhere, and this question is answered by the work of Fatou* and Plessner. It is equivalent to the question whether Abel's limit for the allied series exists for almost all values of x; the existence of this limit being, for points of the Lebesgue set, a necessary and sufficient condition for the existence of the integral (1). This connection was first remarked by Fatou as holding at all points of continuity of $f(x)$,† and is established in full generality in Plessner's dissertation. An equivalent condifion is the summability $(C, 1)$ of the allied series. This may be inferred from the work of Young, and is proved directly and explicitly by Plessner; and the theorem quoted in the last paragraph shows that we may replace $(C, 1)$ by (C, δ). To settle the main question, however, Plessner has recourse again to Abel's limit, and proves, by a direct adaptation of one of Fatou's theorems, that this limit does in fact exist (though not necessarily in the Lebesgue set) for almost all values of x. Combining these results we reach our final conclusion, that *the allied series is almost everywhere summable* (C, δ) *for every positive* δ.

We should refer in conclusion to two notes of Priwaloff.‡ The subject matter of these notes is extremely interesting, but the indications of demonstrations are insufficient.

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* P. Fatou, *Acta Math.*, Vol. 30 (1906), pp. 335–400.

† *Loc. cit.*, p. 360.

‡ I. Priwaloff, *Comptes Rendus*, Vol. 162 (1916), pp. 123–126, and Vol. 165 (1917), pp. 96–99.

COMMENTS

The results described here are proved in full in 1926, 4. It should be noted that in 1926, 4 a different, though equivalent, definition of the statement

$$\int_0^a f(t)\,dt = A(C)$$

is adopted.

THE ALLIED SERIES OF A FOURIER SERIES

By G. H. Hardy *and* J. E. Littlewood.

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[*Extracted from the Proceedings of the London Mathematical Society*, *Ser.* 2, *Vol.* 24, *Part* 3.]

1. *Introduction.*

1.1. Suppose that $f(t)$ is integrable in the sense of Lebesgue or, as we shall say, integrable (L), and that its Fourier series is

$$\tfrac{1}{2}a_0+\sum_1^\infty(a_n\cos nt+b_n\sin nt) = A_0+\sum_1^\infty A_n = \Sigma A_n. \tag{1.11}$$

Then the series

$$\sum_1^\infty(b_n\cos nt-a_n\sin nt) = \sum_1^\infty B_n = \Sigma B_n \tag{1.12}$$

is called the *allied* or *conjugate* series of the Fourier series.

The allied series of a Fourier series is not necessarily itself a Fourier series. Thus

$$\sum_2^\infty \frac{\cos nx}{\log n}$$

is a Fourier series, while
$$\sum_2^\infty \frac{\sin nx}{\log n},$$

although convergent for all values of x, is not a Fourier series, since its sum function is not integrable (L) in any interval which includes the origin.*

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* It has been proved by Young [W. H. Young, "The Fourier series of bounded functions", *Proc. London Math. Soc.* (2), 12 (1913), 41-70 (46)], that $\Sigma a_n \cos nx$ is a Fourier series if a_n is convex (*i.e.* $a_n-2a_{n+1}+a_{n+2}\geqslant 0$) and tends to zero. S. Szidon ["Reihentheoretische Sätze und ihre Anwendungen in der Theorie der Fourierschen Reihen", *Math. Zeitschrift*, 10 (1921), 121-127] has shown that Young's condition cannot be replaced by $a_n-a_{n+1}\geqslant 0$. Both Young and Szidon give a number of other interesting theorems of a similar character.

The fact that this particular sine-series is not a Fourier series was first noticed by Fatou, and is referred to in Lebesgue's *Leçons sur les séries trigonométriques*, 124. See also O. Perron, "Einige elementare Funktionen, welche sich in eine trigonometrische, aber nicht Fouriersche Reihe entwickeln lassen", *Math. Annalen*, 87 (1922), 84-89; and E. C. Titchmarsh, "Principal value Fourier series", *Proc. London Math. Soc.* (2), 23 (1925), xli-xliii (*Records* for March 13, 1924).

The formula for the sum of the allied series is

$$(1.13) \qquad g(x) = \frac{1}{2\pi}\int_0^\pi \{f(x+t)-f(x-t)\} \cot \tfrac{1}{2}t \, dt$$

or

$$(1.14) \qquad g(x) = \frac{1}{\pi}\int_0^\infty \frac{f(x+t)-f(x-t)}{t}\, dt,$$

it being, of course, understood that, in (1 . 14), $f(x)$ is defined by periodicity outside its original interval of definition. In these formulae the integrals which occur are generally not Lebesgue integrals, but "generalized" or, as we shall say, "Cauchy" integrals, defined as limits of the type

$$\lim_{\epsilon\to 0}\int_\epsilon^\pi, \quad \lim_{\epsilon\to 0,\, T\to\infty}\int_\epsilon^T.$$

1 . 2. The first accurate discussion of a convergence problem for the allied series is due to Pringsheim.* Pringsheim bases his work upon the *absolute* convergence of the integral (1 . 13), and his criteria are therefore much less general than those with which we shall be concerned. The more delicate theory of the convergence or summability of the series was initiated in two papers of Young,† though there is an implicit discussion of one problem in Fatou's well known memoir,‡ to which we shall refer further in a moment. Young proves (i) that the series is summable $(C, 1)$ if the integral (1 . 13) exists, in Cauchy's sense, and

$$(1.21) \qquad \int_0^u |f(x+t)-f(x-t)|\, dt = o(u);$$

and (ii) that the series is convergent if (1 . 13) exists and

$$\frac{1}{u}\int_0^u \{f(x+t)-f(x-t)\}\, dt$$

* A. Pringsheim, "Ueber das Verhalten von Potenzreihen auf dem Convergenzkreise", *Münchener Sitzungsberichte*, 30 (1900), 37–100 (79–100).

† W. H. Young, "Konvergenzbedingungen für die verwandte Reihe einer Fourierschen Reihe", *Münchener Sitzungsberichte*, 41 (1911), 361–371; and "On the convergence of a Fourier series and of its allied series", *Proc. London Math. Soc.* (2), 10 (1912), 254–272. See also a later paper, "On the convergence of the derived series of Fourier series", *ibid.*, 17 (1918), 195–236.

‡ P. Fatou, "Séries trigonométriques et séries de Taylor", *Acta Math.*, 30 (1906), 335–400.

is of bounded variation, a theorem corresponding to de la Vallée-Poussin's theorem* for Fourier series and including the tests which correspond to the classical tests, for Fourier series, of Jordan and Dini.

1.3. The condition (1.21) is certainly satisfied for all values of x in which $|f(x)-q|$ is the derivative of its integral whatever be q. We shall call this set the *Lebesgue set*. The set complementary to the Lebesgue set has measure zero, so that (1.21) is true almost always.

In the Lebesgue set the Fourier series is summable $(C, 1)$,† and indeed (C, δ) for every positive δ,‡ so that the series is summable (C, δ) almost always. It is natural to ask whether the allied series possesses the same property. The answer is less immediate, but is to be found, so far as summability $(C, 1)$ is concerned, in a recent dissertation of Plessner.§ The first step in this direction was taken by Fatou, who proved|| that, if x is a point of continuity of $f(x)$, the integral (1.13) exists, as a Cauchy integral, if and only if the harmonic function

$$\Sigma(b_n \cos nx - a_n \sin nx)\, r^n$$

tends to a limit when $r \rightarrow 1$; or, in other words, if and only if the allied series is summable by Abel's limit, or summable (A). Plessner shows that Fatou's conclusion remains valid in all points of the Lebesgue set, and so almost always; and that it holds for summability $(C, 1)$ as well as for summability (A). The problem is thus reduced to that of proving that the integral (1.13) exists for almost all values of x.

This is in fact true; but it is not necessarily true in the Lebesgue set, or even at a point of continuity of $f(x)$;¶ and it has never been proved generally and directly. It has been proved directly by Besikovitch** for functions of integrable square, and by M. Riesz and

* Ch. J. de la Vallée-Poussin, "Un nouveau cas de convergence des séries de Fourier", *Rend. di Palermo*, 31 (1911), 296–299; see also his *Cours d'analyse*, 2 (ed. 2), 149.

† H. Lebesgue, "Recherches sur la convergence des séries de Fourier", *Math. Annalen*, 61 (1905), 251–280.

‡ G. H. Hardy, "On the summability of Fourier's series", *Proc. London Math. Soc.* (2), 12 (1913), 365–372.

§ A. Plessner, "Zur Theorie der konjugierten trigonometrischen Reihen", *Mitteilungen des Math. Seminars der Univ. Giessen*, 10 (1923), 1–36.

|| *Loc. cit.*, 360.

¶ Example to the contrary: $x = 0$, $f(x) = (\log x)^{-1}$ for $x > 0$, $f(x) = 0$ for $x \leqslant 0$.

** A. Besikovitch, "Sur la nature des fonctions à carré sommable et des ensembles mesurables", *Fundamenta Math.*, 4 (1923), 172–195. The actual theorem, for functions of integrable square, is due to Lusin: see N. Lusin, "Sur la convergence des séries trigonométriques de Fourier", *Comptes Rendus*, June 2, 1913.

Titchmarsh* for functions of which any power higher than the first is integrable; but Plessner's proof is indirect and based upon the theory of analytic functions, and no alternative has yet been found.

1.4. In § 2 we prove first that, *in the Lebesgue set, the allied series is either summable by every Cesàro mean of positive order or summable by no Cesàro mean* (nor indeed by Abel's limit). There is nothing very novel in such a theorem, nor in the method of proof, but it is one which, as the natural analogue for the allied series of the generalizations of Fejér's theorem, should be stated explicitly and proved. We have only to combine this theorem with Plessner's to reach the main conclusion of this section, viz. that *the allied series is summable (C, δ) for every positive δ and almost all values of x.*

In §§ 3–4, which form the most substantial part of the paper, we solve for the allied series the problem which we solved for Fourier series in a recent memoir,† viz. that of finding the necessary and sufficient conditions for summability of the series by the aggregate of Cesàro's means. Suppose that $\phi(t)$ is integrable (L) in (ϵ, a) for every positive ϵ, and that

$$\chi(\epsilon) = \int_\epsilon^a \phi(t)\, dt. \tag{1.41}$$

Then we say that
$$\int_0^a \phi(t)\, dt$$
exists in the Cesàro sense, or *is summable* (C), if $\chi(\epsilon)$ tends to a limit in the Cesàro sense when $\epsilon \to 0$, *i.e.* if

$$\frac{1}{\epsilon}\int_0^\epsilon \frac{d\epsilon_1}{\epsilon_1}\int_0^{\epsilon_1} \frac{d\epsilon_2}{\epsilon_2} \cdots \int_0^{\epsilon_{r-2}} \frac{d\epsilon_{r-1}}{\epsilon_{r-1}}\int_0^{\epsilon_{r-1}} \chi(\epsilon_r)\, d\epsilon_r \tag{1.42}$$

tends to a limit for some value of r.‡ This being so, we prove that

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* In investigations as yet unpublished.

† G. H. Hardy and J. E. Littlewood, "Solution of the Cesàro summability problem for power series and Fourier series", *Math. Zeitschrift*, 19 (1923), 67–96. We refer to this memoir as F.S.

‡ We alter the form of the definition which we gave in our preliminary notice of this paper [*Proc. London Math. Soc.* (2), 22 (1924), xliii–xlv, *Records* for June 14, 1923], in order to avoid the questions discussed briefly in our note "The equivalence of certain integral means" (*ibid.* xl–xliii). The theorem is true with either definition.

the necessary and sufficient condition that the allied series should be summable (*C*) *is that the integral* (1 . 13) *should be summable* (*C*).

Finally, in § 5, we discuss, less systematically, certain questions concerning the integral

$$\int_0^a \frac{f(x+t)-f(x)}{t}\,dt. \tag{1.43}$$

There is a striking contrast between this integral and the integral

$$\int_0^a \frac{f(x+t)-f(x-t)}{t}\,dt, \tag{1.44}$$

which dominates the discussion of the allied series. The latter is convergent, for almost all x, for every integrable f, whereas (1 . 43) may diverge for almost all x even when f is continuous.

The integral (1 . 43) is connected with the series

$$\Sigma \frac{s_n - s}{n}, \tag{1.45}$$

where s_n is the sum of the first $n+1$ terms of the Fourier series of $f(x)$, and s is an appropriate number independent of n, in much the same way that (1 . 44) is connected with the allied series. We give one or two theorems to illustrate this connexion, but have not thought it worth while to prove a theorem analogous to that of § 3, though such a theorem is doubtless true. We conclude by giving a sufficient condition that (1 . 43) should converge almost everywhere, and represent a function of integrable square, viz. that

$$\Sigma(a_n^2+b_n^2)(\log n)^2 \tag{1.46}$$

should be convergent.

2. *The summability* ($C\delta$) *of the allied series.*

2 . 1. THEOREM 1.—*In a point of the Lebesgue set, the allied series is either summable by every Cesàro mean of positive order, or summable by no Cesàro mean.*

It is convenient to work not with the actual Cesàro means, but with

the equivalent Rieszian means.* We write

$$B_\omega^\delta = \sum_{1 \leqslant n < \omega} \left(1 - \frac{n}{\omega}\right)^\delta B_n, \tag{2.11}$$

$$\mathfrak{B}_\omega^\delta = \sum_{1 \leqslant n < \omega} \left(1 - \frac{n}{\omega}\right)^\delta nB_n; \tag{2.12}$$

and we have to show that, if x is in the Lebesgue set, and ΣB_n is summable by some Cesàro or Rieszian mean, then

$$B_\omega^\delta \to f(x), \tag{2.13}$$

when $\omega \to \infty$, for every positive δ.

2.2.—Lemma α.—*If* $\delta > 0$ *and* x *is in the Lebesgue set, then*

$$\mathfrak{B}_\omega^\delta = o(\omega). \tag{2.21}$$

We may plainly suppose that $\delta \leqslant 2$. We have

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\} \sim \Sigma B_n \sin nt, \tag{2.22}$$

the series being the Fourier series of the function in t. We multiply (2.22) by

$$\gamma(t) = \frac{d}{dt}\{t^{-1-\delta} C_{1+\delta}(\omega t)\}, \tag{2.23}$$

where

$$C_r(t) = \frac{t^r}{\Gamma(r+1)}\left\{1 - \frac{t^2}{(r+1)(r+2)} + \frac{t^4}{(r+1)\dots(r+4)} - \dots\right\}, \tag{2.24}$$

and integrate term by term over the interval $(0, \infty)$. When $\omega t = u$ is large, we have

$$u^{-1-\delta} C_{1+\delta}(u) = \frac{A}{u^2} + O\left(\frac{1}{u^3}\right) + \frac{B \sin(u - \frac{1}{2}\delta\pi)}{u^{1+\delta}} + O\left(\frac{1}{u^{2+\delta}}\right),$$

where A and B are constants, while asymptotic formulae for the

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* See M. Riesz, "Une méthode de sommation équivalent à la méthode des moyennes arithmétiques", *Comptes Rendus*, 22 Nov., 1909. Riesz does not give all the details of the proof: a complete account will appear in the second volume of the second edition of Hobson's *Theory of functions of a real variable.*

derivatives of the function can be found by formal differentiation.* It follows that

$$(2.25)\qquad \gamma(t) = \omega^{2+\delta}\frac{d}{du}\{u^{-1-\delta}C_{1+\delta}(u)\}$$

$$= \omega^{2+\delta}\left\{-\frac{2A}{u^3}+O\left(\frac{1}{u^4}\right)+\frac{B\cos(u-\frac{1}{2}\delta\pi)}{u^{1+\delta}}+O\left(\frac{1}{u^{2+\delta}}\right)\right\},$$

and that its derivative is of the form obtained by formal differentiation. Hence $\gamma(t)$ and $\gamma'(t)$ both possess absolutely convergent integrals up to infinity, and the legitimacy of the integration results from a theorem of Young.†

We obtain on integration

$$(2.26)\qquad \int_0^\infty \gamma(t)\psi(t)\,dt = \Sigma B_n \int_0^\infty \gamma(t)\sin nt\,dt$$

$$= -\Sigma nB_n \int_0^\infty t^{-1-\delta}C_{1+\delta}(\omega t)\cos nt\,dt$$

$$= -\frac{\pi}{2\Gamma(1+\delta)}\sum_{n\leqslant\omega}(\omega-n)^\delta nB_n,$$

since‡

$$\int_0^\infty t^{-q}C_q(t)\cos xt\,dt = \frac{\pi}{2\Gamma(q)}(1-x)^{q-1}\quad (x\leqslant 1),\qquad =0\quad (x>1).$$

Hence

$$(2.27)\qquad \mathfrak{B}_\omega^\delta = -\frac{2\Gamma(1+\delta)}{\pi\omega^\delta}\int_0^\infty \gamma(t)\psi(t)\,dt.$$

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* The asymptotic expansions for $C_r(u)$ and associated functions, for real values of u, are easily derived, by standard methods, from the formula

$$C_r(u) = \frac{u^r}{\Gamma(r)}\int_0^1 \cos ux\,(1-x)^{r-1}\,dx.$$

† W. H. Young, "On the integration of Fourier series", *Proc. London Math. Soc.* (2), 9 (1911), 449–462 (454). See also G. H. Hardy, "Notes on some points in the integral calculus (55)", *Messenger of Math.*, 51 (1922), 186–192; and other papers of Young there referred to.

‡ W. H. Young, "On infinite integrals involving a generalization of the sine and cosine functions", *Quart. J. of Math.*, 53 (1912), 161–177 (166).

2.3. We write

$$(2.31)\quad \frac{1}{\omega^\delta}\int_0^\infty \gamma(t)\psi(t)\,dt = \frac{1}{\omega^\delta}\int_0^{1/\omega} + \frac{1}{\omega^\delta}\int_{1/\omega}^1 + \frac{1}{\omega^\delta}\int_1^\infty = J_1 + J_2 + J_3,$$

say; and

$$(2.32)\quad \chi(t) = \int_0^t |\psi(u)|\,du = o(t)$$

(since x is in the Lebesgue set). In J, $\omega t < 1$ and $\gamma(t) = O(\omega^{3+\delta}t)$,* so that

$$(2.33)\quad J_1 = O\left(\frac{1}{\omega^\delta}\int_0^{1/\omega} \omega^{2+\delta}\,.\,\omega t\,.\,|\psi(t)|\,dt\right)$$

$$= O\left\{\omega^2\chi\left(\frac{1}{\omega}\right)\right\} = o(\omega).$$

In J_2, $u = \omega t$ is greater than 1, and so†

$$\gamma(t) = O\{\omega^{2+\delta}(\omega t)^{-1-\delta}\} = O(\omega t^{-1-\delta}),$$

by (2.25). Hence

$$(2.34)\quad J_2 = O\left(\omega^{1-\delta}\int_{1/\omega}^1 |\psi|\, t^{-1-\delta}\,dt\right)$$

$$= O\left\{\omega^{1-\delta}\left[\chi(1) - \omega^{1+\delta}\chi\left(\frac{1}{\omega}\right) + (1+\delta)\int_{1/\omega}^1 \chi(t)\,t^{-2-\delta}\,dt\right]\right\}$$

$$= O(\omega^{1-\delta}) + o(\omega) + o\left(\omega^{1-\delta}\int_{1/\omega}^1 t^{-1-\delta}\,dt\right)$$

$$= O(\omega^{1-\delta}) + o(\omega) = o(\omega).$$

Finally

$$(2.35)\quad J_3 = O(\omega^{1-\delta})\int_1^\infty |\psi|\,t^{-1-\delta}\,dt = O(\omega^{1-\delta}) = o(\omega).$$

From (2.27), (2.31), (2.32), (2.33), and (2.34) we deduce

$$\mathfrak{B}_\omega^\delta = o(\omega),$$

which proves the lemma.

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* Since $t^{-1-\delta}C_{1,\delta}(\omega t) = A\omega^{1+\delta} + B\omega^{3+\delta}t^2 + \dots$.

† It is here that we suppose $\delta \leqslant 2$.

2.4. LEMMA β.—*If ΣB_n is summable (C, r), where $r \geqslant 1$, and*

$$\mathfrak{B}_{\omega}^{r-1} = o(\omega), \tag{2.41}$$

then ΣB_n is summable $(C, r-1)$.

For
$$B_{\omega}^{r} - B_{\omega}^{r-1} = -\frac{1}{\omega}\mathfrak{B}_{\omega}^{r-1} = o(1).$$

2.5. Theorem 1 follows at once from Lemmas α and β. And if we combine Theorem 1 with the theorem of Plessner, that the allied series is summable $(C, 1)$ for almost all values of x, we obtain

THEOREM 2.—*The allied series is almost always summable (C, δ) for every positive δ.*

We add one remark before leaving this part of the subject. Theorem 1 does not contain quite all that we stated at the beginning of § 1.4, since we have said nothing about summability by Abel's limit. But here there is in reality nothing new to prove, since, after Plessner, summability by Abel's limit, in a point of the Lebesgue set, implies and is implied by the existence of the integral (1.13), and this, after Young, implies summability $(C, 1)$.

3. *Necessary and sufficient conditions for the summability (C) of the allied series of a function of integrable square.*

3.1. We pass to the problem of finding necessary and sufficient conditions for summability by *some* Cesàro mean at a particular point; and, as in our treatment of the same problem for the Fourier series, we consider first the case in which f^2 is integrable.*

THEOREM 3.—*The necessary and sufficient condition that the allied series should be summable (C), for a particular value of x, is that the integral* (1.13) *should be summable (C). The sum of the series is in this case equal to the value of the integral.*

As we explained in § 1.4, we say that

$$\chi(\epsilon) = \int_{\epsilon}^{a} \phi(t)\,dt \rightarrow A \quad (C, r) \tag{3.11}$$

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* Only trivial changes are needed when we are given that $f^{1+\delta}$ $(\delta > 0)$ is integrable.

or that

$$\int_0^u \phi(t)\,dt = A \quad (C, r), \tag{3.12}$$

or that the integral is summable (C, r), to sum A, if

$$\chi_r(\epsilon) = \frac{1}{\epsilon}\int_0^\epsilon \frac{d\epsilon_1}{\epsilon_1}\int_0^{\epsilon_1}\frac{d\epsilon_2}{\epsilon_2}\dots\int_0^{\epsilon_{r-1}}\chi(\epsilon_r)\,d\epsilon_r \to A. \tag{3.13}$$

We shall say that the integral (3 . 12) is *convergent*, or that $\phi(t)$ possesses a "Cauchy integral" down to 0, or is integrable (C), if $\chi(\epsilon)$ tends to a limit in the ordinary sense. It is to be understood, in all these definitions, that $\phi(t)$ is integrable (L) in any interval which does not include the origin. Finally, we shall say that $\chi \to A$ $(C, -1)$, or that the integral (3 . 12) is equal to A $(C, -1)$, if the integral is convergent and

$$\phi(t) = o\left(\frac{1}{t}\right).$$

Since $\Sigma B_n \sin nt$ is the Fourier series of the function $\psi(t)$ of § 2 . 2, our problem may be restated as follows: we have to prove that, *if* $\Sigma B_n \sin nt$ *is the Fourier series of an odd function* $\psi(t)$, *then the necessary and sufficient condition for the summability of* ΣB_n *is that the integral*

$$\frac{1}{\pi}\int_0^\pi \psi(t)\cot\tfrac{1}{2}t\,dt \tag{3.14}$$

should be summable (C), *and that the sum of the series is then equal to the value of the integral.*

We reduce the problem thus, and then write $f(t)$ for $\psi(t)$ and a_n for B_n, so that now

$$f(t) \sim \Sigma a_n \sin nt. \tag{3.15}$$

3 . 2. LEMMA γ.—*If*

$$\frac{1}{\pi}\int_0^\pi f(t)\cot\tfrac{1}{2}t\,dt \tag{3.21}$$

is summable $(C, -1)$, *i.e. if* (i) *it is convergent and* (ii) $f(t) = o(1)$, *then* Σa_n *is summable* $(C, 1)$; *and its sum is equal to the value of the integral.*

This lemma, which plays the part played by Fejér's theorem in the corresponding problem for the Fourier series, is included in the theorem of Young already referred to.*

* P. 212, f.n. †.

LEMMA δ.*—*If*

$$\frac{2}{\pi}\int_0^\infty \frac{f(t)}{t}\,dt$$

is convergent, as

$$\lim_{\epsilon\to 0,\ T\to\infty}\int_\epsilon^T,$$

then the integral (3 . 21) *is also convergent (as a Cauchy integral), and the two integrals are equal.*

We have

$$\int_0^\infty \frac{f(t)}{t}\,dt = \tfrac{1}{2}\int_{-\infty}^{\infty}\frac{f(t)}{t}\,dt = \tfrac{1}{2}\int_{-\pi}^{\pi}\frac{f(t)}{t}\,dt+\tfrac{1}{2}\sum_{-\infty}^{\infty}{}'\int_{(2k-1)\pi}^{(2k+1)\pi}\frac{f(t)}{t}\,dt$$

$$= \tfrac{1}{2}\int_{-\pi}^{\pi}\frac{f(t)}{t}\,dt+\tfrac{1}{2}\sum_{-\infty}^{\infty}{}'\int_{-\pi}^{\pi}\frac{f(t)}{t+2k\pi}\,dt,$$

$\sum_{-\infty}^{\infty}{}'$ denoting $\lim_{K\to\infty}\left(\sum_{-K}^{-1}+\sum_{1}^{K}\right).$

This series converges uniformly in $(-\pi, \pi)$ to the sum

$$\tfrac{1}{2}\cot\tfrac{1}{2}t-\frac{1}{t}.$$

We may therefore integrate term by term, when we obtain the result of the lemma.

LEMMA ϵ.—*If Σa_n is summable* $(C, -1)$, *then the integral* (3 . 21) *is convergent, and equal to the sum of the series.*

We have

$$\int_\epsilon^\infty \frac{f(t)}{t}\,dt = \Sigma a_n\int_\epsilon^\infty \frac{\sin nt}{t}\,dt = \Sigma a_n\int_{n\epsilon}^\infty\frac{\sin w}{w}\,dw, \tag{3.22}$$

the term-by-term integration being justified by the theorems proved in the papers of Young and Hardy referred to on p. 217.† Now

$$S(\epsilon) = \Sigma a_n\int_0^{n\epsilon}\frac{\sin w}{w}\,dw = \Sigma a_n\rho(n\epsilon)$$

exists for every positive ϵ, and tends to zero with ϵ.

* The result of this lemma is, of course, familiar.

† See footnote †.

For, in the first place,

$$\rho(n\epsilon) = \tfrac{1}{2}\pi - \int_{n\epsilon}^{\infty} \frac{\sin w}{w}\, dw = \tfrac{1}{2}\pi + O\left(\frac{1}{n}\right)$$

if ϵ is fixed, and

$$\tfrac{1}{2}\pi \Sigma a_n, \quad \Sigma a_n O\left(\frac{1}{n}\right) = \Sigma o\left(\frac{1}{n^2}\right)$$

are convergent. Hence $S(\epsilon)$ exists for every $\epsilon > 0$. Finally

$$S(\epsilon) = \sum_{n \leqslant 1/\epsilon} o\left(\frac{1}{n}\right) O(n\epsilon) + \tfrac{1}{2}\pi \sum_{n > 1/\epsilon} a_n + \sum_{n > 1/\epsilon} o\left(\frac{1}{n}\right) O\left(\frac{1}{n\epsilon}\right)$$

$$= \epsilon \sum_{n \leqslant 1/\epsilon} o(1) + o(1) + \frac{1}{\epsilon} \sum_{n > 1/\epsilon} o\left(\frac{1}{n^2}\right) = o(1).$$

Thus $S(\epsilon)$ exists and tends to zero. It follows that the left-hand side of (3 . 22) tends to a limit, and that

$$(3\,.\,23) \qquad \int_0^{\infty} \frac{f(t)}{t}\, dt = \Sigma a_n \int_0^{\infty} \frac{\sin w}{w}\, dw = \tfrac{1}{2}\pi \Sigma a_n.$$

The lemma follows from (3 . 23) and Lemma δ.

This lemma plays the part played in our discussion of the Fourier series by a theorem of Fatou.*

3 . 3. LEMMA ζ.—*Suppose that* $\phi(t)$ *and*

$$(3\,.\,31) \qquad \phi^{(1)} = \frac{\phi_1}{t} = \frac{1}{t}\int_0^t \phi\, du, \quad \phi^{(2)} = \frac{(\phi^{(1)})_1}{t}, \quad \phi^{(3)} = \frac{(\phi^{(2)})_1}{t}, \quad \ldots,$$

are integrable (C). *Then the existence of any one of the integrals*

$$\int_0 \frac{\phi}{t}\, dt \ (C, r), \quad \int_0 \frac{\phi^{(1)}}{t}\, dt \ (C, r-1),$$

$$\ldots, \quad \int_0 \frac{\phi^{(r)}}{t}\, dt \ (C, 0), \quad \int_0 \frac{\phi^{(r+1)}}{t}\, dt \ (C, -1)$$

implies that of the remainder.

In the case at present under consideration ϕ, $\phi^{(1)}$, ... are all integrable (L), and indeed functions of integrable square.† We state the

* See F.S., Lemma 12.
† See F.S., Lemma 10.

lemma more generally in view of § 4. It is plainly sufficient to prove that, if $r \geqslant 0$, the existence of either of the first two integrals implies that of the other.

Let

$$(3.32) \qquad j(\epsilon) = \int_\epsilon^a \frac{\phi}{t}\,dt, \qquad k(\epsilon) = \int_\epsilon^a \frac{\phi^{(1)}}{t}\,dt.$$

Then $$\epsilon j(\epsilon) = \epsilon \int_\epsilon^\delta \frac{\phi}{t}\,dt + \epsilon \int_\delta^a \frac{\phi}{t}\,dt = \int_\epsilon^\theta \phi\,dt + \epsilon \int_\delta^a \frac{\phi}{t}\,dt,$$

where $0 < \epsilon < \theta < \delta < a$. Choosing first δ and then ϵ appropriately, we see that

$$(3.33) \qquad \epsilon j(\epsilon) = o(1).$$

Next, if we suppose $0 < \eta < \epsilon$, and integrate by parts, we obtain

$$j_1(\epsilon, \eta) = \int_\eta^\epsilon j(u)\,du = \int_\eta^\epsilon du \int_u^a \frac{\phi}{w}\,dw = \epsilon \int_\epsilon^a \frac{\phi}{w}\,dw - \eta \int_\eta^a \frac{\phi}{w}\,dw + \int_\eta^\epsilon \phi\,dw.$$

Making $\eta \to 0$, and using (3.33), we obtain

$$(3.34) \qquad j_1(\epsilon) = \int_0^\epsilon j(u)\,du = \epsilon j(\epsilon) + \phi_1(\epsilon).$$

Finally

$$(3.35) \qquad k(\epsilon) = \int_\epsilon^a \frac{\phi_1}{t^2}\,dt = -\frac{\phi_1(a)}{a} + \frac{\phi_1(\epsilon)}{\epsilon} + \int_\epsilon^a \frac{\phi}{t}\,dt;$$

and the combination of (3.34) and (3.35) gives

$$(3.36) \qquad k(\epsilon) = \frac{j_1(\epsilon)}{\epsilon} - \frac{\phi_1(a)}{a}.$$

It follows from (3.36) that the assertions

$$k(\epsilon) \to A \quad (C, r-1), \qquad j(\epsilon) \to A + \frac{\phi_1(a)}{a} \quad (C, r)$$

are equivalent, so long as $r \geqslant 1$; and this proves the lemma except when $r = 0$.

If $j(\epsilon)$ tends to a limit, so does $j_1(\epsilon)/\epsilon$ and therefore, by (3.36), $k(\epsilon)$. Also $\phi_1(\epsilon) = o(\epsilon)$, by (3.34), and so

$$\text{(3.37)} \qquad \frac{\phi_1(t)}{t^2} = o\left(\frac{1}{t}\right).$$

That is to say, $k(\epsilon)$ tends to a limit $(C, -1)$. Finally, if $k(\epsilon)$ tends to a limit $(C, -1)$, then $j_1(\epsilon)/\epsilon$ tends to a limit by (3.36), and $j(\epsilon)$ tends to a limit by (3.34) and (3.37). This completes the proof.

It will be observed that, in Lemma ζ, there is no reference to the actual values of the various integrals considered. For this reason we shall find it necessary to modify the form of the lemma in a moment.

3.4. We proceed to the proof of Theorem 3, for a function of integrable square.

We introduce the functions

$$\text{(3.41)} \quad \psi_0 = f, \quad \psi_1 = \tfrac{1}{2}\cot\tfrac{1}{2}t\int_0^t \psi_0(u)\,du + \gamma_1 \sin^3 t,$$

$$\psi_2 = \tfrac{1}{2}\cot\tfrac{1}{2}t\int_0^t \psi_1(u)\,du + \gamma_2 \sin^3 t, \quad \ldots,$$

the γ's being constants whose values will be fixed in a moment. These functions correspond to the functions

$$f, \; f^{(1)}, \; f^{(2)}, \; \ldots,$$

defined as in Lemma ζ. If any ψ_s exists, so does the corresponding $f^{(s)}$, and conversely, and the difference between the two functions is $o(t)$.* We may therefore apply the results of Lemma ζ to either set of functions indifferently.

In the special case now considered, all the functions ψ_s are of integrable square. We have

$$\text{(3.42)} \quad \sum_n^N \frac{a_\nu}{\nu} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sum_n^N \frac{\sin \nu t}{\nu}\,dt = -\frac{1}{\pi}\int_{-\pi}^{\pi} f_1(t) \sum_n^N \cos \nu t\,dt,$$

where
$$f_1(t) = \int_0^t f(u)\,du,$$

* See F.S., 85. It is important to observe that this argument holds even when f is merely integrable (C).

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so that $f_1(-\pi) = f_1(\pi)$. Thus

(3 . 43)

$$\sum_n^N \frac{a_\nu}{\nu} = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(N+\frac{1}{2})t - \sin(n-\frac{1}{2})t}{\sin\frac{1}{2}t} f_1(t)\,dt$$

$$\to \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin nt \cot\tfrac{1}{2}t f_1(t)\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(t)\cos nt\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \psi_1 \sin nt\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(t)\cos nt\,dt - \frac{\gamma_1}{\pi}\int_{-\pi}^{\pi} \sin^3 t \sin nt\,dt$$

$$= a_n^1 + b'_{n,1} + b''_{n,1} = a_n^1 + b_{n,1},$$

say; so that
$$a_{n,1} = \sum_1^\infty \frac{a_\nu}{\nu} = a_n^1 + b_{n,1}.$$

Here a_n^1 is the Fourier sine coefficient of the odd function $\psi_1(t)$. As regards $b_{n,1}$, we observe first that $b'_{n,1}$ (defined for $n \geqslant 1$) is the Fourier cosine coefficient of the even function $-\frac{1}{2}f_1(t)$. This function is an integral, and vanishes for $t = 0$; and its Fourier series is

$$-\tfrac{1}{2}\sum_1^\infty \frac{a_n}{n}(1-\cos nt) = -\tfrac{1}{2}\sum_1^\infty \frac{a_n}{n} + \tfrac{1}{2}\sum_1^\infty \frac{a_n}{n}\cos nt,$$

so that its first coefficient $b'_{0,1}$ is $-\Sigma\, a_n/n$. It follows that

$$\tfrac{1}{2}b'_{0,1} + \sum_1^\infty b'_{n,1} = 0 \quad (C, -1),$$

(3 . 44)
$$\sum_1^\infty b'_{n,1} = \tfrac{1}{2}\sum_1^\infty \frac{a_n}{n} \quad (C, -1).$$

Next, $b''_{n,1}$ vanishes except when $n = 1$ and $n = 3$, while $b''_{1,1} = -\frac{3}{4}\gamma_1$ and $b''_{3,1} = \frac{1}{4}\gamma_1$. Hence $\Sigma\, b''_{n,1}$ is summable $(C, -1)$, to sum $-\frac{1}{2}\gamma_1$. If then we choose γ_1 so that

(3 . 45)
$$\gamma_1 = \sum_1^\infty \frac{a_n}{n},$$

the series $\Sigma\, b_{n,1}$ will be summable $(C, -1)$ to sum zero. In other words, when γ_1 is defined by (3 . 45), $a_{n,1}$ differs from a_n^1, the Fourier coefficient of ψ_1, by the general term of a series summable $(C, -1)$ to sum 0.

We may now (as in F.S., 87–88) repeat the argument, choosing $\gamma_2, \gamma_3, \dots$ appropriately at successive stages,* and we conclude that, if $a_{n,s}$ is defined as there, then *$a_{n,s}$ differs from a_s^n, the Fourier coefficient of ψ_s, by the general term of a series summable $(C, -1)$ to sum 0.*

3.5. In order to complete the proof, we require one further lemma (the modified form of Lemma ζ referred to at the end of § 3.3). We state it, like Lemma ζ, in a more general form than that in which it is required immediately.

LEMMA η.—*Suppose that $\psi_0, \psi_1, \psi_2, \dots$ are defined by* (3.41) *(the γ's being chosen as in* § 3.4*), and that $\psi_0 = f$ is integrable* (C). *Then the existence of any one of the integrals*

$$\frac{1}{\pi}\int_0^\pi \psi_s(t)\cot\tfrac{1}{2}t\,dt \quad (C,\, r-s;\;\; 0 \leqslant s \leqslant r+1)$$

implies that of the remainder. Further, the values of all the integrals are equal.

As regards the *existence* of the integrals, there is nothing new to prove, as Lemma ζ, and the remark at the beginning of § 3.4, contain all that we require. It is therefore only necessary to verify that

$$\frac{1}{\pi}\int_0^\pi \psi_0(t)\cot\tfrac{1}{2}t\,dt = \frac{1}{\pi}\int_0^\pi \psi_1(t)\cot\tfrac{1}{2}t\,dt.\dagger \tag{3.51}$$

If

$$j(\epsilon) = \int_\epsilon^\pi \tfrac{1}{2}\cot\tfrac{1}{2}t\,f(t)\,dt, \quad k(\epsilon) = \int_\epsilon^\pi \tfrac{1}{2}\cot\tfrac{1}{2}t\,\psi_1(t)\,dt, \tag{3.52}$$

we have

$$j_1(\epsilon) = \int_0^\epsilon j(t)\,dt = \int_0^\epsilon \sec^2\tfrac{1}{2}t\,j(t)\,dt - \int_0^\epsilon \tan^2\tfrac{1}{2}t\,j(t)\,dt. \tag{3.53}$$

Now $tj(t) = o(1)$, as in § 3.3. The second term on the right-hand side of (3.53) is therefore $o(\epsilon^2)$. The first term is

$$2\int_0^\epsilon \frac{d}{dt}(\tan\tfrac{1}{2}t)\,dt\int_t^\pi f(u)\,\tfrac{1}{2}\cot\tfrac{1}{2}u\,du = 2\tan\tfrac{1}{2}\epsilon\int_\epsilon^\pi f(t)\,\tfrac{1}{2}\cot\tfrac{1}{2}t\,dt + \int_0^\epsilon f(t)\,dt\,;$$

---

* Thus $$\gamma_2 = \Sigma\frac{a_n^1}{n}.$$

† The first integral being summable (C, r), the second $(C, r-1)$.

and so

(3.54) $$j_1(\epsilon) = 2\tan\tfrac{1}{2}\epsilon\int_\epsilon^\pi f(t)\tfrac{1}{2}\cot\tfrac{1}{2}t\,dt+\int_0^\epsilon f(t)\,dt+o(\epsilon^2).$$

On the other hand

$$k(\epsilon) = \int_\epsilon^\pi \tfrac{1}{4}\cot^2\tfrac{1}{2}t\,f_1(t)\,dt+\tfrac{1}{2}\gamma_1\int_\epsilon^\pi \cot\tfrac{1}{2}t\sin^3 t\,dt$$

$$= \int_\epsilon^\pi \tfrac{1}{4}\operatorname{cosec}^2\tfrac{1}{2}t\,f_1(t)\,dt-\tfrac{1}{4}\int_\epsilon^\pi f_1(t)\,dt+\tfrac{1}{2}\gamma_1\int_\epsilon^\pi \cot\tfrac{1}{2}t\sin^3 t\,dt.$$

The second term here is

$$-\tfrac{1}{4}\int_0^\pi f_1(t)\,dt+o(\epsilon) = -\tfrac{1}{4}\pi\Sigma\frac{a_n}{n}+o(\epsilon),$$

and the third is

$$\tfrac{1}{2}\gamma_1\int_0^\pi \cot\tfrac{1}{2}t\sin^3 t\,dt+o(\epsilon) = \tfrac{1}{4}\pi\gamma_1+o(\epsilon).$$

These two terms, in virtue of (3.45), yield $o(\epsilon)$. Hence

(3.55) $$k(\epsilon) = \int_\epsilon^\pi \tfrac{1}{4}\operatorname{cosec}^2\tfrac{1}{2}t\,f_1(t)\,dt+o(\epsilon)$$

$$= \tfrac{1}{2}\cot\tfrac{1}{2}\epsilon\int_0^\epsilon f(t)\,dt+\int_\epsilon^\pi \tfrac{1}{2}\cot\tfrac{1}{2}t\,f(t)\,dt+o(\epsilon).$$

Combining (3.54) and (3.55), we obtain

(3.56) $$k(\epsilon)-\tfrac{1}{2}\cot\tfrac{1}{2}\epsilon\; j_1(\epsilon) = o(1)$$

and therefore

(3.57) $$k(\epsilon)-\frac{j_1(\epsilon)}{\epsilon} = o(1).$$

This equation takes the place of (3.36); and the remainder of the proof of Lemma ζ now applies to the present lemma. In the present case, however, we can assert what we could not assert before, viz. the equality of the values of all the integrals considered.

3.6. We can now complete the proof of Theorem 3, arguing substantially as in F.S.,* and using the notation and the substance of the fundamental arithmetic theorem (Theorem A 1) proved there, in combination now, however, with Lemmas γ, ϵ, and η. If A is summable (C, r), then A_{r+1} is summable $(C, -1)$, and so A^{r+1} is summable $(C, -1)$.† Hence, by Lemma ϵ,

$$\frac{1}{\pi}\int_0^\pi \psi_{r+1}(t) \cot \tfrac{1}{2}t\, dt \tag{3.61}$$

is summable $(C, 0)$; and therefore, by Lemma η,

$$\frac{1}{\pi}\int_0^\pi \psi_0(t) \cot \tfrac{1}{2}t\, dt \tag{3.62}$$

is summable $(C, r+1)$. Further, the values of all the series and integrals are the same. On the other hand, if (3.62) is summable (C, r), then (3.61) is summable $(C, -1)$, by Lemma η. Hence A^{r+1} is summable $(C, 1)$, by Lemma γ, and therefore A_{r+1} is summable $(C, 1)$; and so A is summable $(C, r+2)$. This completes the proof of Theorem 3, for functions of integrable square.

4. *Proof of Theorem 3 in the general case.*

4.1. In the general case, as in the corresponding investigation of the Fourier series, more delicate arguments are required. We begin, as there, by establishing the *sufficiency* of our criterion.

LEMMA θ.—*Suppose that ϕ is integrable (L), that*

$$j(t) = \int_t^a \frac{\phi}{u}\, du \quad (t > 0),$$

that $$j^{(1)}(t) = \frac{1}{t}\int_0^t j(u)\, du, \quad j^{(2)}(t) = \frac{1}{t}\int_0^t j^{(1)}(u)\, du,$$

$$\ldots, \quad j^{(r)}(t) = \frac{1}{t}\int_0^t j^{(r-1)}(u)\, du$$

---

* F.S., 88.

† See the end of § 3.4.

exist as Cauchy integrals, and that

$$j^{(r)}(t) \to A, \tag{4.11}$$

when $t \to 0$. *Then the functions* $\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(r)}$ *of Lemma* ζ *exist, and are integrable* (L), *and*

$$\phi^{(r+1)}(t) \to 0, \tag{4.12}$$

when $t \to 0$.

We have

$$\frac{d}{dt}(tj) = j + tj' = j - \phi \quad (u > 0). \tag{4.13}$$

Now ϕ is by hypothesis integrable (L) down to 0; and so also is j, since (i) j is by hypothesis integrable (C), and (ii)

$$\int_0^a |j|\,dt \leqslant \int_0^a dt \int_t^a \frac{|\phi|}{u}\,du = \int_0^a \frac{|\phi|}{u}\,du \int_0^u dt = \int_0^a |\phi|\,du.$$

Finally $tj \to 0$, by (3.23).* Hence, integrating (4.13), we find

$$tj(t) = \int_0^t j\,du - \int_0^t \phi\,du = j_1(t) - \phi_1(t),$$

or

$$\phi^{(1)} = j^{(1)} - j. \tag{4.14}$$

Again, j and $j^{(1)}$ are, by hypothesis, integrable (C). Hence $\phi^{(1)}$ is integrable (C). Integrating, and dividing by t, we obtain

$$\phi^{(2)} = j^{(2)} - j^{(1)}.$$

Plainly we may repeat the argument. All of the functions

$$\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(r)}$$

are integrable (C), and

$$\phi^{(s)} = j^{(s)} - j^{(s-1)} \quad (1 \leqslant s \leqslant r). \tag{4.15}$$

---

* This is so, in fact, whenever ϕ is integrable (C).

Finally

$$\phi^{(r+1)} = \frac{1}{t}\int_0^t j^{(r)}(u)\,du - j^{(r)}(t) \to 0 \tag{4.16}$$

when $t \to 0$, which is (4.12).

We can now appeal to Lemma 13 of F.S., which shows that $\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(r)}$ are integrable (L); and the lemma is completely established.

4.2. There is now no difficulty in proving our criterion sufficient. If

$$\int_0 \frac{f(t)}{t}\,dt$$

is summable (C), the conditions of Lemma θ are satisfied, when $\phi = f$, for some value of r. The functions $f^{(1)}, f^{(2)}, \ldots$ are therefore integrable (L), and so also are the functions $\psi_1, \psi_2, \ldots$ of § 3.4. The argument of §§ 3.4 and 3.5, in so far as the sufficiency of the criterion is concerned, is therefore valid as it stands.

4.3. We proceed to the proof of the *necessity* of the condition, for which additional lemmas are required.

We shall, as in the corresponding argument of F.S., use the terms "Fourier series", "Fourier constant", in a slightly extended sense. If (i) $f(t)$ is odd, and possesses a Cauchy integral over $(0, \pi)$, so that

$$a_n = \int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

exists also as a Cauchy integral*

(ii) $$a_n = o(1),$$

we shall say that a_n is the *Fourier coefficient*, and $\Sigma a_n \sin nt$ the *Fourier series* of $f(t)$, and write

$$f(t) \sim \Sigma a_n \sin nt.$$

* As $$\lim\left(\int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi}\right) = 2\lim\int_{\epsilon}^{\pi}.$$

These conditions are, of course, satisfied in particular if $f(t)$ is integrable (L).

LEMMA ι.—*If* (i) $\Sigma a_n \sin nt$ *is the Fourier series of* $f(t)$ (*in the sense just explained*),

(ii) $$f_1(t) = \int_0^t f(u)\,du, \quad f_2(t) = \int_0^t f_1(u)\,du, \;\ldots,$$

(iii) Σa_n *is summable* (C, r),

then

$$f_{r+2}(t) = o(t^{r+2}). \tag{4.31}$$

As a change in a_1 alters $f(t)$ by a multiple of $\sin t$, we may suppose without loss of generality that $\Sigma a_n = 0$. We have

$$f_1(t) = \Sigma \frac{a_n}{n}(1-\cos nt),$$

this series being, as the Fourier series of a continuous function, uniformly summable $(C, 1)$, and so, since $a_n = o(1)$, uniformly convergent. By further integrations we obtain

$$f_2(t) = \Sigma \frac{a_n}{n^2}(nt - \sin nt), \quad f_3(t) = \Sigma \frac{a_n}{n^3}(\cos nt - 1 + \tfrac{1}{2}n^2t^2),$$

and

$$f_{r+2}(t) = \Sigma \frac{a_n}{n^{r+2}} s_{r+2}(nt), \tag{4.32}$$

where $$s_\nu(t) = \left(\int_0^t du\right)^\nu \sin u.$$

It may be verified at once that

$$s_{r+2}(u) \sim Au^{r+3}$$

for small u, and $$s_{r+2}(u) \sim Au^{r+1}$$

for large u; and that

$$\Delta^k s_{r+2}(nt) = O(t^{r+3} n^{r+3-k}) \quad (nt \leqslant 1,\ 0 \leqslant k \leqslant r+3), \tag{4.33}$$

$$\Delta^k s_{r+2}(nt) = O(t^{r+1} n^{r+1-k}) \quad (nt \geqslant 1,\ 0 \leqslant k \leqslant r+1). \tag{4.34}$$

Now

$$f_{r+2}(t) = \Sigma \frac{a_n}{n^{r+2}} s_{r+2}(nt) = \Sigma A_n \Delta\left(\frac{s_{r+2}(nt)}{n^{r+2}}\right) \tag{4.35}$$

$$= \Sigma A_n^1 \Delta^2 \left(\frac{s_{r+2}(nt)}{n^{r+2}}\right) = \dots = \Sigma A_n^r \Delta^{r+1}\left(\frac{s_{r+2}(nt)}{n^{r+2}}\right),$$

provided only

$$A_n^k \Delta^k \left(\frac{s_{r+2}(nt)}{n^{r+2}}\right) \to 0 \quad (0 \leqslant k \leqslant r) \tag{4.36}$$

when t is fixed and $n \to \infty$. Now $A_n^k = o(n^{k+1})$, since $a_n = o(1)$. Hence $A_n^k \Delta^k$ is, by (4 . 34), a sum of terms of the type

$$o(n^{k+1})\, O(n^{-r-2-\lambda})\, O(n^{r+1-k+\lambda}) = o(1),$$

and the condition (4 . 36) is satisfied.

We have therefore

$$f_{r+2}(t) = \Sigma A_n^r \Delta^{r+1}\left(\frac{s_{r+2}(nt)}{n^{r+2}}\right) = \sum_{k=0}^{r+1} c_{r,k} \sum_n A_n^r \Delta^k \big(s_{r+2}(nt)\big)\, \Delta^{r+1-k}\big((n+k)^{-r-2}\big),$$

the c's being functions of r and k only. The typical series here is, by (4 . 33) and (4 . 34),

$$\sum_{nt \leqslant 1} o(n^r)\,.\,O(t^{r+3} n^{r+3-k})\, O(n^{-2r-3+k}) + \sum_{nt>1} o(n^r)\,.\,O(t^{r+1} n^{r+1-k})\,.\,O(n^{-2r-3+k})$$

$$= o\left(t^{r+3} \sum_{nt \leqslant 1} 1\right) + o\left(t^{r+1} \sum_{nt>1} \frac{1}{n^2}\right) = o(t^{r+2}),$$

which proves the lemma.

4 . 4.—LEMMA κ.—*If*

$$f(t) \sim \Sigma a_n \sin nt,$$

in the sense of § 4 . 3, *and* Σa_n *is summable* (C), *then*

$$\int_0 f^{(1)}(t)\, dt = \int_0 \frac{f_1(t)}{t}\, dt$$

is convergent.

For

$$\int_\epsilon^a \frac{f_1(t)}{t}\,dt = \frac{f_2(a)}{a} - \frac{f_2(\epsilon)}{\epsilon} + \int_\epsilon^a \frac{f_2(t)}{t^2}\,dt \tag{4.41}$$

$$= \frac{f_2(a)}{a} + \frac{f_3(a)}{a^2} - \frac{f_2(\epsilon)}{\epsilon} - \frac{f_3(\epsilon)}{\epsilon^2} + 2\int_\epsilon^a \frac{f_3(t)}{t^3}\,dt$$

$$= \ldots = F(a) - F(\epsilon) + (r+1)!\int_\epsilon^a \frac{f_{r+2}(t)}{t^{r+2}}\,dt,$$

where

$$F(\epsilon) = \sum_{k=1}^{r+1} (k-1)!\,\frac{f_{k+1}(\epsilon)}{\epsilon^k}.$$

Since $f_1(t)$ is continuous, every term of $F(\epsilon)$ tends to zero; and the last integral in (4 . 41) tends to a limit in virtue of Lemma ι.

4 . 5. We return to the argument of §§ 3 . 4 and 3 . 5. We assume (i) that $\Sigma a_n \sin nt$ is the Fourier series of $f(t)$ in the sense of § 4 . 3, and (ii) that Σa_n is summable (C), and we consider where the argument requires reconsideration.

We observe first that, by Lemma θ, $f^{(1)}(t)$, and therefore $\psi_1(t)$, is integrable (C). Hence (3 . 43) will be established if we can prove that

$$J(N) = \int_{-\pi}^{\pi} \frac{\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t}\, f_1(t)\,dt \to 0 \tag{4.51}$$

when $N \to \infty$. As $\psi_1(t)$ is no longer given as integrable (L), this is no longer a consequence of the theorem of Riemann-Lebesgue.

The integral $J(N)$ is substantially the partial sum of the Fourier series of $f_1(t)$ for $t=0$. As $f_1(t)$ is continuous, and equal to 0, for $t=0$, it follows from Fejér's theorem that

$$J(N) \to 0 \quad (C,\ 1).$$

But the series in question has coefficients $o(1/n)$, since $a_n = o(1)$, and is therefore, if summable, convergent. Hence (4 . 51) holds in the ordinary sense.

Next, our conclusions concerning $\Sigma b_{n,1}$ retain their validity, since $f_1(t)$, though no longer a Lebesgue integral, is continuous, and has

Fourier coefficients of the order required.* We conclude then, as in §3.4, that $a_{n,1}$ *differs from the Fourier coefficient of* $\psi_1(t)$ *by the general term of a series summable* $(C, -1)$ *to sum* 0.

It remains to show that the argument may be repeated. For this we must show that $\psi_1(t)$ possesses the relevant properties of $\psi_0(t)$ or $f(t)$. And this is in fact clear from what precedes. For $\psi_1(t)$ is, as we have seen, integrable (C), so that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \psi_1(t) \sin nt \, dt$$

is convergent; and being, as we have seen, substantially equal to $a_{n,1}$, it tends to zero. Thus $\psi_1(t)$ has a Fourier series $\Sigma a_n^1 \sin nt$ in the sense of § 4.3. Since Σa_n is summable (C), $\Sigma a_{n,1}$ is summable, and therefore Σa_n^1 is summable. These are the properties assumed for $f(t)$, so that our argument is one capable of repetition. This once established, the proof of the necessity of our condition in § 3.5 stands, and Theorem 3 is proved in full generality.

5. *The series* $\Sigma \frac{s_n - s}{n}$ *and the integral* $\int \frac{\phi(t)}{t}\,dt$.

5.1. In this section we consider the series

$$S = \Sigma \frac{s_n - s}{n}, \tag{5.11}$$

where

$$s_n = \sum_0^n A_\nu. \tag{5.12}$$

The interest of this series lies in its relations to the integral

$$\int_0 \frac{\phi(t)}{t}\,dt, \tag{5.13}$$

* We take this opportunity of making a small correction of F.S., 94-95. We say there (94, 10 lines from below) that "the proof that $b_{n,1} = o(1/n)$ is unchanged." This proof should in fact be replaced by that stated in the text, as the original proof (though actually valid for $b_{n,1}$) would not, since it depends on the integrability (L) of $\phi(t)$, be suitable for repetition at subsequent stages of the argument.

where

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\}, \tag{5.14}$$

these relations being very similar to those between the allied series and the integral

$$\int_0 \frac{\psi(t)}{t}\,dt = \tfrac{1}{2}\int_0 \frac{f(x+t)-f(x-t)}{t}\,dt. \tag{5.15}$$

We do not propose to work out these relations in detail, but confine ourselves to a few selected theorems sufficient to illustrate the likenesses and differences between the two problems.

5.2. The most striking difference between the two problems is embodied in

THEOREM 4.—*There are continuous functions $f(t)$ for which the integral* (5.13) *diverges, whatever be s, for almost all values of x.*

It is plain that, if $f(t)$ is continuous, (5.13) can converge only if $s = f(x)$; and that, since (5.15) is almost always convergent, the integrals (5.13) and (1.43) converge or diverge together for almost all x.

We take

$$f(t) = \Sigma\, b^{-\nu} \cos a^{a^{\nu}} t, \tag{5.21}$$

where a and b are integers and $a > b > 1$; so that

$$a_n = b^{-\nu} \quad (n = a^{a^{\nu}} = \lambda_\nu), \qquad a_n = 0 \quad (n \neq \lambda_\nu). \tag{5.22}$$

Then

$$\phi(t) = -2\Sigma\, b^{-\nu} \cos \lambda_\nu x \sin^2 \tfrac{1}{2}\lambda_\nu t, \tag{5.23}$$

and, if $c > 0$,

$$\chi(\epsilon) = \int_\epsilon^c \frac{\phi(t)}{t}\,dt = -2\Sigma\, b^{-\nu} \cos \lambda_\nu x \int_\epsilon^c \frac{\sin^2 \frac{1}{2}\lambda_\nu t}{t}\,dt \tag{5.24}$$

$$= -2\Sigma\, b^{-\nu} \cos \lambda_\nu x\, \omega_\nu(\epsilon),$$

where

$$\omega_\nu(\epsilon) = \int_\epsilon^c \frac{\sin^2 \frac{1}{2}\lambda_\nu t}{t}\,dt = \int_{\frac{1}{2}\lambda_\nu \epsilon}^{\frac{1}{2}\lambda_\nu c} \frac{\sin^2 u}{u}\,du. \tag{5.25}$$

If $\frac{1}{2}\lambda_\nu\epsilon \leqslant 1 < \frac{1}{2}\lambda_\nu c$, we have

$$\omega_\nu(\epsilon) = \int_{\frac{1}{2}\lambda_\nu\epsilon}^{1} \frac{\sin^2 u}{u}\,du + \tfrac{1}{2}\int_1^{\frac{1}{2}\lambda_\nu c} \frac{1-\cos 2u}{u}\,du \tag{5.26}$$

$$= \tfrac{1}{2}\log\lambda_\nu + O(1) = \tfrac{1}{2}a^\nu \log a + O(1),$$

while if $\frac{1}{2}\lambda_\nu\epsilon > 1$ we have

$$\omega_\nu(\epsilon) = \tfrac{1}{2}\int_{\frac{1}{2}\lambda_\nu\epsilon}^{\frac{1}{2}\lambda_\nu c} \frac{du}{u} + O(1) = \tfrac{1}{2}\log\frac{c}{\epsilon} + O(1) = \tfrac{1}{2}\log\frac{1}{\epsilon} + O(1). \tag{5.27}$$

It follows that

$$\chi(\epsilon) = -\log a \sum_{\frac{1}{2}\lambda_\nu \leqslant 1/\epsilon} \left(\frac{a}{b}\right)^\nu \cos\lambda_\nu x - \log\frac{1}{\epsilon} \sum_{\frac{1}{2}\lambda_\nu > 1/\epsilon} b^{-\nu}\cos\lambda_\nu x + O(1) \tag{5.28}$$

$$= S_1 + S_2 + O(1),$$

say.

5.3. We take

$$\frac{1}{\epsilon} = \tfrac{1}{2}a^{\mu+\frac{1}{2}} = \tfrac{1}{2}\lambda_{\mu+\frac{1}{2}}. \tag{5.31}$$

Then S_1 is numerically less than

$$\left(\frac{a}{b}\right)^\mu \log a\left(1+\frac{b}{a}+\frac{b^2}{a^2}+\ldots\right) = \left(\frac{a}{b}\right)^\mu \log a\,\frac{a}{a-b};$$

and the sum of the terms of S_2, other than the first (for which $\nu = \mu+1$), is numerically less than

$$(a^{\mu+\frac{1}{2}}\log a - \log 2)\, b^{-\mu-1}\left(\frac{1}{b}+\frac{1}{b^2}+\ldots\right) = (a^{\mu+\frac{1}{2}}\log a - \log 2)\,\frac{b^{-\mu-1}}{b-1}.$$

We have therefore

(5.32)

$$\chi(\epsilon) = -\left(\frac{a}{b}\right)^\mu \log a\left\{\frac{\sqrt{a}}{b}\cos\lambda_{\mu+1}x + \frac{\sqrt{a}}{b}\left[\frac{1}{b-1}\right] + \left[\frac{a}{a-b}\right]\right\} + O(1),$$

where $[\alpha]$ denotes generally a number whose modulus is less than α.

Suppose now that b and $\sqrt{a/b}$ are large. Then it is plain from (5 . 32) that, if μ and x have values such that

$$|\cos \lambda_{\mu+1} x| > \tfrac{1}{2},$$

$\chi(\epsilon)$ will be large numerically, and will have the sign opposite to that of $\cos \lambda_{\mu+1} x$. Now the set of limiting values of this function, when $\mu \to \infty$, is, for all values of x which do not lie in a certain set of measure zero, everywhere dense in the interval $(-1, 1)$.* It follows that we can, for almost all values of x, so choose ϵ that $\chi(\epsilon)$ is large, and of either sign: and plainly this establishes Theorem 4.

It may be proved directly that in this case the series (5 . 11) is divergent for almost all x.

We add that Mr. E. C. Titchmarsh has gone further in this direction, and has proved, by direct construction and without using the theory of trigonometrical series, that, *if* $\lambda(t)$ *decreases steadily with* t *and the integral*

$$\int_0 \frac{dt}{\lambda(t)}$$

is divergent, then continuous functions $f(t)$ *exist for which*

$$\int_0 \frac{f(x+t)-f(x)}{\lambda(t)}\, dt$$

is almost always divergent. He has also given examples which show that

$$\int_0 \frac{|f(x+t)-f(x-t)|}{\lambda(t)}\, dt$$

may diverge almost everywhere; and that so also may

$$\int_0 \frac{f(x+t)-f(x-t)}{t\mu(t)}\, dt$$

if $\mu(t)$ is any function which tends steadily to zero with t, so that the results of Besikovitch and Plessner are the best possible of their kind.

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* See G. H. Hardy and J. E. Littlewood, "Some problems of diophantine approximation", *Acta Math.*, 37 (1914), 155-190 (181).

5.4. THEOREM 5.—*If the integral* (5.13) *is convergent, then the series* (5.11) *is summable* $(C, 1)$, *to sum*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\pi-t}{2}\cot\tfrac{1}{2}t-\log|2\sin\tfrac{1}{2}t|\right)\phi(t)\,dt. \tag{5.41}$$

The Fourier series of $f(t)$, for $t=x$, is identical with that of $\phi(t)+s$ for $t=0$. It is therefore sufficient to suppose that

$$f(t) \sim \tfrac{1}{2}a_0+\Sigma a_n\cos nt,$$

and that f/t is integrable (C), and to prove that

$$S=\sum_1^\infty \frac{s_n}{n}=\sum_1^\infty \frac{\tfrac{1}{2}a_0+a_1+\ldots+a_n}{n} \tag{5.42}$$

$$=\frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi-t}{2}\cot\tfrac{1}{2}t-\log|2\sin\tfrac{1}{2}t|\right)f(t)\,dt \quad (C,\ 1).$$

We have

$$s_n=\frac{1}{\pi}\int_0^{\pi}\frac{\sin(n+\tfrac{1}{2})t}{\sin\tfrac{1}{2}t}f(t)\,dt$$

$$=\frac{1}{\pi}\int_0^{\pi}f(t)\cot\tfrac{1}{2}t\sin nt\,dt+\frac{1}{\pi}\int_0^{\pi}f(t)\cos nt\,dt,$$

$$\sum_1^n\frac{s_\nu}{\nu}=\frac{1}{\pi}\int_0^{\pi}f(t)\cot\tfrac{1}{2}t\sum_1^n\frac{\sin\nu t}{\nu}\,dt+\frac{1}{\pi}\int_0^{\pi}f(t)\sum_1^n\frac{\cos\nu t}{\nu}\,dt \tag{5.43}$$

$$=S_1+S_2,$$

say. We consider S_2 first. Integrating by parts we have

$$S_2=\frac{1}{\pi}\left\{f_1(\pi)\sum_1^n\frac{(-1)^\nu}{\nu}+\int_0^{\pi}f_1(t)\sum_1^n\sin\nu t\,dt\right\}=S_2'+S_2'', \tag{5.44}$$

say.

In the first place

$$S_2'\to-\frac{1}{\pi}f_1(\pi)\log 2. \tag{5.45}$$

Next, we observe that

$$f_1(t) = \int_0^t \frac{f(u)}{u} \,.\, u \,.\, du = t \int_{t'}^t \frac{f(u)}{u}\, du \quad (0 < t' < t),$$

so that $f_1(t) = o(t)$. Hence

$$(5\,.\,46) \quad S_2'' = \frac{1}{2\pi} \int_0^\pi f_1(t) \frac{\cos \frac{1}{2}t - \cos (n+\frac{1}{2})\, t}{\sin \frac{1}{2}t}\, dt$$

$$= \frac{1}{2\pi} \int_0^\pi f_1(t) \cot \tfrac{1}{2}t\, dt - \frac{1}{2\pi} \int_0^\pi f_1(t) \cot \tfrac{1}{2}t \cos nt\, dt + \frac{1}{2\pi} \int_0^\pi f_1(t) \sin nt\, dt$$

$$\to \frac{1}{2\pi} \int_0^\pi f_1(t) \cot \tfrac{1}{2}t\, dt.$$

Finally, writing

$$(5\,.\,47) \qquad \chi(t) = \int_0^t f(u) \cot \tfrac{1}{2}u\, du,$$

and again integrating by parts, we obtain

$$S_1 = -\frac{1}{\pi} \int_0^\pi \chi(t) \sum_1^n \cos \nu t\, dt$$

$$= -\frac{1}{2\pi} \int_0^\pi \chi(t) \frac{\sin (n+\frac{1}{2})\, t - \sin \frac{1}{2}t}{\sin \frac{1}{2}t}\, dt$$

$$= \frac{1}{2\pi} \int_0^\pi \chi(t)\, dt - \frac{1}{2\pi} \int_0^\pi \chi(t) \cot \tfrac{1}{2}t \sin nt\, dt - \frac{1}{2\pi} \int_0^\pi \chi(t) \cos nt\, dt.$$

The last term tends to zero, since $\chi(t)$ is continuous; and the second term tends to zero $(C, 1)$, by Fejér's theorem, since $\chi(t)$ is continuous and vanishes for $t = 0$. Hence

$$(5\,.\,48) \qquad S_1 \to \frac{1}{2\pi} \int_0^\pi \chi(t)\, dt \quad (C,\ 1).$$

From (5 . 43)–(5 . 48) we obtain

$$(5\,.\,49) \quad S = -\frac{1}{\pi} f_1(\pi) \log 2 + \frac{1}{2\pi} \int_0^\pi f_1(t) \cot \tfrac{1}{2}t\, dt + \frac{1}{2\pi} \int_0^\pi \chi(t)\, dt \quad (C,\ 1).$$

If now we make the transformations

$$-\frac{1}{\pi}f_1(\pi)\log 2 = -\frac{1}{\pi}\log 2\int_0^\pi f(t)\,dt,$$

$$\frac{1}{2\pi}\int_0^\pi f_1(t)\cot\tfrac{1}{2}t\,dt = \frac{1}{\pi}\int_0^\pi f_1(t)\frac{d}{dt}\log|\sin\tfrac{1}{2}t|\,dt$$

$$= -\frac{1}{\pi}\int_0^\pi \log|\sin\tfrac{1}{2}t|\,f(t)\,dt,$$

$$\frac{1}{2\pi}\int_0^\pi \chi(t)\,dt = \frac{1}{2\pi}\int_0^\pi dt\int_0^t f(u)\cot\tfrac{1}{2}u\,du$$

$$= \tfrac{1}{2}\int_0^\pi f(u)\cot\tfrac{1}{2}u\,du - \frac{1}{2\pi}\int_0^\pi u\,f(u)\cot\tfrac{1}{2}u\,du,$$

we are led to (5 . 42). These partial integrations present no difficulty.

5 . 5. THEOREM 6.—*If the series* (5 . 11) *is summable* $(C, -1)$, *then the integral* (5 . 13) *is summable* $(C, 1)$.

To say that (5 . 11) is summable $(C, -1)$ is plainly equivalent to asserting its convergence and also that of the Fourier series of $f(t)$.
If $\epsilon > 0$, we have

$$J(\epsilon) = \int_\epsilon^\pi \phi(t)\cot\tfrac{1}{2}t\,dt = \sum_0^\infty A_n\int_\epsilon^\pi \cot\tfrac{1}{2}t\cos nt\,dt - s\int_\epsilon^\pi \cot\tfrac{1}{2}t\,dt$$

$$= \sum_0^\infty s_n\int_\epsilon^\pi \cot\tfrac{1}{2}t\,\{\cos nt - \cos(n+1)t\}\,dt + 2s\log\sin\tfrac{1}{2}\epsilon,$$

since

$$s_n\int_\epsilon^\pi \cot\tfrac{1}{2}t\cos nt\,dt \to 0$$

when ϵ is fixed and $n\to\infty$. Hence

$$J(\epsilon) = \sum_0^\infty s_n\int_\epsilon^\pi \{\sin(n+1)t + \sin nt\}\,dt + 2s\log\sin\tfrac{1}{2}\epsilon \tag{5.51}$$

$$= s_0\cos\epsilon + \sum_1^\infty s_n\left\{\frac{\cos n\epsilon}{n} + \frac{\cos(n+1)\epsilon}{n+1}\right\}$$

$$+ s_0 - \sum_1^\infty s_n\frac{(-1)^n}{n(n+1)} + 2s\log\sin\tfrac{1}{2}\epsilon.$$

If in (5 . 51) we replace s_n and s_0 by s, it is easily verified that the coefficient of s reduces to zero, so that

$$J(\epsilon) = J_1(\epsilon)+J_2(\epsilon), \tag{5.52}$$

where

$$J_1(\epsilon) = (s_0-s)\cos\epsilon+\sum_1^\infty (s_n-s)\left\{\frac{\cos n\epsilon}{n}+\frac{\cos(n+1)\epsilon}{n+1}\right\}, \tag{5.53}$$

and

$$J_2(\epsilon) = (s_0-s)-\sum_1^\infty (s_n-s)\,\frac{(-1)^n}{n(n+1)} \tag{5.54}$$

is independent of ϵ.

Now

$$\frac{1}{\epsilon}\int_0^\epsilon J_1(\eta)\,d\eta = (s_0-s)\,\frac{\sin\epsilon}{\epsilon}+\sum_1^\infty (s_n-s)\left\{\frac{\sin n\epsilon}{n^2\epsilon}+\frac{\sin(n+1)\epsilon}{(n+1)^2\epsilon}\right\}$$

$$= \sum_1^\infty (s_n-s)\,\frac{\sin n\epsilon}{n^2\epsilon}+\sum_0^\infty (s_n-s)\,\frac{\sin(n+1)\epsilon}{(n+1)^2\epsilon}.$$

Each of these series is of the form

$$\sum_1^\infty t_n\,\frac{\sin n\epsilon}{n\epsilon},$$

where Σt_n is summable $(C, -1)$; and so each tends to the limit Σt_n, by a theorem of Fatou.* Hence

$$J_1(\epsilon)\to\sum_1^\infty\frac{s_n-s}{n}+\sum_0^\infty\frac{s_n-s}{n+1} = s_0-s+2\sum_1^\infty\frac{s_n-s}{n}-\sum_1^\infty\frac{s_n-s}{n(n+1)} \quad (C,\ 1);$$

$$J(\epsilon)\to 2(s_0-s)+2\sum_1^\infty\frac{s_n-s}{n}-\sum_1^\infty (s_n-s)\,\frac{1+(-1)^n}{n(n+1)} \quad (C,\ 1); \tag{5.55}$$

which proves the theorem.

We note an obvious corollary which will be useful later.

LEMMA λ.—*If $f(t)$ is of integrable square, and $s = f(x)$, and the series (5 . 11) is almost always summable $(C, -1)$, and to a function of integrable square, then the integral (5 . 13) is almost always summable $(C, 1)$, and represents a function of integrable square.*

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* F.S., 88 (Lemma 12).

This is obvious from (5 . 55), since the three terms on the right-hand side are of integrable square.

5 . 6. Theorems 5 and 6 are plainly the first of two scales of theorems, the next theorem of the scale begun by Theorem 6, for example, being that, *if the series* (5 . 11) *is convergent, then* (5 . 13) *is summable* $(C, 2)$. It is doubtless true that *the necessary and sufficient condition for the summability of* (5 . 11) *is the summability of* (5 . 13); but we have not attempted to work out the proof in detail.

5 . 7. We have seen that (5 . 13) does not necessarily exist almost always even when $f(t)$ is continuous, and we know of no very illuminating additional criterion, sufficient to ensure that it shall do so, which is not obviously trivial.* It is, however, comparatively easy to find a significant criterion that (5 . 13) shall converge almost everywhere *to a function of integrable square.*

THEOREM 7.—*A sufficient condition that the integral* (5 . 13) *should exist as a Cauchy integral, with* $s = f(x)$, *for almost all values of* x, *and represent a function of integrable square, is that the series*

$$\Sigma(|a_n|^2+|b_n|^2)(\log n)^2 \tag{5.71}$$

should be convergent.

We require three additional lemmas.

LEMMA μ.—*If* Σa_n *is convergent*† *to the sum* s, *and* $\Sigma a_n \log n$ *is summable* $(C, 1)$, *then*

$$\Sigma \frac{s_n - s}{n},$$

where $s_n = a_1 + a_2 + \ldots + a_n$, *is convergent.*

* Obviously, if $f(t)$ satisfies a Lipschitz condition $f(t+h)-f(t) = O(|h|^\alpha)$, where $\alpha > 0$, uniformly in t, the integral is uniformly convergent (as a Lebesgue integral). We mean by an "illuminating" criterion one which would throw light on the possibilities of non-absolute convergence.

† This condition is really unnecessary; but it simplifies the proof, and the lemma is as it stands sufficient for our purpose. If Σa_n is not convergent, it is any rate summable $(C, 1)$, and s must be taken to be its Cesàro sum.

We may plainly suppose, without loss of generality, that $a_1 = 0$, $s = 0$. We write

$$t_n = a_2 \log 2 + \ldots + a_n \log n, \quad T_n = t_2 + \ldots + t_n,$$

$$\mathbf{T}_n = \sum_2^{n-1} t_\nu \Delta \frac{1}{\log \nu},^* \qquad \tau_n = \frac{s_2}{2} + \ldots + \frac{s_n}{n}.$$

We have then, by partial summation,

$$s_n = \sum_2^n \frac{t_\nu - t_{\nu-1}}{\log \nu} = \mathbf{T}_n + \frac{t_n}{\log n} \tag{5.72}$$

and

$$\tau_n = s_n \log n - t_n + \sum_2^n s_\nu \omega_\nu, \tag{5.73}$$

where
$$\omega_\nu = \frac{1}{\nu} + \Delta \log \nu = O\left(\frac{1}{\nu^2}\right).$$

From (5 . 72) and 5 . 73) we deduce

$$\tau_n = \mathbf{T}_n \log n + \sum_2^{n-1} s_\nu \omega_\nu. \tag{5.74}$$

The last series is plainly convergent when continued to infinity, so that what we have to prove is that $\mathbf{T}_n \log n$ tends to a limit.

We observe first that

$$\mathbf{T}_n = \sum_2^{n-1} t_\nu \Delta \frac{1}{\log \nu} = \sum_2^{n-2} T_\nu \Delta^2 \frac{1}{\log \nu} + T_{n-1} \Delta \frac{1}{\log (n-1)}. \tag{5.75}$$

Now $T_n \sim \mathbf{a}n$, where $\mathbf{a}$ is the $(C, 1)$ sum of $\Sigma a_n \log n$. Hence the second term on the right of (5 . 75) tends to zero, and the first series is convergent when continued to infinity; so that

$$\mathbf{T}_n \to \sum_2^\infty t_\nu \Delta \frac{1}{\log \nu} = \mathbf{T},$$

say. But $\mathbf{T}$ must be 0, by (5 . 72); for $s_n \to 0$, and $t_n \to \mathbf{a}$ $(C, 1)$, so that

$$\frac{t_n}{\log n} \to 0 \quad (C, 1),$$

* The operator Δ is defined by the equation

$$\Delta f(\nu) = f(\nu) - f(\nu + 1).$$

from which it follows that $\mathbf{T}_n \to 0$ $(C, 1)$, or that $\mathbf{T} = 0$. We may therefore write (5 . 75) in the form

$$\mathbf{T}_n = -\sum_n^\infty t_\nu \Delta \frac{1}{\log \nu} = T_{n-1} \Delta \frac{1}{\log n} - \sum_n^\infty T_\nu \Delta^2 \frac{1}{\log \nu}.$$

Here the first term is $o(1/\log n)$, while the second is asymptotic to

$$-\mathbf{a} \sum_{n-1}^\infty \nu \Delta^2 \frac{1}{\log \nu} \sim -\mathbf{a} \sum_{n-1}^\infty \frac{1}{\nu (\log \nu)^2} \sim -\frac{\mathbf{a}}{\log n}.$$

Then $\mathbf{T}_n \log n$ tends to a limit, and the lemma is proved.

5 . 8. LEMMA ν.—*In order that*

$$\mathbf{S}_n = \sum_1^n \frac{s_\nu - f(x)}{\nu},$$

where s_ν is the sum of the first $\nu+1$ terms of the Fourier series of $f(t)$, for $t = x$, should converge in mean to a function of integrable square, it is necessary and sufficient that the series (5 . 71) *should be convergent.*

A function $\chi_n(x)$, defined for $a \leqslant x \leqslant b$, is said to converge in mean if

$$\lim \int_a^b \{\chi_\mu(x) - \chi_\nu(x)\}^2 dx = 0$$

when μ and ν tend to infinity. There is then a function $\chi(x)$, of integrable square, such that

$$\lim \int_a^b \{\chi_n(x) - \chi(x)\}^2 dx = 0$$

when $n \to \infty$. And it is possible to choose a sequence $n_1, n_2, \ldots$ of values of n so that $\chi_{n_i}(x) \to \chi(x)$, when $i \to \infty$, for almost all values of x.*

The problem is immediately reducible to the corresponding problems for an even and odd function, which may be treated similarly. We suppose then that $f(x)$ is even and, as plainly we may without loss of

* The notion of convergence in mean is due to E. Fischer ("Sur la convergence en moyenne", *Comptes Rendus*, 13 May, 1907). For a simple account of the fundamentals of the theory see M. Plancherel, "Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies", *Rend. di Palermo*, 30 (1910), 289-335.

generality, that $a_0 = 0$, so that

$$f \sim \sum_1^\infty a_n \cos nx = \sum_1^\infty A_n, \qquad s_n = A_1 + A_2 + \ldots + A_n.$$

We have

$$\sum_\mu^\nu \frac{s_n - f(x)}{n} = \frac{A_1 + \ldots + A_\mu}{\mu} + \ldots + \frac{A_1 + \ldots + A_\nu}{\nu} - f\sigma_{\mu,\nu}$$

$$= A_1 \sigma_{\mu,\nu} + \ldots + A_\mu \sigma_{\mu,\nu} + A_{\mu+1}\sigma_{\mu+1,\nu} + \ldots + A_\nu \sigma_{\nu,\nu} - f\sigma_{\mu,\nu},$$

where
$$\sigma_{\mu,\nu} = \sum_\mu^\nu \frac{1}{n}.$$

Squaring, integrating, and performing some simple reductions, we obtain

$$(5.81)\quad \frac{1}{\pi}\int_{-\pi}^{\pi} (\mathbf{S}_\nu - \mathbf{S}_{\mu-1})^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\sum_\mu^\nu \frac{s_n - f(x)}{n}\right)^2 dx$$

$$= a_{\mu+1}^2 \sigma_{\mu,\mu}^2 + a_{\mu+2}^2 \sigma_{\mu,\mu+1}^2 + \ldots + a_\nu^2 \sigma_{\mu,\nu-1}^2 + (a_{\nu+1}^2 + a_{\nu+2}^2 + \ldots)\sigma_{\mu,\nu}^2.$$

Suppose first that (5 . 71) is convergent. Then

$$a_{\nu+1}^2 + a_{\nu+2}^2 + \ldots = o\left(\frac{1}{(\log \nu)^2}\right), \qquad \sigma_{\mu,\nu} = O(\log \nu),$$

so that the last term in (5 . 81) tends to zero. The remainder of the right-hand side of (5 . 81) is

$$O\left\{\sum_{\mu+1}^\nu a_n^2 \left(\log \frac{n}{\mu}\right)^2\right\} = O\left(\sum_{\mu+1}^\nu a_n^2 (\log n)^2\right) \to 0.$$

Hence the integral (5 . 81) tends to 0, and the sufficiency of our criterion is proved. Its necessity is obvious from (5 . 81), since $\sigma_{\mu,\lambda} \sim \log \lambda$ when μ is fixed and $\lambda \to \infty$.

LEMMA o.—*If x is in the Lebesgue set, and the integral* (5 . 13) *is summable* $(C, 1)$, *then it is convergent as a Cauchy integral.*

It is plainly enough to show that, if

$$j(t) = \int_t^a \frac{\phi(t)}{t}\, dt,$$

then
$$j(t) - \frac{1}{t}\int_0^t j(u)\, du = j(t) - \frac{j_1(t)}{t} \to 0,$$

when x is in the Lebesgue set; and this follows at once from (3 . 34), since

$$\phi_1(t) = \int_0^t \phi(u)\,du = O\left(\int_0^t |\phi(u)|\,du\right) = o(t).*$$

5 . 9. We can now prove Theorem 7. If the series (5 . 71) is convergent, then $\Sigma A_n \log n$ is almost always summable $(C, 1)$, and ΣA_n is almost always convergent to $f(x)$.†

Hence first, by Lemma μ, the series

$$\mathbf{S} = \sum_1^\infty \frac{s_\nu - f(x)}{\nu}$$

is almost always convergent. It converges in mean, by Lemma ν, to a function of integrable square, so that the square of its sum function $\sigma(x)$ is integrable. Since $s_\nu \to f(x)$ almost always, it is almost always summable $(C, -1)$; and hence, by Theorem 6, the integral (5 . 13), with $s = f(x)$, is almost always summable $(C, 1)$; and therefore, by Lemma o, almost always convergent. Finally, by Lemma λ, its value is of integrable square.

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* Obviously we do not use the full force of the condition that x is in the Lebesgue set; and the lemma might be generalized in many ways.

† That ΣA_n converges almost always follows from a theorem proved by Hardy (*loc. cit.* p. 213, f.n.‡ Recently Kolmogoroff and Seliverstoff ("Sur la convergence des séries de Fourier", *Comptes Rendus*, January 14th, 1924), have proved more, viz. that $\Sigma(|a_n|^2 + |b_n|^2)(\log n)^{1+\delta}$ is convergent for any positive δ.

CORRECTIONS

p. 215, *line* 5 *from below*. Read (C, δ) for $(C\delta)$.

p. 218, *line* 5. For J read J_1.

p. 229, *line* 12. For (3.23) read (3.33).

COMMENTS

§§ 1–4. In the following comments we suppose $f \in L(-\pi, \pi)$, and we write

$$\psi(t) = \psi_0(t) = \tfrac{1}{2}\{f(\theta+t)-f(\theta-t)\},$$

$$\chi(t) = \chi_0(t) = \frac{1}{\pi}\int_t^\pi \psi(t)\cot\tfrac{1}{2}t\,dt,$$

$$\xi(t) = \xi_0(t) = \frac{2}{\pi}\int_t^\infty \frac{\psi(t)}{t}\,dt.$$

We write also ψ_α for the (C,α) mean of ψ, where $\alpha \geqslant 0$, and similarly for χ and ξ.

The statement of Theorem 3 (p. 219) says only that some Cesàro mean of the conjugate series of f at θ converges to the sum s if and only if some Cesàro mean of χ has the limit s at 0. What is actually proved in the text is as follows:

(i) *If $\chi_\alpha(t) \to s$ as $t \to 0+$, where α is a non-negative integer, then the conjugate series of f at θ is summable (C,δ) to the sum s for $\delta = \alpha+2$.*

(ii) *If the conjugate series of f at θ is summable (C,δ) to the sum s, where δ is a non-negative integer, then $\chi_\alpha(t) \to s$ as $t \to 0+$ for $\alpha = \delta+1$.*

These results were improved and completed by R. E. A. C. Paley, *Proc. Camb. Phil. Soc.* 26 (1930), 173–203, who proved:

(iii) *If $\alpha \geqslant 0$, and $\xi_\alpha(t) \to s$ and $\psi_\alpha(t) \to 0$ as $t \to 0+$, then the conjugate series of f at θ is summable (C,δ) to the sum s for all $\delta > \alpha$.*

(iv) *If $\delta \geqslant -1$, and the conjugate series of f at θ is summable (C,δ) to the sum s, then $\xi_\alpha(t) \to s$ and $\psi_\alpha(t) \to 0$ as $t \to 0+$ for all $\alpha > \delta+1$.*

The case $\alpha = 0$ of (iii) was already known, this being essentially the result of W. H. Young quoted as Lemma γ (p. 220).

It can be shown that if $\alpha \geqslant 1$, then the conditions that $\xi_\alpha(t) \to s$ and $\psi_\alpha(t) \to 0$ are together equivalent to the condition that $\xi_{\alpha-1}(t) \to s$, and that this is in turn equivalent to the condition that $\chi_{\alpha-1}(t) \to s$. The results (i) and (ii) of Hardy and Littlewood are therefore included in (iii) and (iv).

Paley showed also that the bound for δ in the case $\alpha = 0$ of (iii) and the bound for α in the case $\delta = 0$ of (iv) are best possible.

An alternative proof of the case $\alpha \geqslant 1$ of (iii) and the case $\delta \geqslant 0$ of (iv), using a method similar to that of Wiener for the Fourier series case (see the comments on 1924, 1), has been given by G. I. Sunouchi, *Tôhoku Math. J.* (2), 1 (1950), 167–85.

p. 213. Direct proofs of the existence p.p. of the integral (1.13) do now exist. See, for example, Z I, p. 131.

p. 235. The result of Theorem 4 was strengthened by Kaczmarz and Mazurkiewicz, who showed that there exist continuous f such that the integral (5.13) diverges for all x (see Z I, pp. 133–4, 378, and P. L. Ul′yanov, *Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him.* 1959, no. 5, 33–42).

NOTES ON THE THEORY OF SERIES—II: THE FOURIER SERIES OF A POSITIVE FUNCTION

G. H. HARDY *and* J. E. LITTLEWOOD*.

[*Extracted from the Journal of the London Mathematical Society, Vol.* 1, *Part* 3.]

1. In a memoir published in 1924 in the *Mathematische Zeitschrift*† we proved that the "summability problem" for Fourier series admits, when appropriately defined, of a definite solution. If we ask whether the Fourier series of $f(t)$ is summable, when $t = x$, not by a particular Cesàro mean (such as the first mean), but by the aggregate of Cesàro means, that is to say if we ask not whether the series is summable (C, k) but whether it is summable (C), then there is a complete answer to the question. The series is summable (C), to sum s, if and only if some one of the functions $\phi_r(t)$ defined by

$$(1.1) \qquad \phi(t) = \phi_0(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\},$$

$$(1.2) \qquad \phi_1(t) = \frac{1}{t}\int_0^t \phi_0(u)\,du, \qquad \phi_2(t) = \frac{1}{t}\int_0^t \phi_1(u)\,du, \qquad \ldots$$

tends to zero when $t \to 0$.

For special classes of functions, and in particular for bounded functions, we can go further. If $f(t)$ is bounded in a neighbourhood of $t = x$, and in particular, of course, if it is bounded in $(-\pi, \pi)$, then the summability problem can be solved in whatever sense it is proposed. In this case all Cesàro means are equivalent; the series is either summable (C, δ) for every positive δ, or summable by no Cesàro mean; and the necessary and sufficient condition for summability, by any mean or by all, is that $\phi_1(t) \to 0$‡.

In the present note we extend this theorem to functions *bounded on one side* only, *i.e.* functions which satisfy one or other of the conditions

$$f(t) \geqslant -A, \quad f(t) \leqslant A,$$

where A is a constant. The condition need naturally be satisfied only in a neighbourhood of $t = x$. The special interest of the extension lies in the fact that the class of functions considered includes the class of *positive* functions.

* Received 28 February, 1926; read 11 March, 1926.

† Hardy and Littlewood, **5** (Theorem C). The functions ϕ_2, ϕ_3, ... are defined, in the first instance, by means of "Cauchy" integrals of the type $\lim\limits_{\epsilon \to 0}\int_\epsilon^t$; but we show that, when ϕ_1 is integrable, in the sense of Lebesgue, and (as in the cases which we have to consider) some ϕ_r is continuous, then every ϕ_r is also integrable in the sense of Lebesgue.

‡ Hardy and Littlewood, **5** (Theorem C 1).

1926, 10 (with J. E. Littlewood) *Journal of the London Mathematical Society*, 1, 134–8.

THEOREM A. *Suppose that $f(t)$ is integrable, and that*

$$f(t) \geqslant -A, \tag{1.3}$$

in a neighbourhood of $t = x$. Then the Fourier series of $f(t)$, for $t = x$, is either summable by every Cesàro mean of positive order or summable by no Cesàro mean. The necessary and sufficient condition for summability, to sum s, is that

$$\phi_1(t) \to 0. \tag{1.4}$$

2. We base the proof of the theorem on certain preliminary propositions, in part known, which we state in the form of lemmas.

We shall say that $\quad \phi(t) = o(1) \quad (C, r)$

if $\phi_r(t) \to 0$, and that $\quad \phi(t) = O(1) \quad (C, r)$

if $\phi_r(t) = O(1)$.

LEMMA 1. *If*

$$\phi(t) \geqslant -A, \qquad \phi(t) = o(1) \quad (C, 1),$$

then $$|\phi(t)| = O(1) \quad (C, 1).$$

We write $$\phi = \phi^+ - \phi^-, \qquad |\phi| = \phi^+ + \phi^-,$$

so that ϕ^+ and ϕ^- are non-negative functions of which always one is $|\phi|$ and the other zero. Plainly $\phi^- \leqslant A$, and so

$$\int_0^t |\phi|\, du = \int_0^t \phi\, du + 2\int_0^t \phi^-\, du \leqslant 2At + o(t),$$

which proves the lemma.

LEMMA 2. *If $\phi(t) \geqslant -A$ and*

$$\phi(t) = o(1) \quad (C, r)$$

for some value of r, then $$\phi(t) = o(1) \quad (C, 1).$$

We may suppose $r \geqslant 2$, and it is plainly sufficient to prove that

$$\phi(t) = o(1) \quad (C, r-1),$$

or that $$\phi_{r-2}(t) = o(1) \quad (C, 1).$$

Now $$\phi_{r-2}(t) = o(1) \quad (C, 2),$$

and the equations (1 . 2) show that

$$\phi_1(t) \geqslant -A, \quad \phi_2(t) \geqslant -A, \quad \ldots, \quad \phi_{r-2}(t) \geqslant -A.$$

It is therefore sufficient to prove that the lemma is true for $r = 2$.

If $r=2$, and $\phi_1(t)$ does not tend to zero, there is a positive H such that one of the inequalities

$$\phi_1(t) \geqslant H, \quad \phi_1(t) \leqslant -H, \tag{2.1}$$

is true for a sequence of values of t whose limit is zero. If the first inequality is true for such a sequence, and

$$0 < \delta \leqslant \operatorname{Min}\left(\tfrac{1}{2}, \frac{H}{2A}\right),$$

then
$$\phi_1(t+\eta t) = \frac{1}{t+\eta t}\int_0^{t+\eta t} \phi\, du = \frac{1}{t+\eta t}\left(t\phi_1(t)+\int_t^{t+\eta t} \phi\, du\right)$$
$$\geqslant \frac{(H-A\eta)t}{t+\eta t} \geqslant \frac{H}{2(1+\delta)} \geqslant \tfrac{1}{3}H,$$

when $0 \leqslant \eta \leqslant \delta$ and t belongs to the sequence. Hence

$$(t+\delta t)\,\phi_2(t+\delta t) - t\phi_2(t) = \int_t^{t+\delta t} \phi_1 du \geqslant \tfrac{1}{3}H\delta t,$$

which contradicts $\phi_2(t) = o(1)$. Similarly, considering an interval $(t-\delta t, t)$, we can prove the impossibility of the second inequality (2.1). It follows that $\phi_1(t) \to 0$, which proves the lemma.

LEMMA 3. *If every Cesàro mean of a series Σc_n, of positive order, is bounded, and the series is summable by some Cesàro mean, then it is summable by every Cesàro mean of positive order.*

This is well known*.

3. The preceding lemmas are of a quite general character. In what follows $\phi(t)$ is the function (1.1), and the series is the Fourier series of $f(t)$ for $t = x$.

LEMMA 4. *If*

$$|\phi(t)| = O(1) \quad (C, 1), \tag{3.1}$$

then every Cesàro mean of the series, of positive order, is bounded.

This is also well known†.

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* See Andersen, 1, 56–59. So far as means of integral order are concerned, the result is included in Theorem 19 of our paper 3 (since there β may be 0).

† Hardy, 2. The result is proved there in the sharper form that the series is summable when $|\phi| = o(1)$ $(C, 1)$.

LEMMA 5. *The necessary and sufficient condition that the series should be summable* (C, k), *for some* k, *is that*

$$\phi(t) = o(1) \quad (C, r) \tag{3.2}$$

for some r.

This is the principal theorem of our memoir in the *Mathematische Zeitschrift* already referred to.

4. We can now prove Theorem A. If the series is summable, by any mean, then (3 . 2) is true, for some r, by Lemma 5. But $f \geqslant -A$ involves $\phi \geqslant -A-s = -A_1$, say, and therefore (3 . 2) is true for $r=1$, by Lemma 2. Hence, by Lemma 1, (3 . 1) is true, and therefore, by Lemma 4, every Cesàro mean of the series is bounded. And therefore, by Lemma 3, the series is summable by every Cesàro mean of positive order. Thus all such Cesàro means are equivalent to one another. Finally, the necessary and sufficient condition for summability is that (3 . 2) be true for some r, in which case, as we have seen already, it is necessarily true for $r=1$.

5. We state without proof another theorem which gives a generalization of our earlier result* in a different direction.

THEOREM B. *If* $p \geqslant 1$ *and* $|\phi(t)|^p = O(1)$ $(C, 1)$, *i.e. if*

$$\int_0^t |\phi(u)|^p \, du = O(t), \tag{5.1}$$

then the Fourier series of $f(t)$, *for* $t = x$, *is either summable by every Cesàro mean of positive order or summable by no Cesàro mean. The necessary and sufficient condition for summability is*

$$\phi(t) \to 0 \quad (C, r), \tag{5.2}$$

where r *is any number such that*

$$r > 1/p \quad (p > 1), \quad r \geqslant 1 \quad (p = 1). \tag{5.3}$$

Here (5 . 2) means

$$t^{-r} \int_0^t (t-u)^{r-1} \phi(t) \, dt \to 0 \dagger.$$

* Theorem C 1 of our memoir **5**.

† The definition agrees with our previous definition when r is a positive integer. This is not obvious and, so far as we know, no general proof has been published, though our note **4** contains a statement of the facts concerning the equivalence of these and other integral means. The question is not important here, since the only case in which the definition (5 . 2) is really interesting for our present purpose is that in which $r < 1$.

In particular we may take $r=1$, when (5 . 2) reduces to (1 . 4); when $r<1$, (5 . 2) states more.

The case $p=1$, $r=1$ is particularly interesting, as the theorem then also generalizes the criterion of Lebesgue* and Hardy† for summability, viz.

$$|\phi(t)| = o(1) \quad (C, 1).$$

It is easy to show by examples that the bounds assigned for r in (5 . 3) cannot be improved; while in Theorem A we cannot replace (1 . 4) by (5 . 2), with any value of $r<1$.

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5. G. H. Hardy and J. E. Littlewood, "Solution of the Cesàro summability problem for power-series and Fourier series", *Math. Zeitschrift*, 19 (1924), 67–96.

6. H. Lebesgue, "Recherches sur la convergence des séries de Fourier", *Math. Annalen*, 61 (1905), 251–280.

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* Lebesgue, **6**, 274–276.

† Hardy, **2**.

CORRECTIONS

p. 137. The inequalities (5.3) should be replaced by $r > 1/p$ (for all $p \geqslant 1$). A proof of Theorem B (as corrected) is given in 1931, 5.

p. 138, *line* 1. After 'particular' insert 'if $p > 1$'.

p. 138, *lines* 3–6. This passage should be deleted.

Notes on the theory of series (III): *On the summability of the Fourier series of a nearly continuous function.* By Mr G. H. HARDY and Mr J. E. LITTLEWOOD.

[*Received* 4 January, *read* 31 January 1927.]

1. The theorem which we prove here seems obvious enough when stated, but it appears to have been overlooked by the numerous writers who have discussed the subject, and the proof is less immediate than might be expected.

We say that $\phi(t)$ is continuous (C, α), where $\alpha > 0$, if

$$\frac{1}{\Gamma(\alpha) t^{\alpha}} \int_0^t \phi(u)(t-u)^{\alpha-1}\, du$$

tends to a finite limit l when $t \to 0$. We shall also say that $\phi(t) \to l\,(C, \alpha)$. To assert continuity (C, α), for a small α, is to assert a little less than ordinary continuity.

If $f(t)$ is periodic and integrable, and

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2s\} \to 0,$$

then the Fourier series of $f(t)$, for $t = x$, is summable (C, δ) to sum s for every positive δ.* If $\phi(t) \to 0\,(C, 1)$, then the series is summable $(C, 1+\delta)$.† It is therefore natural to expect the truth of the following theorem.

THEOREM. *If $\phi(t) \to 0\,(C, \alpha)$, then the Fourier series of $f(t)$, for $t = x$, is summable $(C, \alpha+\delta)$ to sum s for every positive δ.*

We prove the theorem in the most interesting case when $0 < \alpha < 1$. The proof for $\alpha > 1$ will differ in complication but not in principle.

2. We suppose β and δ positive and $\beta + \delta < 1$. The $(\beta+\delta)$-th Rieszian mean of the Fourier series, for terms whose rank does not exceed ω, differs from s by

$$\frac{2\Gamma(1+\beta+\delta)}{\pi} J_{\beta+\delta}(\phi, \omega) = \frac{2\Gamma(1+\beta+\delta)}{\pi} \int_0^{\infty} t^{-1-\beta-\delta}\, C_{1+\beta+\delta}(t)\, \phi\left(\frac{t}{\omega}\right) dt,$$

where $C_q(t)$ is Young's function‡

$$C_q(t) = \frac{t^q}{\Gamma(1+q)} \left\{1 - \frac{t^2}{(1+q)(2+q)} + \frac{t^4}{(1+q)\dots(4+q)} - \dots\right\}.$$

Since $\quad C_{1+\beta+\delta}(\omega u) = \dfrac{\omega^{\beta}}{\Gamma(\beta)} \displaystyle\int_0^u C_{1+\delta}(\omega v)(u-v)^{\beta-1}\, dv,$

* M. Riesz, 4, 5; Chapman, 1. See also Hobson, 3.
† Young, 6, 8. ‡ Young, 7.

1927, 2 (with J. E. Littlewood) *Proceedings of the Cambridge Philosophical Society*, 23, 681–4.

we have

$$J_{\beta+\delta}(\phi, \omega) = \omega^{-\beta-\delta} \int_0^\infty u^{-1-\beta-\delta} C_{1+\beta+\delta}(\omega u)\, \phi(u)\, du$$

$$= \frac{\omega^{-\delta}}{\Gamma(\beta)} \int_0^\infty u^{-1-\beta-\delta} \phi(u)\, du \int_0^u C_{1+\delta}(\omega v)(u-v)^{\beta-1} dv$$

$$= \frac{\omega^{-\delta}}{\Gamma(\beta)} \int_0^\infty C_{1+\delta}(\omega v)\, dv \int_v^\infty u^{-1-\beta-\delta}(u-v)^{\beta-1} \phi(u)\, du$$

$$= \frac{\omega^{-\delta}}{\Gamma(\beta)} \int_0^\infty v^{-1-\delta} C_{1+\delta}(\omega v)\, \psi(v)\, dv = \frac{1}{\Gamma(\beta)} J_\delta(\psi, \omega),$$

where $$\psi(v) = v^{1+\delta} \int_v^\infty u^{-1-\beta-\delta}(u-v)^{\beta-1} \phi(u)\, du.$$

There is no difficulty in the inversion of the order of integration, since $C_{1+\delta}(t)$ is bounded, and $O(t^{1+\delta})$ for small t, and the double integral is absolutely convergent.

Assume for the moment that it has been proved that $\psi(v)$ tends to zero with v. Then $J_\delta(\psi, \omega) \to 0$ when $\omega \to \infty$, this being indeed the kernel of the proof that the Fourier series of a continuous function is summable (C, δ).* Hence $J_{\beta+\delta}(\phi, \omega) \to 0$, and the Fourier series of $f(t)$ is summable $(C, \beta+\delta)$ for $t = x$. The theorem will accordingly be proved if we can shew that $\psi(v) \to 0$ with v whenever $\beta > \alpha$ and $\delta > 0$.

3. We may suppose without loss of generality that $x = 0$, that $s = 0$, and that the Fourier series of $f(t)$ has no constant term, so that the mean value of $\phi(t)$ is zero. It is in fact easily verified that $\psi(v) \to 0$ with v when $f(t)$ is $f_0(t) = a_0 - (a_0 - s)\cos t$, and we may consider $f - f_0$ instead of f. Our hypothesis is that

$$\phi_\alpha(t) = A \int_0^t \phi(u)(t-u)^{\alpha-1} du = o(t^\alpha), \quad \text{......(3·1)}$$

A being, here and in the sequel, a constant (a different constant in different places). From (3·1) it follows that

$$\phi(t) = A \frac{d}{dt} \int_0^t \frac{\phi_\alpha(u)}{(t-u)^\alpha} du = \phi_1'(t), \quad \text{.........(3·2)}$$

* See Young, 6, 7; Hardy, 2. Here ψ will play the part of the ϕ of the ordinary proof. The genesis of ψ is really irrelevant, and it is not important that ψ is not periodic; but we can, if we please, replace ψ by the periodic function ψ^* equal to ψ in $(0, 2\pi)$, it being easily proved that the difference between $J_\delta(\psi, \omega)$ and $J_\delta(\psi^*, \omega)$, for any ψ integrable in $(0, 2\pi)$, is $O(\omega^{-\delta})$ and therefore trivial.

say, for almost all t.* Also

$$\phi_1(t) = \int_0^t \frac{o(u^\alpha)}{(t-u)^\alpha}\, du = o(t) \quad \text{..............(3·3)}$$

for small t, while, being the integral of $\phi(t)$, it is $O(1)$ for large t.

We write

$$\psi(v) = v^{1+\delta}\int_v^\infty = v^{1+\delta}\int_v^{2v} + v^{1+\delta}\int_{2v}^\infty = \psi_1 + \psi_2,$$

and we dispose first of ψ_2. This is

$$-v^{1+\delta}(2v)^{-1-\beta-\delta}v^{\beta-1}\phi_1(2v) - v^{1+\delta}\int_{2v}^\infty \phi_1(u)\frac{d}{du}\{u^{-1-\beta-\delta}(u-v)^{\beta-1}\}\, du.$$

The first term is $o(1)$, by (3·3). The second is

$$v^{1+\delta}\int_{2v}^1 o(u)\, O\{u^{-2-\beta-\delta}(u-v)^{\beta-1}\}\, du$$

$$+ v^{1+\delta}\int_{2v}^1 o(u)\, O\{u^{-1-\beta-\delta}(u-v)^{\beta-2}\}\, du$$

$$+ v^{1+\delta}\int_1^\infty O(1)\, O(u^{-3-\delta})\, du$$

$$= o(1) + o(1) + O(v^{1+\delta}) = o(1).$$

Hence $$\psi_2 \to 0.$$

4. It remains to prove that $\psi_1 \to 0$. We observe first that if $0 < \xi \leqq \eta \leqq 2\xi$, then

$$\int_\xi^\eta \phi(u)\, du = o\{\xi^\alpha(\eta-\xi)^{1-\alpha}\}.$$

We have in fact

$$\phi_1(\eta) - \phi_1(\xi) = A\int_\xi^\eta \frac{\phi_\alpha(t)}{(\eta-t)^\alpha}\, dt - A\int_0^\xi \phi_\alpha(t)\left\{\frac{1}{(\xi-t)^\alpha} - \frac{1}{(\eta-t)^\alpha}\right\} dt.$$

The first term here is

$$o\left\{\xi^\alpha \int_\xi^\eta \frac{dt}{(\eta-t)^\alpha}\right\} = o\{\xi^\alpha(\eta-\xi)^{1-\alpha}\};$$

and the second is

$$o\left[\xi^\alpha \int_0^\xi \left\{\frac{1}{(\xi-t)^\alpha} - \frac{1}{(\eta-t)^\alpha}\right\} dt\right] = o[\xi^\alpha\{(\eta-\xi)^{1-\alpha} - \eta^{1-\alpha} + \xi^{1-\alpha}\}],$$

which is of the same form; and this establishes our assertion.

* This is simply the solution of 'Abel's integral equation'.

We have now

$$\int_v^{2v} u^{-1-\beta-\delta}(u-v)^{\beta-1}\phi(u)\,du = \left[u^{-1-\beta-\delta}(u-v)^{\beta-1}\left(\int_v^u \phi(t)\,dt\right)\right]_v^{2v}$$

$$-\int_v^{2v}\left[\frac{d}{du}\{u^{-1-\beta-\delta}(u-v)^{\beta-1}\}\int_v^u \phi(t)\,dt\right]du$$

$$= o\,(v^{\alpha}\,.\,v^{1-\alpha}\,.\,v^{-1-\beta-\delta}\,.\,v^{\beta-1})$$

$$+\int_v^{2v} O\,\{u^{-2-\beta-\delta}(u-v)^{\beta-1}+u^{-1-\beta-\delta}(u-v)^{\beta-2}\}\,o\,\{v^{\alpha}(u-v)^{1-\alpha}\}\,du$$

$$=\int_v^{2v} o\,\{v^{-2-\beta-\delta+\alpha}(u-v)^{\beta-\alpha}+v^{-1-\beta-\delta+\alpha}(u-v)^{\beta-\alpha-1}\}\,du$$

$$= o\,(v^{-2-\beta-\delta+\alpha}\,.\,v^{\beta-\alpha+1}+v^{-1-\beta-\delta+\alpha}\,.\,v^{\beta-\alpha}) = o\,(v^{-1-\delta}),$$

or $\psi_2 = o(1)$; which completes the proof.

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COMMENT

See the comments on 1924, 1.

NOTES ON THE THEORY OF SERIES (IV): ON THE STRONG SUMMABILITY OF FOURIER SERIES

By G. H. HARDY *and* J. E. LITTLEWOOD.

[Received 5 May, 1926.—Read 29 April, 1926.]

[*Extracted from the Proceedings of the London Mathematical Society, Ser.* 2, *Vol.* 26, *Part* 4.]

1. We proved in 1913* that *the Fourier series of an integrable function $f(t)$ is strongly summable* $(C, 1)$ *at any point of continuity* x, *i.e.* that, if $s_m = s_m(x)$ is the sum of the first $m+1$ terms of the series for $t = x$, then

$$(1.1) \qquad \sum_0^n |s_m - f(x)| = o(n).$$

More generally and more precisely, we proved that, if $f(t)$ is of integrable square in a neighbourhood of x, and

$$(1.2) \qquad \int_0^t \{\phi(u)\}^2 du = o(t),$$

where
$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\},$$

then

$$(1.3) \qquad \sum_0^n (s_m - s)^2 = o(n),$$

and *a fortiori*

$$(1.31) \qquad \sum_1^n |s_m - s| = o(n).$$

In particular when $f(t)$ is of integrable square in $(-\pi, \pi)$ the series is strongly summable, to sum $f(x)$, for almost all values of x.

The classical theorem of Fejér and its generalization by Lebesgue† show that, in these circumstances,

$$\sum_0^n (s_m - s) = o(n).$$

* Hardy and Littlewood, **4**.
† Lebesgue, **9**.

1927, 3 (with J. E. Littlewood) *Proceedings of the London Mathematical Society* (2), 26, 273–86.

The interest of our theorem is that it shows (for example) that, when the Fourier series of a continuous function is not convergent, its summability is not merely a consequence of the cancelling of the various deviations summed in Fejér's mean, but rather of the comparative smallness of the deviations.

Our original results were completed and simplified in various ways by Fejér* and Fekete†. A wider extension was made in 1922 by Carleman‡, who showed that, if

$$\int_0^t \{\phi(u)\}^2 du = O(t) \tag{1.4}$$

and

$$\int_0^t |\phi(u)|\, du = o(t), \tag{1.5}$$

then

$$\sum_0^n |s_m - s|^q = o(n) \tag{1.6}$$

for all positive values of q. The conditions (1.4) and (1.5) are certainly satisfied if (1.2) is satisfied, and (1.6), by Hölder's inequality, says the more the larger q is, so that Carleman's theorem generalizes ours in every direction. His proof is also more direct.

More recently, Sutton§ succeeded in simplifying Carleman's proof considerably by the use of certain inequalities due to Hausdorff. He also replaced Carleman's condition (1.4) by the less restrictive condition

$$\int_0^t |\phi(u)|^p du = O(t), \tag{1.7}$$

where p may be any number greater than 1. He also stated another theorem, viz. that, if Carleman's second condition (1.5) is replaced by

$$\int_0^t \phi(u)\, du = o(t), \tag{1.8}$$

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* Fejér, 2.
† Fekete, 3.
‡ Carleman, 1.
§ Sutton, 12.

then

$$\sum_0^n |\sigma_m - s|^q = o(n) \tag{1.9}$$

for every q, σ_m being now the m-th Fejér mean.

The state in which the theorem has been left by all these investigations is, however, still not entirely satisfactory. In § 2 we prove a theorem which has more the air of finality. Roughly, we show that Sutton's most general hypotheses (1.7) and (1.8) involve actually not merely (1.9), but Carleman's conclusion (1.6). We then, in § 3, prove the corresponding theorem for the allied series. Finally, in §§ 4–7, we prove two theorems which may be called the "reciprocals" or "transforms" of the two earlier theorems.

2. THEOREM 1. *If* $p > 1$ *and*

$$\int_0^t |\phi(u)|^p du = O(t), \quad \int_0^t \phi(u)\, du = o(t), \tag{2.1}$$

then

$$\sum_0^n |s_m - s|^q = o(n)$$

for every positive q.

We begin by making certain standard simplifications of the data. We suppose that

$$f(t) \sim \sum_1^\infty a_n \cos nt$$

is an even function with zero mean value, and that $x = 0$, $s = 0$, so that

$$\phi(t) = f(t), \quad s_m = a_1 + a_2 + \ldots + a_m.$$

If the theorem is proved in this case it will (as is usual in the theory of Fourier series) be true generally.

Following Carleman and Sutton, we write

$$\begin{aligned} \pi s_n &= \int_0^\pi \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2}t} f(t)\, dt \\ &= \int_0^{k/n} \sin nt \cot \tfrac{1}{2}t\, f(t)\, dt + \int_{k/n}^\pi \sin nt \cot \tfrac{1}{2}t\, f(t)\, dt + \int_0^\pi \cos nt\, f(t)\, dt \\ &= \alpha_n + \beta_n + \gamma_n, \end{aligned} \tag{2.2}$$

say. It is plain that $\gamma_n \to 0$, so that

$$(2.3) \qquad \left(\sum_0^n |\gamma_m|^q\right)^{1/q} < \epsilon n^{1/q},$$

for $n > n_1(\epsilon)$.

Next, if
$$\eta = \frac{k}{n}, \quad F(t) = \int_0^t f(u)\,du,$$
we have

$$a_n = \sin n\eta \cot \tfrac{1}{2}\eta\, F(\eta) - \int_0^\eta (n \cos nt \cot \tfrac{1}{2}t - \tfrac{1}{2} \sin nt \operatorname{cosec}^2 \tfrac{1}{2}t)\, F(t)\, dt$$

$$= o(1) + \int_0^\eta o\left\{\left(\frac{n}{t} + \frac{nt}{t^2}\right)t\right\} dt = o(1) \to 0,$$

when k is fixed. It follows that, when k is fixed, we have

$$(2.4) \qquad \left(\sum_0^n |a_m|^q\right)^{1/q} < \epsilon n^{1/q}$$

for $n > n_2(\epsilon) = n_2(\epsilon, k)$. So far we have made no use of the first of our hypotheses (2.1).

In discussing β_n we assume that*

$$q > 2, \quad q' = \frac{q}{q-1} < p,$$

this being legitimate because the conclusion is stronger the larger be q. We denote by $c_n(\tau)$ the n-th Fourier sine coefficient of the odd function $\chi(t)$ which is equal to $f(t)$ in $(0, \tau)$ and to zero in (τ, π). We have then

$$(2.5) \quad \beta_n = \int_\eta^\pi \sin nt \cot \tfrac{1}{2}t\, f(t)\, dt = \int_\eta^\pi \cot \tfrac{1}{2}t \left(\frac{d}{dt}\int_0^t \sin mu\, f(u)\, du\right) dt$$

$$= -\tfrac{1}{2}\pi \cot \tfrac{1}{2}\eta\, c_m(\eta) + \tfrac{1}{4}\pi \int_\eta^\pi \operatorname{cosec}^2 \tfrac{1}{2}t\, c_m(t)\, dt.$$

From (2.5) it follows, by Minkowski's inequality, that

$$\left(\sum_1^n |\beta_m|^q\right)^{1/q} \leqslant \tfrac{1}{2}\pi \cot \tfrac{1}{2}\eta \left(\sum_1^n |c_m(\eta)|^q\right)^{1/q} + \tfrac{1}{4}\pi \int_\eta^\pi \operatorname{cosec}^2 \tfrac{1}{2}t \left(\sum_1^n |c_m(t)|^q\right)^{1/q} dt.$$

* We write generally r' for $r/(r-1)$ when $r > 1$.

But, by Hausdorff's inequality*,

$$\left(\sum_1^n |c_m(t)|^q\right)^{1/q} \leqslant \left(\frac{1}{\pi}\int_{-\pi}^{\pi} |\chi(u)|^{q'} du\right)^{1/q'}$$
$$= \left(\frac{1}{\pi}\int_{-t}^{t} |f(u)|^{q'} du\right)^{1/q'} \leqslant A t^{1/q'},$$

where A is a constant, since the first condition (2 . 1) is satisfied *a fortiori* when p is replaced by the smaller index q'. Hence

$$(2\,.\,6) \qquad \left(\sum_1^n |\beta_m|^q\right)^{1/q} \leqslant \frac{A}{\eta}\,.\,\eta^{1/q'} + A\int_\eta^\pi \frac{1}{t^2}\,.\,t^{1/q'}\,dt \leqslant A\eta^{-1/q}$$
$$= Ak^{-1/q}n^{1/q} = \epsilon_k n^{1/q},$$

where ϵ_k is a function of k only which tends to zero when $k \to \infty$.

From (2 . 2), (2 . 3), (2 . 4), and (2 . 6) it follows that

$$\pi\left(\sum_1^n |s_n|^q\right)^{1/q} \leqslant \left(\sum_1^n |\alpha_n|^q\right)^{1/q} + \left(\sum_1^n |\beta_n|^q\right)^{1/q} + \left(\sum_1^n |\gamma_n|^q\right)^{1/q}$$
$$< (\epsilon_k + 2\epsilon)\,n^{1/q},$$

if n is greater than n_1 or n_2, and our conclusion follows by choice first of k and then of n.

3. The corresponding theorem for the allied series is:

THEOREM 2. *If $\bar{s}_m$ is the sum of the first $m+1$ terms of the series allied to the Fourier series of $f(t)$; if*

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\},$$

and

$$\int_0^t |\psi(u)|^p\,du = O(t) \quad (p > 1);$$

and if

$$\frac{1}{\pi}\int_0^\pi \psi(t)\cot\tfrac{1}{2}t\,dt$$

exists, as a Cauchy integral†, *and has the value $\bar{s}$; then*

$$\sum_0^n |\bar{s}_m - \bar{s}|^q = o(n)$$

for every positive q.

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* Hausdorff, **8**.

† See Hardy and Littlewood, **7**, 212.

We may reduce the problem in the standard manner to the special case in which

$$f(t) \sim \sum_1^\infty b_n \sin nt$$

is an odd function, $x = 0$, and $\bar{s} = 0$. The allied series is then Σb_n, and $\psi(t) = f(t)$. We have

$$\pi \bar{s}_n = \pi(b_1+b_2+\ldots+b_n) = \tfrac{1}{2}\int_{-\pi}^{\pi} \frac{\cos\frac{1}{2}t - \cos(n+\frac{1}{2})t}{\sin\frac{1}{2}t} f(t)\,dt$$

$$= -\int_0^{\pi} \frac{\cos(n+\frac{1}{2})t}{\sin\frac{1}{2}t} f(t)\,dt$$

$$= -\int_0^{k/n} \cos nt \cot\tfrac{1}{2}t\, f(t)\,dt - \int_{k/n}^{\pi} \cos nt \cot\tfrac{1}{2}t\, f(t)\,dt + \int_0^{\pi} \sin nt\, f(t)\,dt$$

$$= \alpha_n + \beta_n + \gamma_n,$$

say. It is plain that $\gamma_n \to 0$.

Writing, as before, η for k/n, we observe first that $\cos nt$ has less than Ak turning values in $(0, \eta)$, so that we may divide the first interval into less than Ak pieces, in each of which $\cos nt$ is monotonic. If we do this, and apply the second mean value theorem to the integral over each piece, we see that $|\alpha_n|$ is not greater than the sum of Ak terms of the type

$$\left| \int_\zeta^{\zeta'} \cot\tfrac{1}{2}t\, f(t)\,dt \right|,$$

where $0 \leqslant \zeta < \zeta' \leqslant \eta$. It follows that $\alpha_n \to 0$ when k is fixed and $n \to \infty$. Thus α_n and γ_n behave as in § 2. The remaining term β_n may be dealt with just as in § 2, the only difference being that $c_n(\tau)$ must now be defined as the Fourier cosine coefficient of an even function; so that the theorem requires no further proof. In particular, the allied series is strongly summable at any point at which $f(t)$ is continuous and the integral which expresses its sum exists.

4. We shall now apply to Theorems 1 and 2 a process of formal transformation which is fruitful in the generation of new theorems in the theory of trigonometrical series. The genesis of the principle is naturally to be found in the theory of "Fourier transforms". Taking Theorem 1 in the reduced form

(A) "*if* $\int_0^t |f|^p\,du = O(t)$, $F(t) = \int_0^t f\,du = o(t)$, *then* $\sum_1^n |s_m|^q = o(n)$",

we replace t by n, $f(t)$ by a_n, integration over $(0, t)$ by summation over $(1, n)$, and conversely, so that s_n becomes $F(t)$. We thus obtain

(B) "*if* $\sum_1^n |a_m|^p = O(n), \quad \sum_1^n a_m = o(n),$ *then* $\int_0^t |F|^q \, du = o(t)$".

If now $f \sim \Sigma a_n \cos nt$, then

$$F = \Sigma \frac{a_n}{n} \sin nt = \Sigma b_n \sin nt,$$

say; and, when we express (B) in terms of b_n and F, we obtain

(C) "*if* $\sum_1^n (m|b_m|)^p = O(n), \quad \sum_1^n m b_m = o(n),$ *then* $\int_0^t |F|^q \, du = o(t)$".

Finally, passing from a "reduced" to a "general" form, we are led to formulate

THEOREM 3. *If* $p > 1$ *and*

$$B_n = b_n \cos nx - a_n \sin nx, \tag{4.1}$$

$$\sum_1^n (m|B_m|)^p = O(n), \quad \sum_1^n m B_m = o(n), \tag{4.2}$$

then there is a function $f(t)$, *belonging to the class* L^q *for every* q, *whose Fourier series is*

$$\Sigma A_n = \Sigma(a_n \cos nt + b_n \sin nt); \tag{4.3}$$

and $f(t)$ *satisfies the equation*

$$\int_0^t |\psi(u)|^q \, du = \int_0^t |\tfrac{1}{2}\{f(x+u) - f(x-u)\}|^q \, du = o(t) \tag{4.4}$$

for every q.

The last of our tentative enunciations was, in substance, the special case of Theorem 3 in which $x = 0$. The hypotheses of the theorem are fulfilled, for example, whenever $na_n = o(1)$ and $nb_n = o(1)$, or, again, whenever $na_n = O(1)$, $nb_n = O(1)$, and the allied series ΣB_n is convergent or summable.

It may help to elucidate the relation of Theorem 3 to known theorems if we return for a moment to the reduced form (C), and suppose, in particular, that $nb_n = o(1)$, when the conditions are obviously satisfied. The conclusion involves *a fortiori*

$$\int_0^t F \, du = \Sigma b_n \frac{1 - \cos nt}{n} = o(t)$$

or (changing t to $2t$ and writing a_n again for nb_n)

$$\Sigma a_n \frac{\sin^2 nt}{n^2} = o(t).$$

That this is so whenever $a_n = o(1)$ is a familiar theorem of Riemann*. This theorem is the "transform" of Fejér's theorem, and Theorem 3 is a development of the one as Theorem 1 is of the other.

5. In proving Theorem 3 we may confine ourselves to the case in which $x=0$, $B_n = b_n$. We require some simple lemmas.

LEMMA α. *If*

$$\sum_1^n (m\,|\,c_m\,|)^p = O(n) \tag{5.1}$$

for any positive p, then the same equation is true for any smaller positive p.

This is an immediate corollary of Hölder's inequality.

LEMMA β. *If* (5.1) *is true for a $p>1$, then $\Sigma\,|\,c_n\,|^q$ is convergent for every $q>1$, and*

$$\sum_n^\infty |\,c_m\,|^q = O(n^{1-q}) \tag{5.2}$$

for $1<q\leqslant p$.

If
$$\gamma_{n,m} = \sum_n^m (\nu\,|\,c_\nu\,|)^q,$$

then $\gamma_{n,m} = O(m)$ if $q<p$, by Lemma α. And

$$\sum_n^N |\,c_m\,|^q = \sum_n^{N-1} \gamma_{n,m}\,\Delta m^{-q} + \gamma_{n,N} N^{-q}$$
$$= O\left(\sum_n^N m^{-q}\right) + O(N^{1-q}) = O(n^{1-q}).$$

This proves (5.2). If the series is convergent for any q, it is naturally convergent for any larger q.

* See, *e.g.*, E. W. Hobson, *Theory of functions of a real variable*, second edition (1926), 2, 647.

LEMMA γ. *If c_n is real and satisfies* (5 . 1) *for a $p > 1$, then there is an $f(t)$, belonging to every Lebesgue class L^q, of which c_n is the Fourier (sine or cosine) coefficient.*

This is an immediate corollary of Lemma β and Hausdorff's theorem*.

6. Passing to the proof of Theorem 3, we may suppose, as in § 2, that $q > 2$ and $q' < p$ (or $q > p'$). By Lemma γ,

$$f \sim \Sigma b_n \sin nt$$

exists and belongs to L^q for every q. And if

$$f_n = \sum_1^n b_m \sin mt,$$

then, by Hausdorff's inequality and Lemma β,

$$\int_0^\pi |f - f_n|^q \, dt \leqslant A \left(\sum_{n+1}^\infty |b_n|^{q'} \right)^{q-1} \leqslant \frac{A}{n}, \tag{6 . 1}$$

the A's here being positive numbers depending only on q†.

We suppose that $0 \leqslant u \leqslant t$, $0 < h < k$‡, and write

$$n_1 = h/t, \quad n_2 = k/t,$$

$$f(u) = \sum_1^{n_1} b_m \sin mu + \sum_{n_1+1}^{n_2} b_m \sin mu + \{f(u) - f_{n_2}(u)\}$$

$$= S_1 + S_2 + S_3,$$

say. We have first

$$|S_1| \leqslant u \sum_1^{n_1} m |b_m| \leqslant A n_1 u \leqslant Ah,$$

$$\int_0^t |S_1|^q \, du \leqslant A h^q t;$$

and next

$$\int_0^t |S_3|^q \, du \leqslant \frac{A}{n_2} = \frac{At}{k},$$

by (6 . 1). Finally

$$S_2 = u \sum_{n_1+1}^{n_2} m b_m \frac{\sin mu}{mu}.$$

* The original form due to Young (14) would be sufficient.

† It will not be necessary to distinguish between A's which depend or do not depend on q.

‡ h is to be thought of as small, k as large.

The range of integration may be divided into at most Ak pieces in which the last factor is monotonic. Applying the second mean value theorem to each piece, and using the second condition (4 . 2), we see that

$$S_2 = o(un_2) = o(1),$$

when h and k are fixed and $t \to 0$. It follows that

$$|S_2| < \epsilon, \quad \int_0^t |S_2|^q\,du < \epsilon^q t$$

for $0 < t \leqslant t_1 = t_1(\epsilon, h, k)$.

Hence

$$\left(\int_0^t |f(u)|^q\,du\right)^{1/q} \leqslant \left(\int_0^t |S_1|^q\,du\right)^{1/q} + \left(\int_0^t |S_2|^q\,du\right)^{1/q} + \left(\int_0^t |S_3|^q\,du\right)^{1/q}$$
$$< t^{1/q}(Ah + Ak^{-1/q} + \epsilon),$$

for $0 < t \leqslant t_1$. Our conclusion follows by choice of ϵ, h, k, and t_1.

7. The theorem which corresponds to Theorem 2 as Theorem 3 corresponds to Theorem 1 is

THEOREM 4. *If*

$$\sum_1^n (m|A_m|)^p = O(n) \tag{7.1}$$

and ΣA_n *is convergent to sum* s, *then there is an* $f(t)$, *belonging to* L^q *for every* q, *whose Fourier series is* ΣA_n; *and* $f(t)$ *satisfies*

$$\int_0^t |\phi(u)|^q\,du = o(t) \tag{7.2}$$

for every q.

If we suppose that $x = 0$, $A_n = a_n$, and replace (7 . 1) by the stronger hypothesis $na_n = O(1)$ and (7 . 2) by the weaker conclusion

$$\int_0^t \phi(u)\,du = o(t),$$

we obtain the theorem that

"*if* $na_n = O(1)$ *and* Σa_n *is convergent to sum* s, *then*

$$\Sigma\, a_n \frac{\sin nt}{nt} \to s$$

when $t \to 0$".

This theorem, which is in its turn a generalization of a theorem of Fatou, we proved in our paper **5***. It is the "transform" of Young's theorem concerning the summability of the allied series†.

In proving Theorem 4 we need consider only the reduced case in which $x = 0$, $\phi = f$, and we may suppose also $s = 0$. We now write

$$f \sim \Sigma a_n \cos nt, \quad f_n = \sum_1^n a_m \cos mt,$$

$$f(u) = \sum_1^{n_2} a_m - \sum_1^{n_1} a_m(1-\cos mu) - \sum_{n_1+1}^{n_2} a_m(1-\cos mu) + f(u) - f_{n_2}(u)$$

$$= S_0 + S_1 + S_2 + S_3.$$

Here $S_0 = o(1)$ and

$$|S_2| \leqslant \tfrac{1}{2}u^2 \sum_1^{n_1} m^2 |a_m| \leqslant A u^2 n_1^2 \leqslant A h^2;$$

and the rest of the proof follows the same lines as that of Theorem 3.

8. There is another lemma which enables us to generalize the conditions of Theorems 3 and 4, but which we include rather on account of its intrinsic interest.

LEMMA δ. *If c_n satisfies* (5 . 1), *and Σc_n is summable* (C), *then Σc_n is convergent.*

The conclusion is indeed true whenever Σc_n is summable by Abel's limit, and the lemma may be deduced, in this stronger form, from R. Schmidt's extension‡ of Littlewood's converse of Abel's theorem. As stated, it is a generalization of Hardy's theorem concerning Cesàro summability, and it is interesting to give a direct proof. In order that Σc_n should be summable (C, r), where $r \geqslant 0$, it is necessary and sufficient that Σb_n, where now

$$(8.1) \qquad b_n = \frac{c_n}{n} + \frac{c_{n+1}}{n+1} + \dots \quad (C, r-1),$$

* Hardy and Littlewood, **5**.

† See Hardy and Littlewood, **7**, 212.

‡ Schmidt, **11**. For a more direct proof, see Vijayaraghavan, **13**.

should be summable $(C, r-1)$*. In the present case the series (8 . 1) is (absolutely) convergent, and if $q < p$,

$$|b_n| \leqslant \left(\sum_n^\infty |c_m|^q\right)^{1/q} \left(\sum_n^\infty m^{-q'}\right)^{1/q'} = O\left(\frac{1}{n}\right)$$

by Lemma β. Hence Σb_n is convergent and Σc_n is summable $(C, 1)$.

Also, if $s_n = c_1 + \ldots + c_n$, we have

$$|s_{n+\delta n} - s_n| \leqslant \sum_n^{n+\delta n} |c_m| \leqslant \left(\sum_n^{n+\delta n} m^p |c_m|^p\right)^{1/p} \left(\sum_n^{n+\delta n} m^{-p'}\right)^{1/p} < \epsilon_\delta,$$

where ϵ_δ is a function of δ only which tends to zero when $\delta \to 0$. The convergence of Σc_n now follows by the argument usual in the proof of Hardy's theorem. It is of interest to observe that the result is not true when $p = 1$†.

It follows that, in Theorems 3 and 4, we may replace the hypotheses

$$\text{“}\sum_1^n m b_m = o(n)\text{”}, \quad \text{“}\Sigma A_m \textit{ is convergent}\text{”},$$

by "ΣB_m (*or* ΣA_m) *is summable by some Cesàro mean*".

Additional note (8 *March*, 1927).

We have recently found an alternative proof of Theorems 1 and 2, which seems to us in some ways preferable to that given in the text.

We take Theorem 1 in the "reduced" form, and suppose, as we may, that $q > 2$ and $q > p'$. It is plainly sufficient to show that

$$\mathrm{S}_n^q = \sum_1^n |\mathrm{s}_m|^q = o(n),$$

where
$$\mathrm{s}_n = \int_0^\pi \frac{f(t)}{t} \sin nt\, dt.$$

We write‡

$$c_m = |\mathrm{s}_m|^{q-1} \operatorname{sgn} \bar{\mathrm{s}}_m, \quad C_n(t) = \sum_1^n c_m \sin mt, \quad \Gamma_n = \sum_1^n |c_m|.$$

* Hardy and Littlewood, **6**, 75.

† This may be shown by an example due to Neder (**1**, 180).

‡ $\operatorname{sgn} z = z/|z|$ if $z \neq 0$; $\operatorname{sgn} 0 = 0$; and $\bar{z}$ is the conjugate of z. Note that s_n (which alone occurs in the sequel) is a different symbol from s_n.

Then

$$\text{(1)}\qquad S_n^q = \int_0^\pi \frac{f(t)}{t} C_n(t)\,dt = \int_0^{k/n} + \int_{k/n}^\pi = J_1 + J_2,$$

say. We have clearly

$$|C_n(t)| \leqslant \Gamma_n, \quad |C_n(t)| \leqslant nt\Gamma_n, \quad |C_n'(t)| \leqslant n\Gamma_n.$$

Hence

$$\text{(2)}\qquad J_1 = \left[\frac{F(t)\,C_n(t)}{t}\right]_0^{k/n} - \int_0^{k/n} F(t)\left\{\frac{C_n'(t)}{t} - \frac{C_n(t)}{t^2}\right\} dt$$

$$= o(\Gamma_n) + \int_0^{k/n} o(t)\left\{O\left(\frac{n\Gamma_n}{t}\right) + O\left(\frac{nt\Gamma_n}{t^2}\right)\right\} dt$$

$$= o(\Gamma_n) = o(n^{1/q} S_n^{q/q'}),$$

by Hölder's inequality.

On the other hand

$$\text{(3)}\qquad |J_2| \leqslant \left(\int_0^\pi |C_n(t)|^q\,dt\right)^{1/q} \left(\int_{k/n}^\pi \left|\frac{f(t)}{t}\right|^{q'} dt\right)^{1/q'}$$

$$\leqslant \left(\pi \sum_1^n |c_m|^{q'}\right)^{1/q'} \left(\int_{k/n}^\pi \frac{\psi}{t^{q'}}\,dt\right)^{1/q'} \leqslant \pi S_n^{q/q'} I^{1/q'},$$

where $\psi = |f|^{q'}$, so that

$$\Psi(t) = \int_0^t \psi(u)\,du = \int_0^t |f(u)|^{q'}\,du \leqslant At,$$

where A is a constant. But

$$\text{(4)}\qquad I = \int_{k/n}^\pi \frac{\psi}{t^{q'}}\,dt = \left[\frac{\Psi(t)}{t^{q'}}\right]_{k/n}^\pi + q'\int_{k/n}^\pi \frac{\Psi(t)\,dt}{t^{q'+1}}$$

$$\leqslant \frac{\Psi(\pi)}{\pi^{q'}} + \frac{A_{q'}}{q'-1}\left(\frac{n}{k}\right)^{q'-1} < \{o(1)+\epsilon_k\}\, n^{q'-1},$$

where ϵ_k is a function of k only which tends to zero with k. From (1)–(4) we deduce

$$S_n^{1/q} \leqslant |J_1|^{1/q} + |J_2|^{1/q} \leqslant \{\epsilon_k + o(1)\}\, n^{1/q} S_n^{1/q'},$$

$$S_n < \{\epsilon_k + o(1)\}\, n^{1/q},$$

and the theorem follows. Theorem 2 may be proved similarly.

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12. O. G. Sutton, "On a theorem of Carleman", *Proc. London Math. Soc.* (2), 23 (1925), xlviii-li (*Records* for 24 April, 1924).
13. T. Vijayaraghavan, "A Tauberian theorem", *Journal London Math. Soc.*, 1 (1926), 113–120.
14. W. H. Young, "On the determination of the summability of a function by means of its Fourier constants", *Proc. London Math. Soc.* (2), 12 (1912), 71–88.

COMMENTS

§ 1. For a general account of strong Cesàro summability, see T. M. Flett, *Quart. J. of Math.* (2), 10 (1959), 115–39.

§ 2. More precise results than Theorems 1 and 2 can be obtained by the introduction of fractional integrals (see T. M. Flett, *J. London Math. Soc.* 33 (1958), 311–26, and references given there). For example, for the Fourier series Flett has proved:

Let $1 < p \leqslant 2,\ \alpha \geqslant 0$, *and*

$$\beta = 0 \quad (0 \leqslant \alpha < 1-1/p), \qquad \beta > \alpha-1+1/p \quad (\alpha \geqslant 1-1/p).$$

Let also ϕ_α *be the* (C,α) *mean of* $\phi(t) = \frac{1}{2}\{f(\theta+t)+f(\theta-t)\}$, *and* $\sigma_n^\beta(\theta)$ *be the* (C,β) *mean of the Fourier series of* f *at* θ. *If*

$$\int_0^t |\phi_\alpha(u)-s|^p\,du = O(t) \tag{1}$$

as $t \to 0+$, *then either*

$$\sum_{m=0}^{n} |\sigma_m^\beta(\theta)-s|^{p/(p-1)} = o(n), \tag{2}$$

or the Fourier series of f *at* θ *is not summable* (A). *When* (1) *is satisfied, a necessary and sufficient condition that* (2) *holds is that, for some positive* μ, $\phi_\mu(t) \to s$ *as* $t \to 0+$.

The result of Theorem 1 is false when $p = 1$; a counter-example is given in 1935, 5.

NOTES ON THE THEORY OF SERIES (VII): ON YOUNG'S CONVERGENCE CRITERION FOR FOURIER SERIES

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1. The Fourier series of an integrable function $f(t)$ converges to s, for $t = x$, if and only if

$$J(n) = \frac{1}{\pi}\int_0^\pi \phi(t)\,\frac{\sin(n+\frac{1}{2})\,t}{\sin\frac{1}{2}t}\,dt,$$

where
$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\},$$

tends to zero when $n \to \infty$.

One of the most interesting of the more modern criteria for convergence is that of Young: *it is sufficient that*

$$\phi(t) \to 0 \tag{1.1}$$

and

$$\chi(t) = \int_0^t |\,d(u\phi)\,| = O(t) \tag{1.2}$$

for small t [*i.e.* that the variation of $u\phi(u)$ in $(0,\ t)$ is $O(t)$ for small t]. The theorem was proved in this form by Young in the *Proceedings* in 1916†; in an earlier note in the *Comptes rendus* he had stated the criterion a little less generally‡. The condition (1.2), which we shall refer to as Y, may be stated in alternative forms, such as (a) that the variation of $t^c\phi$ is $O(t^c)$, for some (or any) positive c, or (b) that the variation of ϕ in $(t,\ 2t)$ is $O(1)$; and it plainly includes (and is more general than) the ordinary (Jordan's) condition that ϕ is of bounded variation in some interval $(0,\ \delta)$.

† Young, **18**.
‡ Young, **15**.

It is natural to ask whether the condition (1 . 1) cannot be generalized. If we write

(1 . 3) $$\phi_1(t) = \int_0^t \phi(u)\,du, \qquad \phi_2(t) = \int_0^t \phi_1(u)\,du, \qquad \ldots,$$

we may say that

(1 . 4) $$\phi(t) \to 0 \quad (C, r)$$

if

(1 . 5) $$\phi_r(t) = o(t^r).$$

It is stated by Young, in a later note in the *Comptes rendus*†, that (1 . 1) may be replaced by (1 . 4); but apparently no proof of this theorem, in its general form, has been published. The case $r = 1$ has, however, been dealt with recently by Pollard‡, who gives a concise direct proof, and also shows that, in this form, Young's criterion may be deduced from a generalization of the well known criterion of Lebesgue§.

There is another direction in which the criterion may be developed. We may strengthen Y by demanding that (1 . 2) shall hold, not merely for small t, but for all t of $(0, \pi)$: we shall call the more stringent condition Y^*. It is plain that ϕ satisfies Y^* if, and only if (1) it satisfies Y, and (2) it is of bounded variation in any interval $0 < \delta \leqslant t \leqslant \pi$. It was observed by Young|| that Y^*, together with (1 . 1), implies more than convergence, and in fact summability $(C, -1+\delta)$ for every positive δ, though he gives only a very summary indication of the proof.

All these writings have points of contact with our own recent researches. Our first object here is to prove a theorem (Theorem 1) which includes comprehensively all the results to which we have referred. We also prove (Theorem 3) what may be described as the "transform" of Young's theorem in its most general form, and some other theorems which dispose of points of incidental interest.

2. THEOREM 1. *If $\phi(t)$ satisfies Y, then the Fourier series is convergent whenever it is summable by any Cesàro mean. The necessary and sufficient condition for convergence (to sum s) is*

(2 . 1) $$\phi_1(t) = o(t).$$

† Young, **17**.

‡ Pollard, **13**.

§ It had already been shown by Hardy (**3**) that Young's criterion, in its original form, can be deduced from Lebesgue's.

|| Young, **18**, 211.

If, further, $\phi(t)$ satisfies Y^*, then the series is summable $(C, -1+\delta)$, for every positive δ, whenever it is summable by any Cesàro mean.

If ϕ satisfies Y, so that (1 . 2) is true (say) in $(0, \xi)$, f^* is equal to f in $(x-\xi, x+\xi)$ and to 0 outside, and ϕ^* is the ϕ of f^*, then ϕ^* satisfies Y^*. The Fourier series of $f-f^*$ converges to 0 for $t=x$. It is therefore sufficient to consider a function subject to Y^*, and to prove that the series is then summable $(C, -1+\delta)$ if summable at all, and that (2 . 1) is the necessary and sufficient condition for summability.

We use four known theorems which we state as lemmas†.

LEMMA 1. *If a series is summable (C) and bounded $(C, k+\delta)$, where $k > -1$, for every positive δ, then it is summable $(C, k+\delta)$ for every positive δ.*

LEMMA 2. *If $\phi(t)$ is bounded and $\phi(t) \to 0$ (C)‡, then $\phi(t) \to 0$ $(C, 1)$.*

LEMMA 3. *The necessary and sufficient condition that the Fourier series should be summable (C) is that $\phi(t) \to 0$ (C).*

LEMMA 4. *If $-1 < \gamma < 1$, $\phi(t) \sim \frac{1}{2}a_0 + \Sigma a_n \cos nt$, and $s_n^{(\gamma)}$ is the (C, γ) mean of Σa_n, then*

(2 . 2) $$s_n^{(\gamma)} = \frac{1}{\pi}\int_0^\pi \phi(t)\,\Omega(t)\,dt,$$

where

(2 . 3) $$\Omega(t) = \Omega_1(t) + \Omega_2(t),$$

(2 . 4) $$\Omega_1(t) = 2^{-1-\gamma}\,\frac{\Gamma(n+1)\,\Gamma(1+\gamma)}{\Gamma(n+1+\gamma)}\,\frac{\sin\{(n+\frac{1}{2}+\frac{1}{2}\gamma)\,t - \frac{1}{2}\gamma\pi\}}{(\sin\frac{1}{2}t)^{1+\gamma}},$$

(2 . 5) $$|\Omega(t)| \leqslant An, \qquad |\Omega_2(t)| \leqslant \frac{A}{nt^2},$$

and the A's are independent of n and t§.

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† Lemmas 1 and 2 are propositions of a well-known type. Lemma 1 is proved, in the form required here, by Andersen, **1**, 56. For Lemma 2, see Hardy and Littlewood, **8**, 83. The result is proved there for means of $\phi(t)$ of "Hölder" type, but the equivalence of these to those of Cesàro type is familiar. See Landau, **11**: in any case the proof is easily adapted.

Lemma 3 is the main result of our memoir **8**. It is not actually necessary here to use the full force of this theorem; we only need Lebesgue's theorem that $\phi_1 = o(t)$ involves summability $(C, 2)$, and Riemann's theorem that convergence implies $\phi_2 = o(t^2)$.

Finally Lemma 4, which we quote from Zygmund, **19**, is due substantially to Kogbetliantz, **10**. The most natural proof is that of Kogbetliantz, by Cauchy's theorem; but a proof by real analysis is given by Szegö, **14**.

‡ That is to say, $\phi \to 0$ (C, r) for some r.

§ It should be observed that we must not replace $\sin\frac{1}{2}t$ by $\frac{1}{2}t$ in the denominator of $\Omega_1(t)$.

3. We prove first that $s_n^{(\gamma)}$ is bounded for any $\gamma > -1$. We write

$$(3.1) \qquad s_n^{(\gamma)} = \frac{1}{\pi}\int_0^{1/n} \phi\Omega\,dt + \frac{1}{\pi}\int_{1/n}^{\pi} \phi\Omega_1\,dt + \frac{1}{\pi}\int_{1/n}^{\pi} \phi\Omega_2\,dt = J_1+J_2+J_3.$$

Since ϕ satisfies Y^* it is bounded, and so, by (2.5),

$$(3.2) \qquad J_1 = O\left(n\int_0^{1/n} dt\right) = O(1), \qquad J_3 = O\left(\frac{1}{n}\int_{1/n}^{\pi} \frac{dt}{t^2}\right) = O(1).$$

Hence we have only to prove that J_2 is bounded.

We observe first that Y^* is equivalent to

$$\int_0^t \left| d\left(\sin\tfrac{1}{2}u\,\phi(u)\right)\right| = O(t).$$

If now we write

$$\Lambda = \Lambda(n, t) = \int_t^{\pi} \frac{\sin\{(n+\frac{1}{2}+\frac{1}{2}\gamma)\,u - \frac{1}{2}\gamma\pi\}}{(\sin\frac{1}{2}u)^{2+\gamma}}\,du,$$

we have
$$\Lambda = O(n^{-1}t^{-2-\gamma}),$$

by the second mean-value theorem. Hence

$$(3.3) \qquad J_2 = O\left(n^{-\gamma}\left|\int_{1/n}^{\pi} \sin\tfrac{1}{2}t\,\phi(t)\,d\Lambda\right|\right) = O(n^{-\gamma}|K|),$$

say, where

$$K = [\sin\tfrac{1}{2}t\,\phi(t)\,\Lambda]_{1/n}^{\pi} + \int_{1/n}^{\pi} \Lambda\,d(\sin\tfrac{1}{2}t\,\phi) = K' + K''.$$

Here

$$(3.4) \qquad K' = O(n^{-1}) + O(n^{-1}\,.\,n^{-1}n^{2+\gamma}) = O(n^{\gamma}).$$

Also, if we write

$$\sin\tfrac{1}{2}t\,\phi = \psi, \qquad \Psi = \int_0^t |d\psi| = O(t),$$

we have

$$(3.5) \qquad |K''| \leqslant \int_{1/n}^{\pi} |\Lambda|\,|d\psi| \leqslant \frac{A}{n}\int_{1/n}^{\pi}\frac{d\Psi}{t^{2+\gamma}} = \frac{A}{n}\left[\frac{\Psi}{t^{2+\gamma}}\right]_{1/n}^{\pi} + \frac{A}{n}\int_{1/n}^{\pi}\frac{\Psi\,dt}{t^{3+\gamma}}$$
$$\leqslant \frac{A}{n}\frac{\Psi(\pi)}{\pi^{2+\gamma}} + O\left(\frac{1}{n}\int_{1/n}^{\pi}\frac{dt}{t^{2+\gamma}}\right) = O(n^{\gamma}).$$

From (3.3), (3.4), and (3.5) it follows that J_2, and so $s_n^{(\gamma)}$, is bounded.

The proof of Theorem 1 is now immediate. The series is bounded (C, γ) for all $\gamma > -1$, and therefore, by Lemma 1, summable (C, γ) if summable at all. The necessary and sufficient condition for summability (C, γ) is therefore the same as for summability (C). This condition is, by Lemma 3, that $\phi \to 0$ (C). But ϕ is bounded, and therefore, by Lemma 2, it is necessary and sufficient that (2.1) be true.

4. It is of some interest to observe that (2 . 1) is a consequence of the summability of the series by some negative mean, whether Young's condition be satisfied or not. It is known that it is not a consequence of mere convergence†.

THEOREM 2. *If Σa_n is summable $(C, -k)$, where $0<k<1$, to sum s, then*

$$S(t) = \Sigma a_n \frac{\sin nt}{nt} \to s$$

when $t \to 0$.

This is naturally equivalent to (2 . 1) when $\phi(t)$ and a_n are defined as in Theorem 1. We suppose that $s=0$, $a_0=0$, and write

$$c_n = s_n^{(-k)} = o(n^{-k}),$$

$$\gamma_n = \frac{\Gamma(n+k)}{\Gamma(k)\Gamma(n+1)} = O(n^{k-1}),$$

$$\Delta u_n = u_n - u_{n+1}, \qquad \phi_n(t) = \Delta \frac{\sin nt}{nt}.$$

Then‡

$$s_n = a_1+a_2+\ldots+a_n = \sum_{\nu=1}^{n} \gamma_{n-\nu} c_\nu$$

and

$$S(t) = \sum_{n=1}^{\infty} s_n \phi_n(t) = \sum_{n=1}^{\infty} \phi_n(t) \sum_{\nu=1}^{n} \gamma_{n-\nu} c_\nu \tag{4.1}$$

$$= \sum_{\nu=1}^{\infty} c_\nu \sum_{n=\nu}^{\infty} \gamma_{n-\nu} \phi_n(t) = \sum_{\nu=1}^{\infty} c_\nu T_\nu,$$

say, if we may invert the order of summation. This inversion is not entirely trivial, since the double series is not absolutely convergent. We have, however,

$$S(t) = \lim_{N\to\infty} \sum_{n=1}^{N} \sum_{\nu=1}^{n} = \lim_{N\to\infty} \sum_{\nu=1}^{N} \sum_{n=\nu}^{N}.$$

Now, for a fixed $t>0$, we have

$$\sum_{n=N+1}^{\infty} \gamma_{n-\nu} \phi_n(t) = O\left\{(N-\nu+1)^{k-1} \operatorname*{Max}_{m>N} \left| \sum_{N+1}^{m} \phi_n(t) \right| \right\}$$

$$= O\left\{\frac{(N-\nu+1)^{k-1}}{N}\right\}$$

† See Hardy and Littlewood, **4**, 224-227, and **5**, 255-261.

‡ Andersen, **1**, 6.

and

$$\frac{1}{N}\sum_{\nu=1}^{N}|c_\nu|(N-\nu+1)^{k-1}=O\left\{\frac{1}{N}\sum_{\nu=1}^{N}\nu^{-k}(N-\nu+1)^{k-1}\right\}=O\left(\frac{1}{N}\right).$$

Hence
$$\sum_{\nu=1}^{N}\sum_{n=N+1}^{\infty}$$

exists and tends to zero when $N\to\infty$, so that

$$S(t)=\lim_{N\to\infty}\sum_{\nu=1}^{N}\sum_{n=\nu}^{\infty}=\sum_{\nu=1}^{\infty}\sum_{n=\nu}^{\infty}.$$

We have thus proved (4 . 1).

We shall now prove that

(4 . 2) $$T_\nu=O(t^{1-k})\quad(\nu t\leqslant 1),\qquad T_\nu=O(\nu^{-1}t^{-k})\quad(\nu t>1),$$

uniformly in ν and t. If this be granted, the theorem follows, since

$$S(t)=O(t^{1-k})\sum_{\nu\leqslant 1/t}o(\nu^{-k})+O(t^{-k})\sum_{\nu>1/t}o(\nu^{-k-1})=o(1).$$

To prove (4 . 2) we write

$$\rho=\left[\frac{1}{t}\right],\qquad T_\nu=\sum_{\nu}^{\nu+\rho}+\sum_{\nu+\rho+1}^{\infty}=U_\nu+V_\nu.$$

Then, first,

$$V_\nu=\sum_{\nu+\rho+1}^{\infty}\gamma_{n-\nu}\phi_n(t)=\gamma_{\rho+1}\frac{\sin(\nu+\rho+1)t}{(\nu+\rho+1)t}-\sum_{\nu+\rho+1}^{\infty}\frac{\sin(n+1)t}{(n+1)t}\Delta\gamma_{n-\nu}$$

$$=O\left\{\frac{1}{1+\nu t}\left[\rho^{k-1}+\sum_{\nu+\rho+1}^{\infty}(n-\nu+1)^{k-2}\right]\right\}=O\left(\frac{\rho^{k-1}}{1+\nu t}\right)=O\left(\frac{t^{1-k}}{1+\nu t}\right),$$

and so satisfies inequalities like those in (4 . 2). Secondly, in U_ν, we use the inequalities

$$\phi_n(t)=O(n^{-1})\quad(nt\geqslant 1),\qquad \phi_n(t)=O(nt^2)\quad(nt\leqslant 2),$$

according as $\nu t>1$ or $\nu t\leqslant 1$. If $\nu t>1$, then

$$U_\nu=\sum_{\nu}^{\nu+\rho}\gamma_{n-\nu}\phi_n(t)=O\left\{\frac{1}{\nu}\sum_{\nu}^{\nu+\rho}(n-\nu+1)^{k-1}\right\}=O(\nu^{-1}\rho^k)=O(\nu^{-1}t^{-k}).$$

If $\nu t\leqslant 1$, then $(\nu+\rho)t\leqslant 2$ and

$$U_\nu=O\left\{t^2\sum_{\nu}^{\nu+\rho}n(n-\nu+1)^{k-1}\right\}=O\left\{t^2\left(\nu+\frac{1}{t}\right)\rho^k\right\}=O(t^{1-k}).$$

Thus U_ν satisfies (4 . 2), and this completes the proof.

5. We now apply to Young's theorem the heuristic process of "transformation" explained in § 4 of our fourth note†. The theorem to which we are ultimately led is one concerning the summability of the allied series of a Fourier series, viz.

THEOREM 3. *Suppose that the Fourier constants of $f(t)$ satisfy*

$$\sum_1^n \nu(|a_\nu|+|b_\nu|) = O(n). \tag{5.1}$$

Then the series $\Sigma B_n = \Sigma(b_n \cos nx - a_n \sin nx)$, the allied series of the Fourier series of $f(t)$ for $t = x$, is summable (C, δ) for every positive δ or not summable (C), and the necessary and sufficient condition for summability is that

$$\int_0^\pi \frac{f(x+t)-f(x-t)}{t}\,dt \tag{5.2}$$

should exist as a Cauchy integral.

It is sufficient to prove this in the "reduced"‡ form

THEOREM 3A. *If $\psi(t)$ is an odd function whose Fourier series is $\Sigma a_n \sin nt$, and*

$$\sum_1^n \nu|a_\nu| = O(n), \tag{5.3}$$

then Σa_n is summable (C, δ) for every positive δ or not summable (C); and the necessary and sufficient condition for summability is the existence of

$$\int_0^\pi \frac{f(t)}{t}\,dt \tag{5.4}$$

as a Cauchy integral.

We again require certain known theorems, which we state as lemmas.

LEMMA 5. *If Σa_n is summable (C), and satisfies (5.3), then it is bounded, and summable (C, δ)*§.

Let $t_n^{(r)}$ be the r-th Cesàro sum formed from $\Sigma n a_n$. Then (5.3) involves *a fortiori*

$$t_n^{(0)} = \sum_1^n \nu a_\nu = O(n),$$

† Hardy and Littlewood, **7**.

‡ Cf. Hardy and Littlewood, **9**.

§ But not necessarily convergent, as may be shown by an example due to Neder, **12**, 180. On the other hand (as may be seen from the proof which follows) the hypothesis (5.3) of Lemma 5 may be replaced by

$$\sum_1^n \nu a_\nu = O(n).$$

and so $t_n^{(r)} = O(n^{r+1})$ for $r \geqslant 0$. But†

$$\frac{t_n^r}{n^{r+1}} = \frac{n+r+1}{n} \frac{s_n^r}{n^r} - (r+1) \frac{s_n^{r+1}}{n^{r+1}}.$$

Hence Σa_n, if bounded $(C, r+1)$, is bounded (C, r). It is therefore bounded $(C, 0)$, and therefore, by Lemma 1, summable (C, δ).

LEMMA 6. *The necessary and sufficient condition that Σa_n should be summable (C) is that the integral* (5 . 4) *should be summable (C)*‡.

LEMMA 7. *If*

(5 . 5) $$\int_0^t f(u)\, du = o(t),$$

then the integral (5 . 4), *if summable, is convergent.*

We prove first that, if (5 . 4) is summable $(C, 1)$, it is convergent. We write

$$\int_t^\pi \frac{f}{u}\, du = \chi(t).$$

Then, if δ is any number between t and π, we have

$$t\chi = t\int_t^\delta \frac{f}{u}\, du + t\int_\delta^\pi \frac{f}{u}\, du = \int_t^\tau f\, du + t\int_\delta^\pi \frac{f}{u}\, du,$$

where $t \leqslant \tau \leqslant \delta$, and we can make this as small as we please by choice first of δ and then of t. Hence $t\chi \to 0$. It then follows by partial integration that

$$\frac{1}{t}\int_0^t \chi\, du = \frac{1}{t}[u\chi]_0^t + \frac{1}{t}\int_0^t f\, du = \chi + \frac{1}{t}\int_0^t f\, du$$

or

(5 . 6) $$\psi = \frac{1}{t}\int_0^t f\, du = \frac{1}{t}\int_0^t \chi\, du - \chi.$$

Since $\psi \to 0$, by (5 . 5), χ tends to a limit in the ordinary sense if it tends to a limit $(C, 1)$.

It is plain, moreover, that (5 . 6) contains a proof of the lemma in its general form. For from (5 . 6) it follows that

$$\frac{1}{t}\int_0^t \psi\, du = \frac{1}{t}\int_0^t \frac{du}{u}\int_0^u \chi\, dw - \frac{1}{t}\int_0^t \chi\, du,$$

so that χ, if it tends to a limit $(C, 2)$, leads to one $(C, 1)$; and so generally.

† Hardy, **2**, 304.

‡ This is substantially the principal theorem of our paper **6**. See p. 214 for the definition of summability of the integral.

6. Passing to the proof of the theorem, we observe first that the *sufficiency* of the condition results immediately from Lemmas 5 and 6, since Σa_n is summable by Lemma 6 and summable (C, δ) by Lemma 5. We have thus only to establish its *necessity*. If Σa_n is summable, the integral is summable, by Lemma 6, and its convergence will follow from Lemma 7 if we can prove (5 . 5).

Now (5 . 3) and the summability (C) of Σa_n imply (by Lemma 5) the boundedness of s_n and the summability of the series $(C, 1)$; and

$$\int_0^t f\,du = \Sigma a_n \frac{1-\cos nt}{n}.$$

Hence everything is reduced to the proof of the following lemma.

LEMMA 8. *If* $s_n = O(1)$ *and* $s_1+s_2+\ldots+s_n = o(n)$, *then*

$$S(t) = \Sigma a_n \frac{1-\cos nt}{nt} \to 0.$$

7. We choose positive numbers h (small) and H (large). O's are uniform in t, h, H. We write

$$\begin{aligned} S &= \sum_{n \leqslant H/t} + \sum_{n > H/t} \\ &= \sum_{n \leqslant h/t} s_n \Delta \frac{1-\cos nt}{nt} + \sum_{h/t < n < \nu} s_n \Delta \frac{1-\cos nt}{nt} \\ &\qquad + s_\nu \frac{1-\cos \nu t}{\nu t} + \sum_{n > H/t} a_n \frac{1-\cos nt}{nt} = S_1+S_2+S_3+S_4, \end{aligned}$$

say, where $\nu = [H/t]$. Since (5 . 3) involves

$$\sum_n^\infty \frac{|a_\nu|}{\nu} = O\left(\frac{1}{n}\right) \dagger,$$

we have
$$S_4 = O\left(\sum_{n > H/t} \frac{|a_n|}{nt}\right) = O\left(\frac{1}{t}\frac{t}{H}\right) = O\left(\frac{1}{H}\right);$$

and
$$S_3 = O\left(\frac{1}{H}\right)$$

because s_n is bounded. In S_1

$$\Delta \frac{1-\cos nt}{nt} = O(t),$$

and so
$$S_1 = O\left(t \sum_{n \leqslant h/t} 1\right) = O\left(t \cdot \frac{h}{t}\right) = O(h).$$

† See Hardy and Littlewood, **9** (Lemma α.)

We can therefore choose h and H so that $S_1+S_3+S_4$ is as small as we please independently of t, and we have only to show that S_2 tends to zero with t when h and H are fixed.

Now

$$\Delta \frac{1-\cos nt}{nt} = \frac{1}{n+1}\,\frac{1-\cos nt}{nt} - \frac{n}{n+1}(1-\cos t)\frac{\cos nt}{nt} - \frac{n}{n+1}\sin t\,\frac{\sin nt}{nt}.$$

This equation decomposes S_2 into three parts, of which the last is

$$-\sin t \sum_{h/t<n<\nu} \frac{n}{n+1} s_n \frac{\sin nt}{nt} = O\left(t \operatorname{Max}\left|\sum_l^m s_n \frac{\sin nt}{nt}\right|\right),$$

where l and m vary in the range $(h/t,\ H/t)$. We can split up the range of summation $(l,\ m)$ into $O(H)$ pieces in each of which the coefficient of s_n is monotonic. Applying the second mean value theorem to each of them, we obtain

$$O\left(Ht \operatorname{Max}\left|\sum_l^m s_n\right|\right) = o\left(Ht\,.\,\frac{1}{t}\right) = o(1).$$

The same argument is applicable with trivial variations to the other sums. Hence $S_2 = o(1)$, which proves Lemma 8, and so Theorem 3A.

Theorems 3 and 3A should be compared with Theorems 2 and 2A of our note **9**. There more is assumed, and the conclusion (the convergence of the allied series) is correspondingly stronger.

8. We conclude with two remarks of a negative character. It is easy to prove that the Fourier constants of a function subject to Y^* are $O(\log n/n)$. It is natural to ask whether they are not, like those of a function of bounded variation, $O(1/n)$, since in this event Theorem 1 would become much more trivial. A *Gegenbeispiel* is given by the function $f(t)$ which is equal to 1 in the intervals $(\pm 10^{-2k-1}\alpha,\ \pm 10^{-2k}\alpha)$, where $k = 0,\ 1,\ 2,\ \ldots$ and α is chosen appropriately, and otherwise is equal to 0. The Fourier constant a_n of this function is a multiple of

$$\frac{1}{n}\left(\sin n\alpha - \sin\frac{n\alpha}{10} + \sin\frac{n\alpha}{10^2} - \ldots\right),$$

and it is easily proved that, for appropriate α, the series in brackets is not bounded in n.

It is also natural to ask whether the condition (2.1) cannot be replaced by the *continuity of* $\phi(t)$. The answer is easily seen to be negative. An affirmative answer would imply that Y^* and (2.1) necessarily involve the continuity of $\phi(t)$.

But suppose, for example, that δ_ν is the interval

$$(2^{-\nu}-2^{-2\nu},\ 2^{-\nu}) \quad (\nu = 1, 2, \ldots);$$

that $f(x)$ is 0 outside the intervals δ_ν, and that in each interval it rises symmetrically and linearly from 0 at the ends to 1 in the middle. Consider now the point $x = 0$, and suppose that $s = 0$. Then for $t > 0$ we have $\phi(t) = \frac{1}{2}f(t)$. The variation of $\phi(t)$ in $(2^{-\nu-1}, 2^{-\nu})$ is 1, and its variation in any interval $(t, 2t)$ is less than 2. Hence $\phi(t)$ satisfies Y^*. Also

$$\phi_1(t) \leqslant \sum_{2^{-\nu} \leqslant 2t} \tfrac{1}{2} \,.\, 2^{-2\nu} = O(t^2) = o(t),$$

so that $\phi(t)$ satisfies (2 . 1). But $\phi(t)$ is plainly not continuous at the origin. This *Gegenbeispiel* is adapted, by the transformation process, from Neder's, quoted above†.

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† See p. 307, f.n. §.

CORRECTION

§ 6. The condition (5.3) should be added to the hypotheses of Lemma 8.

COMMENT

§ 4. Concerning Theorem 2, see the comments on 1924, 1.

THE SUMMABILITY OF A FOURIER SERIES BY LOGARITHMIC MEANS

By G. H. HARDY

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1. There is a simple necessary and sufficient condition for the Cesàro summability of the Fourier series of a bounded function. Suppose as usual that $f(t)$ is integrable in $(-\pi, \pi)$ and that

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos nt + b_n \sin nt). \tag{1.1}$$

Further, suppose that $f(t)$ is bounded near $t = x$, and that the question is that of the summability of the series, for $t = x$, to sum s. Then the condition

$$\int_0^t \phi(u)\, du = o(t) \tag{1.2}$$

or

$$\phi(t) = o(1) \qquad (C, 1), \tag{1.21}$$

where $t > 0$ and $\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\}$,

is necessary for summability by Cesàro means of any order, and sufficient for summability by Cesàro means of every positive order.*

There are simple bounded functions which do not satisfy this condition. Suppose, for example, that $x = 0$ and

$$f(t) = \cos(a \log |t|), \tag{1.3}$$

where $a > 0$. Then $b_n = 0$, while a_n behaves like

$$\frac{A \cos(a \log n) + B \sin(a \log n)}{n},$$

where A and B are constants, and $\sum a_n$ is not summable (C). The condition (1.2) is naturally not satisfied; all means of $f(t)$, round $t = 0$, oscillate, substantially like $f(t)$ itself.

The series $\sum a_n$ is, however, summable by Riesz's logarithmic means;† if

$$s_n = \tfrac{1}{2}a_0 + a_1 + a_2 + \ldots + a_n, \tag{1.4}$$

then

$$\frac{S_n}{\log n} = \frac{1}{\log n}\left(s_1 + \frac{s_2}{2} + \ldots + \frac{s_n}{n}\right) \tag{1.5}$$

tends to a limit s when $n \to \infty$, in which circumstances we write

* Hardy and Littlewood (**3**: see also **4**).

† For the general properties of Riesz's 'typical means' see the tract by Hardy and Riesz, and various papers by Zygmund in the *Math. Zeitschrift* and the *Bulletin de l'Acad. Polonaise.*

1931, 3 *Quarterly Journal of Mathematics*, 2, 107–12.

$$\tfrac{1}{2}a_0 + \sum a_n = s \qquad (R, 1). \tag{1.6}$$

These facts suggest that we should try to find some general theorem, of the type of that just quoted, concerning the Rieszian summability of the Fourier series of a bounded function. Actually Theorem A below covers a rather wider class of functions.

Some theorems concerning the Rieszian summability of trigonometrical series are known already. In particular Zygmund* has shown that the Fourier series of any integrable function, bounded or not, is summable $(R, 1)$ whenever (1.2) is satisfied; and he and Jacob† have investigated more general theorems of the same type. But none of these theorems covers such a function as (1.3), or is quite of the type desired.‡

2. THEOREM A. *Suppose that*

$$\Phi(t) = \int_0^t |\phi(u)|\, du = o\left(t \log \frac{1}{t}\right) \tag{2.1}$$

(*a condition satisfied whenever* $\phi(t) = o\left(\log \frac{1}{t}\right)$ *and in particular when* $f(t)$ *is bounded near* $t = x$). *Then a necessary and sufficient condition that the series should be summable* $(R, 1)$, *for* $t = x$, *to sum* s, *is that*

$$\psi(t) = \int_t^\pi \frac{\phi(u)}{u}\, du = o\left(\log \frac{1}{t}\right) \tag{2.2}$$

when $t \to 0$.

It is easily verified that (2.2) is satisfied whenever (1.2) is satisfied, and that the function (1.3) satisfies (2.2) but not (1.2). In proving the theorem we may make the usual simplifications, supposing that $f(t)$ is even and that $a_0 = 0$, $x = 0$, $s = 0$, so that $\phi(t) = f(t)$.

* Zygmund (**7**).

† Zygmund (**8**), Jacob (**5**).

‡ Though in other respects they are more general, since they apply to series which are derived series of Fourier series but not Fourier series themselves.

It may be useful to recall that (1.2) is not, when $f(t)$ is otherwise unrestricted, a sufficient condition for summability $(C, 1)$, though it is sufficient for summability $(C, 1+\delta)$, for any positive δ, or (in virtue of Zygmund's theorem) for summability $(R, 1)$. A series summable $(C, 1)$ is necessarily summable $(R, 1)$, but a series summable $(C, 1+\delta)$, for every positive δ, is not, so that Zygmund's theorem is not a corollary of the known results concerning Cesàro summability.

(i) *Proof that the condition is sufficient.* It is known* that if (as here) $f(x)$ is even and integrable, and $a_0 = 0$, then

$$\chi(x) = \int_x^\pi f(u)\,\tfrac{1}{2}\cot\tfrac{1}{2}u\,du \tag{2.3}$$

is also even and integrable, and

$$\chi(x) \sim \tfrac{1}{2}b_0 + \sum b_n \cos nx, \tag{2.31}$$

where
$$b_n = \frac{a_1+a_2+\ldots+a_{n-1}+\frac{1}{2}a_n}{n} \tag{2.32}$$

for $n > 0$. It is plain in the present case, from (2.2), that

$$\chi(x) = o\left(\log\frac{1}{x}\right) \tag{2.33}$$

for small positive x.

We write

$$s_n = a_1+a_2+\ldots+a_n, \qquad \sigma_n = s_1+s_2+\ldots+s_{n-1}, \tag{2.41}$$

$$t_0 = \tfrac{1}{2}b_0, \quad t_n = \tfrac{1}{2}b_0+b_1+\ldots+b_n\ (n>0), \quad \tau_n = t_0+t_1+\ldots+t_{n-1}. \tag{2.42}$$

Then

$$\sigma_n = \frac{1}{\pi}\int_0^\pi f(x)\frac{\sin^2\frac{1}{2}nx}{\sin^2\frac{1}{2}x}\,dx, \qquad \tau_n = \frac{1}{\pi}\int_0^\pi \chi(x)\frac{\sin^2\frac{1}{2}nx}{\sin^2\frac{1}{2}x}\,dx. \tag{2.43}$$

From (2.32) and (1.5)

$$\frac{s_n}{n} = b_n + o\left(\frac{1}{n}\right), \qquad S_n = t_n + o(\log n). \tag{2.5}$$

From (2.33) and (2.43),

$$\tau_n = \frac{1}{\pi}\int_0^{1/n} o\left(\log\frac{1}{x}\right)n^2\,dx + \frac{1}{\pi}\int_{1/n}^\pi o\left(\log\frac{1}{x}\right)\frac{dx}{x^2}$$

$$= o\left(n^2 . \frac{1}{n}\log n\right) + o(n\log n) = o(n\log n).$$

Combining this with (2.42) and (2.5), we obtain

$$S_1+S_2+\ldots+S_n = \tau_n + o(n\log n) = o(n\log n). \tag{2.6}$$

But
$$(n+1)S_n - S_1 - S_2 \ldots - S_n = s_1+s_2+\ldots+s_n,$$

and so, by (2.6),

$$S_n = \frac{S_1+S_2+\ldots+S_n}{n+1} + \frac{\sigma_n}{n+1} = \frac{\sigma_n}{n+1} + o(\log n). \tag{2.7}$$

* Hardy (**2**).

On the other hand

$$\frac{\sigma_n}{n} = \frac{4}{n\pi}\int_0^\pi f(x)\frac{\sin^2\frac{1}{2}nx}{x^2}\,dx + o(1) = J + o(1),$$

say, and

$$|J| \leqslant \frac{4}{n\pi}\int_0^{1/n} |f(x)|\frac{\sin^2\frac{1}{2}nx}{x^2}\,dx + \frac{4}{n\pi}\int_{1/n}^\pi \frac{|f(x)|}{x^2}\,dx = J_1 + J_2,$$

say. Here

$$J_1 \leqslant \frac{n}{\pi}\int_0^{1/n} |f(x)|\,dx = \frac{n}{\pi}\Phi\left(\frac{1}{n}\right) = o(\log n),$$

by (2.1), and

$$J_2 = \frac{4}{n\pi}\int_{1/n}^\pi \frac{\Phi'(x)}{x^2}\,dx < \frac{8}{n\pi}\int_{1/n}^\pi \frac{\Phi(x)}{x^3}\,dx + o(1)$$

$$= \frac{1}{n}\int_{1/n}^\pi \frac{o\left(\log\frac{1}{x}\right)}{x^2}\,dx + o(1) = o(\log n).$$

Collecting our results, we see that

$$\frac{\sigma_n}{n} = o(\log n),$$

and so, from (2.7), $S_n = o(\log n)$, the result required.

3. (ii) *Proof that the condition is necessary.* We now assume that $S_n = o(\log n)$ and deduce (2.2). We begin by showing, without using (2.1), that

$$\chi(x) = o\left(\log\frac{1}{x}\right) \qquad (C, 2), \tag{3.1}$$

i.e. that

$$\chi_2(x) = \int_0^x du \int_0^u \chi(v)\,dv = o\left(x^2 \log\frac{1}{x}\right). \tag{3.11}$$

The proof is much like that of Riemann's classical theorem. We have

$$2\frac{\chi_2(x)}{x^2} = \frac{1}{2}b_0 + \sum_1^\infty b_n\left(\frac{\sin\frac{1}{2}nx}{\frac{1}{2}nx}\right)^2$$

$$= \sum_0^\infty t_n\{g(nx) - g[(n+1)x]\},$$

where

$$g(u) = \left(\frac{\sin u}{u}\right)^2;$$

and so

$$2\frac{|\chi_2(x)|}{x^2} \leqslant \sum_0^\infty |t_n| \int\limits_{nx}^{(n+1)x} |g'(u)|\,du = \sum_0^\infty |t_n| I_n, \tag{3.2}$$

say. But $g'(u)$ is $O(u)$ for small u, and $O(u^{-2})$ for large u; and so

$$I_n = O(nx^2)\ (nx \leqslant 1), \qquad I_n = O\left(\frac{1}{n^2x}\right)\ (nx > 1).$$

Substituting in (3.2) and observing that, after (2.5), $t_n = o(\log n)$, we obtain

$$\frac{\chi_2(x)}{x^2} = \sum_{nx\leqslant 1} o(\log n)O(nx^2) + \sum_{nx>1} o(\log n)O\left(\frac{1}{n^2x}\right)$$

$$= o\left(x^2 \sum_{nx\leqslant 1} n\log n\right) + o\left(\frac{1}{x}\sum_{nx>1}\frac{\log n}{n^2}\right) = o\left(\log\frac{1}{x}\right),$$

which is (3.1).

It follows that
$$\psi_2(x) = o\left(x^2 \log\frac{1}{x}\right) \tag{3.3}$$

(suffixes denoting, as before, integrations from 0 to x). On the other hand it is easily verified* that

$$\tfrac{1}{2}x^2\psi(x) = \psi_2(x) - \tfrac{1}{2}xf_1(x) - \tfrac{1}{2}f_2(x). \tag{3.4}$$

The last two terms are $o\left(x^2\log\frac{1}{x}\right)$, by (2.1); and it therefore follows from (3.3) that

$$\psi(x) = o\left(\log\frac{1}{x}\right),$$

which is (2.2). This completes the proof of the theorem.

4. We might naturally express (2.2) by saying that $f(t)$ is 'continuous $(R, 1)$' for $t = x$; Theorem A then asserts that, if (2.1) is satisfied (and in particular if f is bounded near $t = x$), *continuity* $(R, 1)$ *is a necessary and sufficient condition for summability* $(R, 1)$.

It is natural to ask what happens when the condition (2.1) is dropped, and the answer suggested by the analogy with Cesàro summability is as follows. We must begin by defining summability (R, k) and continuity (R, l). We may then expect a double scale of theorems like those investigated first by Littlewood and myself†, and then, more precisely, by Bosanquet and Paley.‡ None of these theorems, however, will be of the 'necessary and sufficient' type; for

* For example by differentiation.

† Hardy and Littlewood (**3**).

‡ Bosanquet (**1**), Paley (**6**).

that, it will be essential to close the cycle by some such condition as (2.1). It might be worth while to push the analysis a little farther (though hardly to develop it in full detail), but I confine myself to what seems to be the simplest and most interesting case.

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8. ——, 'Sur un théorème de M. Gronwall': ibid. 1925, 207–18.

COMMENT

§ 4. The results conjectured here were subsequently obtained by F. T. Wang, *Tôhoku Math. J.* 40 (1935), 142–59, 274–92, and *Duke Math. J.* 15 (1948), 5–10.

NOTES ON THE THEORY OF SERIES (XIV): AN ADDITIONAL NOTE ON THE SUMMABILITY OF FOURIER SERIES

G. H. Hardy *and* J. E. Littlewood*.

[*Extracted from the Journal of the London Mathematical Society, Vol.* 6, *Part* 1.]

1. We prove here a theorem which we stated in an earlier note†, but without proof, and with a fault in the enunciation which we correct. We use our former notation.

Theorem. *If* $p \geqslant 1$ *and*

$$\int_0^t |\phi(u)|^p \, du = O(t), \tag{1.1}$$

* Received 10 June, 1930; read 19 June, 1930.

† Hardy and Littlewood, **1**.

1931, 5 (with J. E. Littlewood) *Journal of the London Mathematical Society*, 6, 9–12.

then the Fourier series of $f(t)$, for $t = x$, is either summable by every Cesàro mean of positive order or summable by no Cesàro mean. The necessary and sufficient condition for summability is

$$\phi(t) \to 0 \quad (C, r), \tag{1.2}$$

where r is any number greater than $1/p$.*

We use Lemmas 3, 4, and 5 of our earlier note, and another lemma due to M. Riesz†, viz.:

LEMMA A. *If*

$$\phi(t) = O(1) \quad (C, r_1), \qquad \phi(t) = o(1) \quad (C, r_2),$$

where $0 < r_1 < r_2$, then

$$\phi(t) = o(1) \quad (C, r)$$

for $r_1 < r \leqslant r_2$.

Granted these lemmas, the proof of the theorem is simple. If (1.1) is true for a $p > 1$, it is, by Hölder's inequality, true for $p = 1$. It therefore follows in any case, by Lemma 4, that any Cesàro mean of the series of positive order is bounded; and therefore, by Lemma 3, that the series is summable by all Cesàro means of positive order or by none.

If (1.1) is satisfied for a $p > 1$, we have

$$\phi_r(t) = \frac{1}{\Gamma(r)} \int_0^t (t-u)^{r-1} \phi(u)\, du$$

$$= O\left[\left\{\int_0^t |\phi(u)|^p du\right\}^{1/p} \left\{\int_0^t (t-u)^{(r-1)p'} du\right\}^{1/p'}\right],$$

where $p' = p/(p-1)$, and this is

$$O(t^{1/p} \cdot t^{r-1+1/p'}) = O(t^r),$$

if $(r-1)p' > -1$, *i.e.* if $r > 1/p$. Hence

$$\phi(t) = O(1) \quad (C, r) \tag{1.3}$$

for $r > 1/p$. By Lemma 5, the necessary and sufficient condition for the summability of the series is that (1.2) should be true for some r, and then, by (1.3) and Lemma A, it is true for $r > 1/p$.

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* This is Theorem B of 1, except that we there say "$r > 1/p$ if $p > 1$, $r \geqslant 1$ if $p = 1$." We are unable to explain how the error (to which our attention was called by Dr. J. J. Gergen) arose; in our manuscript notes on the theorem it is stated and proved as here, and the case $p = 1$, $r = 1$ is explicitly rejected.

† M. Riesz, 3.

If (1 . 1) is satisfied for $p = 1$, then *a fortiori* (1 . 3) is true for $r > 1$, and the proof may be completed as before.

In particular the conditions

$$|\phi| = O(1) \ (C, 1), \qquad \phi = o(1) \ (C, 1)$$

are sufficient for summability $(C, 1)$. A direct proof of this has been given by Pollard*.

2. We conclude by showing by examples that the condition $r > 1/p$ cannot be improved, whether $p > 1$ or $p = 1$†.

(i) Suppose that $x = 0$, $s = 0$, and $f(t)$ is even, so that $\phi(t) = f(t)$; that $p > 1$, $1/p < a < 1$; and that c_n, ξ_n, δ_n are sequences which tend steadily to 0 when $n \to \infty$; and take

$$f(t) = \Sigma\, c_n \psi_n(t),$$

where

$$\psi_n(t) = |t - \xi_n|^{-1/p} \left| \log \frac{1}{|t-\xi_n|} \right|^{-a}$$

in the intervals $(\xi_n - \delta_n \xi_n, \ \xi_n + \delta_n \xi_n)$ and $\psi_n(t) = 0$ elsewhere. It is plain that

$$\int_0^x |\phi(t)|^p \, dt = o(x),$$

when the sequences tend to zero with sufficient rapidity. *A fortiori* (1 . 1) is satisfied and $\phi(t) \to 0 \ (C, 1)$. But

$$\phi_{1/p}(t) = \frac{1}{\Gamma(1/p)} \int_0^t (t-u)^{1/p-1} \phi(u)\, du = \infty,$$

when $t = \xi_n$, so that (1 . 2) is certainly not true for $r = 1/p$.

(ii) Take $x = 0$, $s = 0$, $f(t)$ even, and

$$f(t) = 2^n \ \ (2^{-n} - 4^{-n} < t < 2^{-n}), \quad f(t) = -2^n \ \ (2^{-n} < t < 2^{-n} + 4^{-n}),$$

and $f(t) = 0$ otherwise. Then

$$\int_0^t |\phi(u)|\, du = O\Big(\sum_{2^{-n} - 4^{-n} \leqslant t} 2^n . 4^{-n} \Big) = O\Big(\sum_{2^{-n} \leqslant t} 2^{-n} \Big) = O(t),$$

so that (1 . 1) is true with $p = 1$.

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* Pollard, 2.

† So that the statement referred to in foot-note *, p. 10, is definitely false.

Next, since $\phi_1(t)$ is zero except in the intervals $(2^{-n}-4^{-n},\ 2^{-n}+4^{-n})$, and does not exceed $2^n 4^{-n}$ in such an interval, we have

$$\phi_2(t) = O\Big(\sum_{2^{-n}-4^{-n}\leqslant t} 2^n.4^{-n}.4^{-n}\Big) = O\Big(\sum_{2^{-n}\leqslant t} 8^{-n}\Big) = O(t^3) = o(t^2),$$

and, by the theorem, the Fourier series is summable.

Finally, when $t = 2^{-n}$, $\phi_1(t)$ is equal to $2^n.4^{-n} = t$. Hence (1.2) is false when $r = 1$, so that (1.2) with $r = 1$ cannot be a necessary condition for the summability of the Fourier series.

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CORRECTION

p. 12. In last line of references read 114–126 instead of 3–15.

NOTES ON THE THEORY OF SERIES (XV): ON THE SERIES CONJUGATE TO THE FOURIER SERIES OF A BOUNDED FUNCTION

G. H. HARDY *and* J. E. LITTLEWOOD†.

[*Extracted from the Journal of the London Mathematical Society, Vol.* 6, *Part* 4.]

1. Mr. Prasad proves‡ that the allied or conjugate series associated with a bounded function $f(x)$ is summable (C, δ), for every positive δ, whenever the integral

$$g(x) = \frac{1}{2\pi}\int_0^\pi \{f(x+t)-f(x-t)\} \cot \tfrac{1}{2}t\, dt \tag{1.1}$$

is convergent§. The point of the theorem is that, when $f(x)$ is unbounded, we can usually only assert summability for $\delta > 1$||.

Theorem A below in a certain sense completes Mr. Prasad's theorem, and corresponds to one concerning the Fourier series of $f(x)$ which we proved some years ago¶.

THEOREM A. *Suppose that $f(t)$ is bounded in the neighbourhood of $t = x$. Then the allied series of $f(t)$, for $t = x$, is either summable by Cesàro means of every positive order, or summable by no Cesàro mean.*

† Received 6 August, 1931; read 12 November, 1931.

‡ See his note preceding this.

§ As a Cauchy integral, *i.e.* as

$$\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_\epsilon^\pi.$$

The sum of the series is, of course, $g(x)$.

|| Paley, **4** (Theorem 2, with $\alpha = 1$).

¶ Hardy and Littlewood, **2**, 71 (Theorem C 1).

1931, 7 (with J. E. Littlewood) *Journal of the London Mathematical Society*, 6, 278–81.

A necessary and sufficient condition for summability is the convergence of the integral (1 . 1).

2. We shall in fact prove rather more than is included in Theorem A. We denote by Λ the set of values of x for which

$$\int_0^t |\psi(u)|\,du = \int_0^t |f(x+u)-f(x-u)|\,du = O(t),$$

and by Λ^* the set in which the same integral is $o(t)$. Plainly Λ includes Λ^*, and Λ^* includes what, in our paper **3**, we called the Lebesgue set, so that all these sets include almost all x. Also x belongs to Λ whenever $f(t)$ is bounded in the neighbourhood of $t = x$.

THEOREM B. (i) *If x is in Λ, then the allied series, if summable* (A), *and a fortiori if summable* (C), *is summable* (C, δ) *for every positive* δ.

(ii) *If x is in Λ, then a necessary and sufficient condition for the summability of the allied series is that the integral* (1 . 1) *should converge in some Cesàro sense.*

(iii) *In particular, if $f(t)$ is bounded near $t = x$, it is necessary and sufficient that* (1 . 1) *should converge in the ordinary sense.*

It is clear that Theorem A is included in Theorem B. Our proof of the latter depends upon a chain of theorems most of which we have used repeatedly before and the remainder of which we state here as lemmas. We use the letters B and $\mathfrak{B}$ in the same senses as in **3**, so that, for example,

$$B_\omega^\delta = \sum_{1 \leq n < \omega} \left(1-\frac{n}{\omega}\right)^\delta B_n,$$

$$\mathfrak{B}_\omega^\delta = \sum_{1 \leq n < \omega} \left(1-\frac{n}{\omega}\right)^\delta nB_n,$$

where $B_n = b_n \cos nt - a_n \sin nt$ is the general term of the allied series of $f(t)$. We also use "summable (C, $r+\delta$)" as meaning "summable by any Cesàro mean of order greater than r".

LEMMA 1. *If x is in Λ^* and $\delta > 0$, then* $\mathfrak{B}_\omega^\delta = o(\omega)$.

This is what we *proved* in Lemma α of **3**, though we stated the result there a little less generally.

LEMMA 2. *If x is in Λ, then*

$$\mathfrak{B}_\omega^\delta = O(\omega).$$

The proof is a trivial simplification of that of Lemma 1.

LEMMA 3. *If ΣB_n is summable (A), and*

$$\beta_n = B_1+2B_2+\ldots+nB_n = O(n) \quad (C,\ 1),$$

then ΣB_n is summable $(C,\ 1+\delta)$.

This (a Tauberian theorem of purely "arithmetic" character) is included in Theorem 3 of the note which follows.

3. We can now prove Theorem B as follows.

(i) If x is in Λ then $\mathfrak{B}_\omega^\delta = O(\omega)$, by Lemma 1. Hence

$$B_\omega^r - B_\omega^{r-1} = -\frac{1}{\omega}\,\mathfrak{B}_\omega^{r-1} = O(1)$$

for $r > 1$. It follows that ΣB_n, if summable (C), is bounded $(C,\ \delta)$. But a series summable (C), and bounded $(C,\ \delta)$, is summable $(C,\ \delta)$†; and therefore ΣB_n, if summable (C), is summable $(C,\ \delta)$.

(ii) A necessary and sufficient condition that ΣB_n should be summable (C) is that stated in clause (ii) of the theorem‡. It follows from (i) that, when x is in Λ, this condition is necessary and sufficient for summability $(C,\ \delta)$.

(iii) If in particular $f(t)$ is bounded near $t = x$, then

$$\{f(x+t)-f(x-t)\}\cot\tfrac{1}{2}t = O\left(\frac{1}{t}\right)$$

when $t \to 0$. In this case (1.1), if convergent in any Cesàro sense, is convergent in the ordinary sense§.

(iv) We have now proved Theorem B except for the reference to summability (A) in clause (1). For this we require Lemma 3. If we take $\delta = 1$, $\omega = n+1$, in Lemma 2, we see that

$$\beta_1+\beta_2+\ldots+\beta_n = O(n^2)$$

† Andersen, **1** (Theorem 8, 56).

‡ Hardy and Littlewood, **3** (Theorem 3).

§ By the integral analogue of the "Cesàro-Tauber" theorem for summable series.

when x is in Λ. This is the second condition of Lemma 3, and it follows that ΣB_n is summable (C).

[We take this opportunity of observing that (as was pointed out to us by Mr. Prasad) the criterion for the summability of a Fourier series proved by Pollard in Vol. 1 of the *Journal* (p. 233), and attributed by him to us (as the case $p = 1$, $r = 1$ of a theorem which we stated on p. 137 of Vol. 1, and proved on p. 10 of Vol. 6), is really due to W. H. Young. See Young's paper "On the convergence of the derived series of Fourier series", *Proc. London Math. Soc.* (2), 17 (1917), 195–236 (207, Corollary 4).]

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4. R. E. A. C. Paley, "On the Cesàro summability of Fourier series and allied series", *Proc. Camb. Phil. Soc.*, 26 (1930), 173–203.

COMMENT

p. 280, *line* 8. The 'note which follows' is 1931, 8.

The strong summability of Fourier series.

By

G. H. Hardy and J. E. Littlewood (Cambridge).

§ 1. Introduction.

1.1. A series

$$A_0 + A_1 + A_2 + \dots$$

may be said to be *strongly summable*, with index k and sum s, if $k > 0$,

$$s_n = A_0 + A_1 + \dots + A_n,$$

and

$$\frac{1}{n+1} \sum_0^n |s_\nu - s|^k \to 0 \tag{1.1.1}$$

when $n \to \infty$. It follows from Hölder's inequality that (1.1.1) says the more the larger k.

Suppose now that $f(t)$ is a periodic function of the class L^r, where $r > 1$, that

$$A_0 + \sum_1^\infty A_n = \frac{1}{2} a_0 + \sum_1^\infty (a_n \cos nt + b_n \sin nt) \tag{1.1.2}$$

is the Fourier series of $f(t)$, that

$$\varphi(x, t) = \frac{1}{2} \{f(x+t) + f(x-t) - 2s\}, \tag{1.1.3}$$

$$\Phi(x, t) = \int_0^t \varphi(x, u)\, du, \tag{1.1.4}$$

and

$$\Phi_r^*(x, t) = \int_0^t |\varphi(x, u)|^r\, du. \tag{1.1.5}$$

1935, 5 (with J. E. Littlewood) *Fundamenta Mathematicae*, 25, 162–89.

If

(1.1.6) $$\Phi(x, t) = o(t)$$

and

(1.1.7) $$\Phi_r^*(x, t) = O(t)$$

for small t, then the series (1.1.2) *is strongly summable, to sum s, for every k* [1]). The conditions (1.1.6) and (1.1.7) are satisfied for almost all x when $s = f(x)$. In particular

(1.1.8) $$\Phi_r^*(x, t) = o(t),$$

which includes both (1.1.6) and (1.1.7), is a sufficient condition for strong summability.

The proof fails when $r = 1$, and this case of the problem has remained unsolved [2]).

1.2. Our main purpose here is to settle this unsolved problem. Our solution is (as was to be expected) negative; (1.1.8) is not sufficient, when $r = 1$, for strong summability with any index k [3]). We prove, however, a good deal more. If $k > 0$, and

(1.2.1) $$\chi = \chi(n) = o(\sqrt{\log n}),$$

then (1.1.8), with $r = 1$, does not imply

(1.2.2) $$\sum_0^n |s_\nu - s|^k = o(n\chi^k).$$

We state the proof primarily for the case $k = 1$.

[1]) Hardy and Littlewood (**4**, Theorem 1). The first theorem of this character appeared in **3**, and the theorem stated here is the result of successive generalisations by Carleman (**1**), Sutton (**8**), and ourselves.

Still further generalisations were made by Paley (**7**). Thus (1.1.6) may be replaced by

$$\varphi(x, t) \to 0 \qquad (C, l)$$

with any l, and (1.1.7) by

$$\int_0^t \psi(|\varphi(x, u)|)\, du = O(t),$$

where $\psi(w)$ may be, for example, any of

$$w(\log^+ w)^{1+\delta}, \quad w \log^+ w (\log^+ \log^+ w)^{1+\delta}, \ldots \qquad (\delta > 0).$$

[2]) See, for example, Zygmund (**9**, 240).

[3]) However small.

This negative result raises a further problem. The proof suggests that the function $\sqrt{\log n}$ should be, in a sense, a „best possible“ function; and in § 3 we prove that, substantially, this is so, since (1.1.8) implies

$$\sum_{0}^{n} |s_\nu - s|^k = o\left(n (\log n)^{\frac{1}{2}k}\right) \tag{1.2.3}$$

at any rate when $k \leqq 2$.

This result completes the main purpose of the paper. In § 4 we discuss, more cursorily, some further problems left open by our work.

§ 2. Negative Theorems.

2.1. In § 2 we suppose that $f(t)$ is even and

$$x = 0, \qquad s = 0,$$

so that

$$\varphi(x, t) = f(t).$$

The numbers

$$A, \; B, \; C, \ldots$$

are positive „world-constants“, which preserve their identity throughout the argument, unless the contrary is stated expressly.

2.2. ***Theorem 1.*** *Suppose that $\chi = \chi(n)$ is an increasing function of n, and that*

$$\chi(n) = o(\sqrt{\log n}). \tag{2.2.1}$$

Then there is an integrable function $f(t)$ for which

$$\int_0^t |f(u)|\, du = o(t) \tag{2.2.2}$$

and

$$\sum_{0}^{n} |s_\nu| \neq o(n\chi). \tag{2.2.3}$$

We begin by transforming the theorem. We may suppose that

$$a_0 = \frac{2}{\pi} \int_0^\pi f(t)\, dt = 0,$$

and then (2.2.3) is equivalent to

$$\sum_{1}^{n} |t_\nu| \neq o(n\chi),$$

where

$$t_n = a_1 + a_2 + \ldots + a_{n-1} + \frac{1}{2} a_n.$$

Now

$$t_n = n\, b_n,$$

where b_n is the Fourier constant

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} \cos nt\, g(t)\, dt$$

of the even and integrable function [4])

$$g(t) = \frac{1}{2} \int_{t}^{\pi} \cot \frac{1}{2} u\, f(u)\, du.$$

Since

$$g'(t) \equiv -\frac{1}{2} \cot \frac{1}{2} t\, f(t),$$

(2.2.2) is equivalent to

(2.2.4) $$\int_{0}^{t} u\, |g'(u)|\, du = o(t).$$

Hence Theorem 1 is equivalent to

Theorem 2. *There is a function*

$$g(b) \sim \frac{1}{2} b_0 + \sum_{1}^{\infty} b_n \cos nt,$$

satisfying (2.2.4), *for which*

(2.2.5) $$\sum_{1}^{n} \nu\, |b_\nu| \neq o(n\chi).$$

It is in this form that we shall consider the theorem.

[4]) See Hardy (2).

Lemmas for Theorems 1 and 2.

2.3. ***Lemma 1.*** *Suppose that*

(2.3.1) $$n = 10^k, \qquad k \geqq 3, \qquad l = [\log k].$$

Then there are positive constants A, B, and an even function

$$f(t) = f_n(t)$$

possessing the following properties.

(a) $f(t) = \varepsilon_r$, *where* $\varepsilon_r = \pm 1$, *in the intervals*

$$\delta_r = \left(\frac{\pi}{10^{r+1}}, \frac{\pi}{10^r}\right) \qquad (l \leqq r < k),$$

and $f(t) = 0$ *if* $t > 10^{-l}\pi$ *or* $0 \leqq t < 10^{-k}\pi$.

(b) $f(t)$ *satisfies*

(2.3.2) $$\int_0^t u\,|df(u)| \leqq A\,t, \qquad (0 \leqq t \leqq \pi),$$

(2.3.3) $$\int_0^t u\,|df(u)| = 0 \qquad \left(0 \leqq t \leqq \frac{\pi}{10^k}\right).$$

(c) *The Fourier constants* $c_\nu = c_\nu(f)$ *of* f *satisfy*

(2.3.4) $$\sum_1^n \nu\,|c_\nu(f)| \geqq B\,n\sqrt{\log n}.$$

This is the critical lemma; when it is proved, the remainder of the proof of Theorems 1 and 2 will be a matter of routine.

In the first place, (2.3.3) is obvious.

Next, $f(t)$ is a step function with discontinuities, of magnitude 2, at some of the ends of the intervals δ_r. If

$$10^{-s-1}\pi \leqq t \leqq 10^{-s}\pi \qquad (l \leqq s < k)$$

then

$$\int_0^t u\,|df(u)| \leqq 2\sum_s^{k-1} \frac{\pi}{10^r} \leqq \frac{2\pi}{10^s}\cdot\frac{10}{9} \leqq \frac{200\,\pi}{9}\,t,$$

which proves (2.3.2).

It remains to prove that, if the ε_r are selected properly, c_ν satisfies (2.3.4).

2.4. Now

$$c_\nu(f) = \sum_{r=l}^{k-1} \varepsilon_r \int_{10^{-r-1}\pi}^{10^{-r}\pi} \cos \nu t\, dt,$$

(2.4.1) $$\nu\, c_\nu(f) = \sum_{r=l}^{k-1} \varepsilon_r \Delta_{r,\nu},$$

where

(2.4.2) $$\Delta_{r,\nu} = \sin \frac{\nu\pi}{10^r} - \sin \frac{\nu\pi}{10^{r+1}}.$$

Suppose, if possible, that

(2.4.3) $$\sum_1^n \nu\, |c_\nu| \leqq Cn \sqrt{\log n}$$

for every set of ε_r; and let

'Av'

denote an average taken over the 2^{k-l} sets of ε_r. Then, after (2.4.3),

(2.4.4) $$\sum_1^n Av\, (\nu\, |c_\nu|) \leqq Cn \sqrt{\log n}.$$

But [5]) there is a D such that

(2.4.5) $$Av(\nu\,|c_\nu|) = Av\left(\left|\sum_{r=l}^{k-1} \varepsilon_r \Delta_{r,\nu}\right|\right) \geqq D \sqrt{\sum_{r=l}^{k-1} \Delta_{r,\nu}^2};$$

and therefore, after (2.4.4),

(2.4.6) $$\sum_{\nu=1}^{n} \sqrt{\sum_{r=l}^{k-1} \Delta_{r,\nu}^2} \leqq \frac{C}{D}\, n \sqrt{\log n}.$$

If we can prove that (2.4.6) is false, for some C, we shall have proved that (2.4.3) is false for at any rate one selection of the ε_r; and this will complete the proof of the lemma.

[5]) Littlewood (6, Lemma 4).

2.5. We suppose now that

$$0 < \nu < n = 10^k, \quad \nu = i_1\, i_2 \dots i_k$$

(in the decimal scale), and consider the conditions under which it is possible that

$$|\Delta| = |\Delta_{r,\nu}| < \frac{1}{100}, \quad r < k.$$

If $r < k$ then

$$\begin{aligned}\Delta &= \sin\frac{\nu\pi}{10^r} - \sin\frac{\nu\pi}{10^{r+1}}\\ &= (-1)^{i_{k-r}} \sin\left(\frac{i_{k-r+1}}{10} + \dots + \frac{i_k}{10^r}\right)\pi\\ &\quad - (-1)^{i_{k-r-1}} \sin\left(\frac{i_{k-r}}{10} + \dots + \frac{i_k}{10^{r+1}}\right)\pi\\ &= \pm \sin\theta \pm \sin\varphi,\end{aligned}$$

say; and $|\sin\theta|$ and $|\sin\varphi|$ differ by more than $\frac{1}{100}$ unless i_{k-r+1} has one of the 6 values

$$i_{k-r}, \quad i_{k-r}-1, \quad i_{k-r}+1, \quad 10-i_{k-r}, \quad 10-i_{k-r}+1, \quad 10-i_{k-r}-1$$ [6]).

Hence

$$|\Delta| > \frac{1}{100}$$

unless i_{k-r+1} satisfies this condition.

We may regard each of the $10^k - 1$ values of ν as determined by successive choice of the digits

$$i_1, \ i_2, \dots, \ i_{k-r}, \ i_{k-r+1}, \dots, \ i_k.$$

In general there are 10 possible choices of i_{k-r+1}; but if i_{k-r+1} is to be chosen so that $|\Delta| < \frac{1}{100}$, then there are at most 6.

[6]) Divide the angle π into ten equal sectors. Then φ lies in a sector which is not the same as or adjacent to that containing θ, nor supplementary to such a sector; and

$$\big||\sin\theta| - |\sin\varphi|\big| > \int_{\frac{1}{2}\pi - \frac{1}{10}\pi}^{\frac{1}{2}\pi} \cos u\, du = 1 - \cos\frac{\pi}{10} > \frac{1}{100}.$$

Suppose now that, for a particular ν,

$$\sum_{r=l}^{k-1} \Delta_{r,\nu}^2 \leqq h^2, \tag{2.5.1}$$

where $h = h(n)$ is a function of n (or of k) to be chosen later. Then

$$\Delta > \frac{1}{100}$$

for at most

$$\lambda = [10^4 h^2]$$

values of r. At most λ of the choices for ν are unrestricted, and the remainder, at least $k - l - \lambda$, restricted. The λ unrestricted choices can correspond to at most

$$\binom{k}{\lambda}$$

sets of positions, and the total number of values of ν which satisfy (2.5.1) is at most

$$m = 10^l \binom{k}{\lambda} 10^\lambda 6^{k-l-\lambda}. \tag{2.5.2}$$

We take

$$h = E\sqrt{\log n}$$

and find an upper bound for m for large n. We have

$$\lambda = [10^4 E^2 \log n] = \gamma k,$$

where

$$\gamma \leqq \frac{10^4 E^2}{\log 10}$$

is small with E. Also

$$\binom{k}{\lambda} = \frac{k(k-1)\dots(k-\lambda+1)}{\lambda!}$$
$$< \frac{k^\lambda}{\lambda!} \leqq \frac{k^{\gamma k} e^{\gamma k}}{(\gamma k)^{\gamma k}} = 10^{\delta k} = n^\delta,$$

where

$$\delta = \frac{\gamma - \gamma \log \gamma}{\log 10}$$

is small with E and γ. We can therefore choose E so that

$$\binom{k}{\lambda} < n^{\frac{1}{30}} \tag{2.5.3}$$

and

(2.5.4) $$10^{\lambda} = 10^{\gamma k} = n^{\gamma} < n^{\frac{1}{30}}$$

for sufficiently large n. Finally

(2.5.5) $$10^{l} = 10^{[\log k]} < n^{\frac{1}{30}}$$

for large n, and

(2.5.6) $$6^{k-l-\lambda} < 6^{k} = n^{\frac{\log 6}{\log 10}} = n^{\cdot 78\cdots} < n^{\frac{8}{10}};$$

and these four inequalities and (2.5.2) show that

$$m < n^{\frac{3}{30}+\frac{8}{10}} = n^{\frac{9}{10}}$$

for large n.

It follows that

$$\sum_{r=l}^{k-1} \Delta_{r,\nu}^{2} \geqq h^{2} = E^{2} \log n$$

for at least

$$n - n^{\frac{9}{10}} > \frac{1}{2} n$$

values of n, and therefore that

$$\sum_{\nu=1}^{n} \sqrt{\sum_{r=l}^{k-1} \Delta_{r,\nu}^{2}} \geqq \frac{1}{2} E n \sqrt{\log n}.$$

This contradicts (2.4.6) when

$$C < \frac{1}{2} D E;$$

and the contradiction proves the lemma.

2.6. ***Lemma 2.*** *Suppose that n, k, l satisfy* (2.3.1). *Then there are positive constants H, K, and an even function*

$$g(t) = g_{n}(t),$$

with the following properties.

(b′) $g(t)$ *satisfies*

(2.6.1) $$\int_{0}^{t} u\,|g'(u)|\,du \leqq Ht \qquad (0 \leqq t \leqq \pi),$$

(2.6.2) $$\int_0^t u\,|g'(u)|\,du = 0 \qquad \left(0 \leqq t \leqq \frac{\pi}{10^k}\right).$$

(c′) *The Fourier constants* $c_\nu = c_\nu(g)$ *of* g *satisfy*

(2.6.3) $$\sum_1^n \nu\,|c_\nu(g)| \geqq Kn\sqrt{\log n}.$$

The f of Lemma 1 has discontinuities at some of the points $10^{-r}\pi$; we construct g by smoothing away these discontinuities appropriately. If, for example, $l < r < k-1$, and f jumps from -1 to $+1$ at $10^{-r}\pi$, then we join the points

$$(10^{-r}\pi - n^{-3},\ -1), \qquad (10^{-r}\pi + n^{-3},\ 1)$$

by a straight line, and take this as part of the definition of g. A jump from $+1$ to -1, or a jump of half the height when r is l or $k-1$, is dealt with similarly. Since g is stationary except along these lines, and

$$\int_{10^{-r}\pi - n^{-3}}^{10^{-r}\pi + n^{-3}} u\,|g'(u)|\,du = \int_{10^{-r}\pi - n^{-3}}^{10^{-r}\pi + n^{-3}} u\,|df(u)|,$$

g has the properties (2.6.1) and (2.6.2). Also f and g differ by at most 1, and that in a set of measure at most

$$\frac{2k}{n^3} < L\frac{\log n}{n^3}.$$

Hence

$$|c_\nu(f) - c_\nu(g)| < L\frac{\log n}{n^3};$$

and

$$\sum_1^n \nu\,|c_\nu(f)|, \qquad \sum_1^n \nu\,|c_\nu(g)|$$

differ by less than

$$L\frac{\log n}{n},$$

so that g has the property (2.6.3).

2.7. In this paragraph we vary our notation a little. M is a world-constant wherever it occurs, but the constants in different places are not the same.

Lemma 3. *The function g of Lemma 2 has also the properties*

$$\sum_{1}^{m} \nu\,|c_\nu(g)| \leqq M m \log n, \tag{2.7.1}$$

$$\sum_{1}^{m} \nu\,|c_\nu(g)| \leqq M m \log m, \tag{2.7.2}$$

for $m > 1$.

(i) Since

$$\nu\,|c_\nu(g)| \leqq M\,V(g),$$

where $V(g)$ is the total variation of g, and $V(g)$ is less than Mk or $M \log n$, g has the property (2.7.1).

(ii) We have

$$|g| \leqq 1, \quad G(t) = \int_0^t u\,|g'(u)|\,du \leqq Ht.$$

Now

$$c_\nu(g) = \frac{2}{\pi}\left(\int_0^{\pi/\nu} + \int_{\pi/\nu}^{\pi}\right) \cos \nu t\, g(t)\,dt = J_1 + J_2, \tag{2.7.3}$$

say. Here

$$|J_1| \leqq \frac{2}{\pi}\int_0^{\pi/\nu} |g(t)|\,dt \leqq \frac{M}{\nu}, \tag{2.7.4}$$

since $|g| \leqq 1$. Also

$$J_2 = \frac{2}{\pi}\int_{\pi/\nu}^{\pi} \cos \nu t\, g(t)\,dt = -\frac{2}{\pi\nu}\int_{\pi/\nu}^{\pi} \sin \nu t\, g'(t)\,dt,$$

$$\begin{aligned} |J_2| &\leqq \frac{M}{\nu}\int_{\pi/\nu}^{\pi} |g'(t)|\,dt = \frac{M}{\nu}\int_{\pi/\nu}^{\pi} \frac{G'(t)}{t}\,dt \\ &= \frac{M}{\nu}\left(\frac{G(\pi)}{\pi} - \frac{G(\pi/\nu)}{\pi/\nu}\right) + \frac{M}{\nu}\int_{\pi/\nu}^{\pi} \frac{G(t)}{t^2}\,dt \\ &\leqq \frac{M}{\nu} + \frac{MH}{\nu}\int_{\pi/\nu}^{\pi} \frac{dt}{t} \leqq M\,\frac{\log \nu}{\nu}. \end{aligned} \tag{2.7.5}$$

From (2.7.3), (2.7.4), and (2.7.5) we deduce

$$|c_\nu(g)| \leqq M\frac{\log\nu}{\nu},$$

and so (2.7.2).

Proof of Theorem 2.

2.8. We can now prove Theorem 2 (and so Theorem 1).

We define $g(t)$ by

$$g(t) = \sum_{s=1}^{\infty} \zeta_s\, g_{n_s}(t) = \sum_{s=1}^{\infty} \frac{g_{n_s}(t)}{\eta_s},$$

where $\eta_s > 0$ and $n_s = 10^{k_s}$ are sequences, to be chosen later, which tend to infinity, when $s \to \infty$, with great rapidity. It is plain that, if the increase of n_s is sufficiently rapid, at most one g_{n_s} differs from 0 for any t. Also $g_{n_s} = 0$ if $t < 10^{-k_s}\pi = \pi/n_s$, and

$$\int_0^t u\,|g'_{n_s}(u)|\,du = O(t)$$

uniformly in s. Hence

$$\int_0^t u\,|g'(u)|\,du \leqq \sum \zeta_s \int_0^t u\,|g'_{n_s}(u)|\,du = O\Bigl(t\sum_{10^{-k_s}\leqq t}\zeta_s\Bigr) = o(t),$$

which is (2.2.4). It remains to verify (2.2.5).

We have

$$c_\nu(g) = \sum_s \zeta_s\, c_\nu(g_{n_s}),$$

$$|c_\nu(g)| \geqq \zeta_\sigma\,|c_\nu(g_{n_\sigma})| - \sum_{s<\sigma}\zeta_s\,|c_\nu(g_{n_s})| - \sum_{s>\sigma}\zeta_s\,|c_\nu(g_{n_s})|,$$

(2.8.1) $$S = \sum_{\nu=1}^{n_\sigma} \nu\,|c_\nu(g)| \geqq \zeta_\sigma \sum_{\nu=1}^{n_\sigma}\nu\,|c_\nu(g_{n_\sigma})|$$

$$- \sum_{s<\sigma}\zeta_s \sum_{\nu=1}^{n_\sigma}\nu\,|c_\nu(g_{n_s})| - \sum_{s>\sigma}\zeta_s\sum_{\nu=1}^{n_\sigma}\nu\,|c_\nu(g_{n_s})|$$

$$= S^* - S_1 - S_2,$$

say.

In the first place, by (2.6.3),

(2.8.2) $$S^* > K\zeta_\sigma n_\sigma \sqrt{\log n_\sigma} \succ n_\sigma \chi(n_\sigma) \text{ [7]}$$

if

(2.8.3) $$\frac{\sqrt{\log n_\sigma}}{\chi(n_\sigma)} \succ \eta_\sigma.$$

Next, in S_1, $s < \sigma$ and, by (2.7.1),

$$\sum_{\nu=1}^{n_\sigma} \nu\,|c_\nu(g_{n_s})| \leqq M n_\sigma \log n_s \leqq M n_\sigma \log n_{\sigma-1},$$

so that

$$S_1 \leqq M n_\sigma \log n_{\sigma-1} \sum_s \zeta_s \leqq M n_\sigma \log n_{\sigma-1},$$

and

(2.8.4) $$S_1 \prec n_\sigma \chi(n_\sigma)$$

if

(2.8.5) $$\log n_{\sigma-1} \prec \chi(n_\sigma).$$

For this it is only necessary that n_σ should tend to infinity with sufficient rapidity.

Finally, in S_2, $s > \sigma$ and, by (2.7.2),

$$\sum_{\nu=1}^{n_\sigma} \nu\,|c_\nu(g_{n_s})| \leqq M n_\sigma \log n_\sigma,$$

Hence

$$S_2 \leqq M n_\sigma \log n_\sigma(\zeta_{\sigma+1} + \zeta_{\sigma+2} + \ldots) \leqq M n_\sigma \log n_\sigma \zeta_{\sigma+1},$$

if ζ_s decreases with sufficient rapidity. It follows that

(2.8.6) $$S_2 \prec n_\sigma \chi(n_\sigma)$$

if

(2.8.7) $$\frac{\log n_\sigma}{\chi(n_\sigma)} \prec \eta_{\sigma+1}.$$

[7]) We write $\varphi \succ \psi$ if $\varphi/\psi \to \infty$ when the variable involved in φ and ψ tends to infinity, and $\varphi \prec \psi$ if $\varphi/\psi \to 0$.

Collecting our results from (2.8.1), (2.8.2), (2.8.4) and (2.8.6), we see that

$$S = \sum_{1}^{n_\sigma} \nu \, |c_\nu(g)| \succ n_\sigma \, \chi(n_\sigma)$$

if n_σ and η_σ tend to infinity both with sufficient rapidity and in such a manner as to satisfy (2.8.3) and (2.8.7). Since we can satisfy these conditions by successive choice of

$$\eta_1, \; n_1, \; \eta_2, \; n_2, \; \eta_3, \ldots,$$

the theorem follows.

We have supposed that the k of § 1 is 1. It is plain that the same argument proves that

$$\sum_{0}^{n} |s_\nu|^k \neq o(n \chi^k)$$

for any positive k.

§ 3. Positive theorems.

3.1. The notation of this section differs from that of § 2.

We suppose that $u(\theta)$ is periodic and integrable; that

$$u(\theta) \sim \frac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = \frac{1}{2} A_0 + \sum_{1}^{\infty} A_n(\theta);$$

that

$$\varphi(\theta, t) = \frac{1}{2} \{u(\theta + t) + u(\theta - t) - 2s\};$$

and that

$$s_n = s_n(\theta) = \frac{1}{2} A_0 + \sum_{1}^{n} A_\nu(\theta).$$

We also suppose that

$$u(r, \theta) = \frac{1}{2} A_0 + \sum_{1}^{\infty} A_n(\theta) \, r^n$$

is the harmonic function associated with $u(\theta)$, or Poisson integral of $u(\theta)$; and that

$$f(z) = f(r\, e^{i\theta}) = \sum c_n z^n = \sum c_n r^n e^{ni\theta},$$

where

$$c_0 = \frac{1}{2} a_0, \qquad c_n = a_n - i\, b_n \qquad (n > 0),$$

is the analytic function, regular for $r < 1$, whose real part is $u(r, \theta)$ and whose imaginary part vanishes at the origin.

The A are positive world-constants, different on different occasions, while C is a positive number which remains the same throughout a proposition and its proof.

Theorem 3. *If*

$$\int_0^t |\varphi(\theta, w)|\, dw = o(t), \tag{3.1.1}$$

then

$$\sum_0^n (s_\nu - s)^2 = o(n \log n) \tag{3.1.2}$$

and

$$\sum_0^n |s_\nu - s| = o(n \sqrt{\log n}). \tag{3.1.3}$$

We need only prove (3.1.2), since (3.1.3) then follows by Cauchy's inequality. The theorem shows that the condition (2.2.1), imposed on $\chi(n)$ in Theorems 1 and 2, is the „best possible" condition of its kind

The series

$$\frac{1}{2} A_0 - s + \sum_1^\infty A_n(\theta) \cos nt$$

is the Fourier series in t of the even function $\varphi(\theta, t)$, and converges to zero when s_n converges to s. We may therefore suppose, without real loss of generality, that $\theta = 0$, $s = 0$, and that $u(t)$ is an even function of t. In these circumstances $\varphi(\theta, t) = u(t)$. The letters θ and φ then disappear from the theorem, and there will be no inconvenience in using them in other senses.

Lemmas for Theorem 3.

3.2. ***Lemma 4.*** *If δ is positive, and θ real, then*

$$\int_{-\infty}^{\infty} \frac{\delta \varphi^2}{(\delta^2 + \varphi^2)(\delta^2 + (\varphi - \theta)^2)}\, d\varphi < A, \tag{3.2.1}$$

$$\int_{-\infty}^{\infty} \frac{\delta |\theta| |\varphi|}{(\delta^2 + \varphi^2)(\delta^2 + (\varphi - \theta)^2)} \, d\varphi < A. \tag{3.2.2}$$

(i) Since $\varphi^2 < \delta^2 + \varphi^2$, we have

$$\int_{-\infty}^{\infty} \frac{\delta \varphi^2}{(\delta^2 + \varphi^2)(\delta^2 + (\varphi - \theta)^2)} \, d\varphi < \int_{-\infty}^{\infty} \frac{\delta \, d\varphi}{\delta^2 + (\varphi - \theta)^2} = \pi.$$

(ii) We may suppose θ positive. It is then sufficient to consider the integral over $(0, \infty)$, the other part being smaller.

We write

$$\int_{0}^{\infty} \frac{\delta \theta \varphi}{(\delta^2 + \varphi^2)(\delta^2 + (\varphi - \theta)^2)} \, d\varphi = \int_{0}^{\frac{1}{2}\theta} + \int_{\frac{1}{2}\theta}^{2\theta} + \int_{2\theta}^{\infty} = J_1 + J_2 + J_3.$$

In J_1,

$$\delta^2 + (\varphi - \theta)^2 > (\varphi - \theta)^2 \geqq \frac{1}{4} \theta^2 \geqq \frac{1}{2} \theta \varphi,$$

so that

$$J_1 < 2 \int_{0}^{\infty} \frac{\delta \, d\varphi}{\delta^2 + \varphi^2} = \pi.$$

In J_2,

$$\delta^2 + \varphi^2 > \varphi^2 \geqq \frac{1}{2} \theta \varphi,$$

so that

$$J_2 < 2 \int_{-\infty}^{\infty} \frac{\delta \, d\varphi}{\delta^2 + (\varphi - \theta)^2} = 2\pi.$$

Finally, in J_3, $\theta \leqq \frac{1}{2} \varphi$ and so

$$\delta^2 + (\varphi - \theta)^2 > \frac{1}{4} (\delta^2 + \varphi^2),$$

$$J_3 < 2 \int_{0}^{\infty} \frac{\delta \varphi^2 \, d\varphi}{(\delta^2 + \varphi^2)^2} = \frac{1}{2} \pi.$$

3.3. ***Lemma 5.*** *If*

$$\left|\int_0^\theta |u(t)|\, dt\right| \leqq C|\theta|, \tag{3.3.1}$$

for all θ, then

$$\left|\int_0^\theta |u(r,t)|\, dt\right| \leqq AC|\theta| \tag{3.3.2}$$

for $r \leqq 1$.

We may suppose that

$$0 < \theta \leqq \frac{1}{2}\pi.$$

We write

$$\delta = 1 - r,$$

so that $0 \leqq \delta \leqq 1$. Then

$$u(r,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)\,u(\varphi)}{1-2r\cos(\varphi - t) + r^2}\, d\varphi,$$

$$|u(r,t)| \leqq A\int_{-\pi}^{\pi} \frac{\delta\,|u(\varphi)|}{\delta^2 + (\varphi - t)^2}\, d\varphi,$$

$$\int_0^\theta |u(r,t)|\, dt \leqq A\int_{-\pi}^{\pi} |u(\varphi)|\,\chi(\varphi, \theta, \delta)\, d\varphi, \tag{3.3.3}$$

where

$$\chi(\varphi, \theta, \delta) = \int_0^\theta \frac{\delta\, dt}{\delta^2 + (\varphi - t)^2}. \tag{3.3.4}$$

It will be enough to prove that

$$J = J(\theta, \delta) = \int_0^\pi |u(\varphi)|\,\chi(\varphi, \theta, \delta)\, d\varphi \leqq AC\theta. \tag{3.3.5}$$

If

$$U(\varphi) = \int_0^\varphi |u(t)|\, dt \leqq C\varphi,$$

then

$$J = \int_0^\pi U'(\varphi)\,\chi(\varphi, \theta, \delta)\, d\varphi = U(\pi)\,\chi(\pi, \theta, \delta) - \int_0^\pi U(\varphi)\,\frac{\partial \chi}{\partial \varphi}\, d\varphi. \tag{3.3.6}$$

The first term is plainly less than $AC\theta$. In the second, we have

$$\frac{\partial\chi}{\partial\varphi} = -\int_0^\theta \frac{d}{dt}\left(\frac{\delta}{\delta^2+(\varphi-t)^2}\right) dt = \frac{\delta}{\delta^2+\varphi^2} - \frac{\delta}{\delta^2+(\varphi-\theta)^2},$$

$$\left|\frac{\partial\chi}{\partial\varphi}\right| \leqq A\,\frac{\delta(\theta\varphi+\theta^2)}{(\delta^2+\varphi^2)(\delta^2+(\varphi-\theta)^2)},$$

$$\left|\int_0^\pi U(\varphi)\frac{\partial\chi}{\partial\varphi}\,d\varphi\right| \leqq AC\int_0^\pi \frac{\delta\theta\varphi^2+\delta\theta^2\varphi}{(\delta^2+\varphi^2)(\delta^2+(\varphi-\theta)^2)}\,d\varphi \leqq AC\theta,$$

by Lemma 4.

3.4. ***Lemma 6.*** *If*

$$\int_0^\theta |u(t)|\,dt = o(|\theta|) \tag{3.4.1}$$

then

$$\int_0^\theta |u(r,t)|\,dt = o(|\theta|), \tag{3.4.2}$$

uniformly for $1-r \leqq |\theta|$.

We may suppose θ positive. The conclusion then asserts that, given a positive ε, we have

$$\int_0^\theta |u(r,t)|\,dt \leqq \varepsilon\,\theta$$

for

$$\delta = 1-r \leqq \theta \leqq \zeta(\varepsilon)\,[8].$$

[8]) We state and prove what we shall actually require. The restrictions on δ and θ are no doubt stronger than is necessary.

It is of course not true that

$$\int_0^\theta |u(r,t)|\,dt = o(|\theta|)$$

uniformly for $r \leqq 1$; in fact

$$\int_0^\theta |u(r,t)|\,dt \sim \theta\,u(r,0),$$

for a fixed r, and $u(r,0)$ is not generally 0.

It is enough to show that the integral $J(\theta, \delta)$ of (3.3.5) satisfies such an inequality.

We transform $J(\theta, \delta)$ as in (3.3.6). Then

$$U(\pi)\,\chi(\pi, \theta, \delta) = U(\pi)\int_0^\theta \frac{\delta\, dt}{\delta^2 + (\pi - t)^2}$$

is less than a multiple of $\delta\theta$, and is $o(\theta)$ in the sense required. It is therefore sufficient to prove that

$$\int_0^\pi \frac{\delta\,\theta\,\varphi\, U(\varphi)}{(\delta^2 + \varphi^2)(\delta^2 + (\varphi - \theta)^2)}\, d\varphi = o(\theta) \tag{3.4.3}$$

and

$$\int_0^\pi \frac{\delta\,\theta^2\, U(\varphi)}{(\delta^2 + \varphi^2)(\delta^2 + (\varphi - \theta)^2)}\, d\varphi = o(\theta). \tag{3.4.4}$$

The argument is, after Lemma 4, the same for either integral. We choose $\eta = \eta(\varepsilon)$ so that

$$U(\varphi) \leqq \varepsilon\,\varphi \qquad (0 < \varphi \leqq \eta).$$

Then (taking the first integral for example) we have

$$\int_0^\eta \frac{\delta\,\theta\,\varphi\, U(\varphi)}{(\delta^2+\varphi^2)(\delta^2+(\varphi-\theta)^2)}\, d\varphi \leqq \varepsilon \int_0^\infty \frac{\delta\,\theta\,\varphi^2}{(\delta^2+\varphi^2)(\delta^2+(\varphi-\theta)^2)}\, d\varphi \leqq A\varepsilon\theta. \tag{3.4.5}$$

But, if $\theta < \frac{1}{2}\eta$, we have also

$$\int_\eta^\pi \frac{\delta\theta\varphi\, U(\varphi)}{(\delta^2+\varphi^2)(\delta^2+(\varphi-\theta)^2)}\, d\varphi < C\delta\theta \int_\eta^\pi \frac{\varphi^2\, d\varphi}{\varphi^2 \cdot \frac{1}{4}\varphi^2} = 4C\delta\theta \int_\eta^\pi \frac{d\varphi}{\varphi^2} = \tag{3.4.6}$$

$$= \frac{4\,C\,\delta\,\theta}{\eta} \leqq \frac{4\,C\,\theta^2}{\eta} = o(\theta);$$

and (3.4.3) follows from (3.4.5) and (3.4.6).

3.5. ***Lemma 7.*** *If $u(t)$ satisfies the condition of Lemma 5, then*

$$\int_0^\theta |f'(r\, e^{it})|\, dt \leqq \frac{A\,C\,|\theta|}{1 - r}. \tag{3.5.1}$$

It is enough to prove that

$$\int_{\delta}^{\theta} |u_r(r,t)|\,dt \leqq \frac{AC\theta}{\delta}, \tag{3.5.2}$$

$$\int_{\delta}^{\theta} |u_t(r,t)|\,dt \leqq \frac{AC\theta}{\delta}, \tag{3.5.3}$$

for $\theta > 0$, $\delta < \frac{1}{2}$. Here the suffixes denote partial differentiations with respect to r and t.

These inequalities are corollaries of those occurring in the proof of Lemma 5. For

$$u_r(r,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\varphi)\frac{\partial}{\partial r}\left(\frac{1-r^2}{1-2r\cos(\varphi-t)+r^2}\right)d\varphi,$$

and

$$\left|\frac{\partial}{\partial r}\left(\frac{1-r^2}{1-2r\cos(\varphi-t)+r^2}\right)\right| = \frac{\left|2(1-r)^2-4(1+r^2)\sin^2\frac{1}{2}(\varphi-t)\right|}{\left((1-r)^2+4r\sin^2\frac{1}{2}(\varphi-t)\right)^2}$$

is less than

$$A\frac{\delta^2+(\varphi-t)^2}{(\delta^2+(\varphi-t)^2)^2} = \frac{A}{\delta^2+(\varphi-t)^2}.$$

Hence

$$\int_{\delta}^{\theta} |u_r(r,t)|\,dt$$

is majorised by an integral which is, apart from a factor δ, that occurring in the proof of Lemma 5. And similarly

$$\left|\frac{\partial}{\partial t}\left(\frac{1-r^2}{1-2r\cos(\varphi-t)+r^2}\right)\right| = \frac{2r(1-r^2)\,|\sin(\varphi-t)|}{(1-2r\cos(\varphi-t)+r^2)^2}$$

is less than

$$A\frac{\delta\,|\varphi-t|}{(\delta^2+(\varphi-t)^2)^2} \leqq \frac{A}{\delta^2+(\varphi-t)^2}.$$

Lemma 8. *If $u(t)$ satisfies the conditions of Lemma 6, then*

$$\int_{0}^{\theta} |f'(re^{it})|\,dt = o\left(\frac{|\theta|}{1-r}\right), \tag{3.5.4}$$

uniformly for $1-r \leqq |\theta|$.

This follows in the same way from the proof of Lemma 6.

3.6. ***Lemma 9.*** *If*

$$\left|\int_0^\theta u(t)\,dt\right| \leqq C|\theta|, \tag{3.6.1}$$

for all θ, *and* a fortiori *if* $u(t)$ *satisfies* (3.3.1) *then*

$$\left|\frac{u(r,\theta)}{1-z}\right| \leqq \frac{AC}{1-r}; \tag{3.6.2}$$

and if

$$\int_0^\theta u(t)\,dt = o(|\theta|), \tag{3.6.3}$$

and a fortiori *if* $u(t)$ *satisfies* (3.4.1), *then*

$$\frac{u(r,\theta)}{1-z} = o\left(\frac{1}{1-r}\right) \tag{3.6.4}$$

when z *tends to* 1 *from inside the unit circle.*

We shall need only one clause of the lemma, viz. that (3.3.1) implies (3.6.2); but it is as easy to prove the stronger form of this proposition, which has some independent interest.

It is familiar that

$$|u(r,\theta)| \leqq \frac{AC}{1-r},$$

and we may therefore suppose, in the first part of the lemma, that $0 < \theta \leqq \frac{1}{2}\pi$. Then, if

$$U_1(\varphi) = \int_0^\varphi u(t)\,dt,$$

we have

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)\,u(\varphi)}{1-2r\cos(\varphi-\theta)+r^2}\,d\varphi$$

$$= \frac{1}{2\pi}\left[\frac{(1-r^2)\,U_1(\varphi)}{1-2r\cos(\varphi-\theta)+r^2}\right]_{-\pi}^{\pi} +$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2r(1-r^2)\sin(\varphi-\theta)\,U_1(\varphi)}{(1-2r\cos(\varphi-\theta)+r^2)^2}\,d\varphi.$$

The first term does not exceed AC. The second has a majorant

$$J = AC\int_{-\pi}^{\pi}\frac{\delta|\varphi||\varphi-\theta|}{(\delta^2+(\varphi-\theta)^2)^2}\,d\varphi \leqq AC\int_{-\pi}^{\pi}\frac{\delta(\varphi-\theta)^2}{(\delta^2+(\varphi-\theta)^2)^2}\,d\varphi +$$

$$+ AC\int_{-\pi}^{\pi}\frac{\delta\theta|\varphi-\theta|}{(\delta^2+(\varphi-\theta)^2)^2}\,d\varphi = J_1 + J_2,$$

say. Here

$$J_1 \leqq AC\int_{-\infty}^{\infty}\frac{\delta\,d\varphi}{\delta^2+(\varphi-\theta)^2} \leqq AC$$

and

$$J_2 \leqq AC\theta\int_{-\infty}^{\infty}\frac{d\varphi}{\delta^2+(\varphi-\theta)^2} \leqq \frac{AC\theta}{\delta}.$$

Hence

$$J \leqq AC\left(1+\frac{\theta}{\delta}\right),$$

and so

$$\left|\frac{u(r,t)}{1-z}\right| \leqq \frac{AC}{|1-z|} + \frac{AC}{\delta} \leqq \frac{AC}{\delta}.$$

This proves the first half of the lemma. The second (with o) is not wanted here, but we sketch the proof for the sake of completeness. It is enough to show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2r(1-r^2)\sin(\varphi-\theta)\,U_1(\varphi)}{(1-2r\cos(\varphi-\theta)+r^2)^2}\,d\varphi = o(1) + o\left(\frac{|\theta|}{\delta}\right).$$

We choose $\zeta = \zeta(\varepsilon)$ so that $|U_1(\varphi)| \leqq \varepsilon|\varphi|$ for $|\varphi| \leqq \zeta$. The argument which precedes then shows that

$$\left|\frac{1}{2\pi}\int_{-\zeta}^{\zeta}\right| \leqq A\varepsilon\int_{-\zeta}^{\zeta}\frac{\delta|\varphi||\varphi-\theta|}{(\delta^2+(\varphi-\theta)^2)^2}\,d\varphi \leqq A\varepsilon\left(1+\frac{|\theta|}{\delta}\right),$$

and the rest of the integral tends to zero with δ and θ, for any fixed ζ.

It will be observed that here we need not make any reservation about the relative magnitudes of θ and δ.

3.7. ***Lemma 10.*** *If* $u(t)$ *satisfies* (3.3.1) *then*

$$\left|\frac{f'(z)}{1-z}\right| \leqq \frac{AC}{(1-r)^2}; \tag{3.7.1}$$

and if $u(t)$ *satisfies* (3.4.1) *then*

$$\frac{f'(z)}{1-z} = o\left(\frac{1}{(1-r)^2}\right). \tag{3.7.2}$$

We prove the first clause only; the second is not required, and the reader will have no difficulty in making the appropriate modifications in the proof.

It is enough to prove that

$$\left|\frac{u_r(r,\theta)}{1-z}\right| \leqq \frac{AC}{\delta^2}, \qquad \left|\frac{u_\theta(r,\theta)}{1-z}\right| \leqq \frac{AC}{\delta^2}.$$

But [9]) neither $|u_r|$ nor $|u_\theta|$ exceeds the sum of

$$A\int_0^\pi \frac{|u(\varphi)|}{\delta^2+(\varphi-\theta)^2}\,d\varphi = A\frac{U(\pi)}{\delta^2+(\pi-t)^2} + 2A\int_0^\pi \frac{U(\varphi)\,|\varphi-\theta|}{(\delta^2+(\varphi-\theta)^2)^2}\,d\varphi$$

and a similar contribution arising from negative values of φ. The integral here has a majorant which differs from that of Lemma 9 only in the absence of a factor δ, and the conclusion follows.

Proof of Theorem 3.

3.8. We may suppose (as we explained in § 3.1) that

$$u(t) \sim \frac{1}{2}a_0 + \sum_1^\infty a_n \cos nt$$

is even, and that $\theta = 0$, $s = 0$.

We write

$$f(z) = \sum c_n z^n, \qquad c_0 = \frac{1}{2}a_0, \qquad c_n = a_n \qquad (n > 0),$$

$$s_n = s_n(f) = c_0 + c_1 + \ldots + c_n, \qquad \sigma_n = \sigma_n(f) = \frac{s_0 + s_1 + \ldots + s_n}{n+1};$$

so that

$$s_n - \sigma_n = \frac{c_1 + 2c_2 + \ldots + nc_n}{n+1} = \frac{s_n(f')}{n+1}.$$

[9]) See § 3.5.

Then

$$\sum_1^n \sigma_\nu^2 = o(n)$$

by the theorem of Fejér and Lebesgue, and it is sufficient to prove that

$$\sum_0^n \left(\frac{s_\nu(f')}{\nu+1}\right)^2 = o(n \log n)$$

or, what is the same thing, that

$$\sum_0^n (s_\nu(f'))^2 = o(n^3 \log n). \tag{3.8.1}$$

Suppose now that

$$\delta = 1 - r = \frac{1}{n}.$$

Then

$$\sum_0^n (s_\nu(f'))^2 \leqq \left(1 - \frac{1}{n}\right)^{-2n} \sum_0^\infty (s_\nu(f'))^2 r^{2\nu}.$$

The first factor is less than A for $n \geqq 2$, and

$$\sum_0^\infty (s_\nu(f'))^2 r^{2\nu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\frac{f'(z)}{1-z}\right|^2 d\theta. \tag{3.8.2}$$

Hence (3.8.1) will follow from

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\frac{f'(z)}{1-z}\right|^2 d\theta = o\left(\frac{1}{\delta^3} \log \frac{1}{\delta}\right). \tag{3.8.3}$$

It will be sufficient to consider the part of the integral in which θ is positive. We write

$$J = \int_0^\pi \left|\frac{f'(z)}{1-z}\right|^2 d\theta = \int_0^\delta + \int_\delta^\pi = J_1 + J_2, \tag{3.8.4}$$

and show that J_1 and J_2 are of the form required.

If $F(\theta) = \int_\delta^\theta |f'(re^{it})|\, dt$ then $F(\theta) = O\left(\frac{\theta}{\delta}\right)$, by Lemma 7;

and $\frac{f'(z)}{1-z} = O\left(\frac{1}{\delta^2}\right)$ by Lemma 10. Hence

$$(3.8.5)\quad J_1 = O\left(\frac{1}{\delta^2}\right)\int_0^\delta \frac{|f'(z)|}{|1-z|}\,d\theta = O\left(\frac{1}{\delta^2}\right)\int_0^\delta \frac{|f'(r\,e^{i\theta})|}{(\delta^2+\theta^2)^{\frac{1}{2}}}\,d\theta$$

$$= O\left(\frac{1}{\delta^2}\right)\left\{\frac{F(\delta)}{(\delta^2+\delta^2)^{\frac{1}{2}}}+\int_0^\delta \frac{\theta\,F(\theta)}{(\delta^2+\theta^2)^{\frac{3}{2}}}\,d\theta\right\}$$

$$= O\left(\frac{1}{\delta^2}\cdot\frac{\delta}{\delta\cdot\delta}\right)+O\left(\frac{1}{\delta^3}\int_0^\delta \frac{\theta^2\,d\theta}{(\delta^2+\theta^2)^{\frac{3}{2}}}\right)$$

$$= O\left(\frac{1}{\delta^3}\right)+O\left(\frac{1}{\delta^3}\int_0^1 \frac{u^2\,du}{(1+u^2)^{\frac{3}{2}}}\right) = O\left(\frac{1}{\delta^3}\right) = o\left(\frac{1}{\delta^3}\log\frac{1}{\delta}\right).$$

On the other hand

$$(3.8.6)\qquad J_2 = O\left(\frac{1}{\delta^2}\right)\int_\delta^\pi \frac{|f'(r\,e^{i\theta})|}{(\delta^2+\theta^2)^{\frac{1}{2}}}\,d\theta.$$

The integral here is

$$(3.8.7)\qquad \frac{F(\pi)}{(\delta^2+\pi^2)^{\frac{1}{2}}}-\frac{F(\delta)}{(2\,\delta^2)^{\frac{1}{2}}}+\int_\delta^\pi \frac{\theta\,F(\theta)}{(\delta^2+\theta^2)^{\frac{3}{2}}}\,d\theta.$$

The first two terms give $O(1)$ and $O(1/\delta)$. In the last, since $\delta \leqq \theta$, we may use (3.5.4), so that

$$\int_\delta^\pi \frac{\theta\,F(\theta)}{(\delta^2+\theta^2)^{\frac{3}{2}}}\,d\theta = o\left(\int_\delta^\pi \frac{\theta^2\,d\theta}{\delta(\delta^2+\theta^2)^{\frac{3}{2}}}\right) = o\left(\frac{1}{\delta}\int_1^{\pi/\delta}\frac{u^2\,du}{(1+u^2)^{\frac{3}{2}}}\right) = o\left(\frac{1}{\delta}\log\frac{1}{\delta}\right).$$

Hence, after (3.8.6) and (3.8.7), we obtain

$$(3.8.8)\qquad J_2 = o\left(\frac{1}{\delta^3}\log\frac{1}{\delta}\right);$$

and the theorem follows from (3.8.5) and (3.8.8).

§ 4. Conclusion.

4. 1. We conclude with a few miscellaneous remarks, in part concerning problems left open by our analysis.

(1) There is no difficult in proving that, under the conditions of Theorem 3,

$$S_n = \sum_0^n |s_\nu - s|^k = o\,(n\,(\log n)^{k-1})$$

for $k \geqq 2$. We replace (3.8.2) by

$$\sum_0^\infty |s_n(f')|^k r^{kn} \leqq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{f'(z)}{1-z}\right|^{k'} d\theta\right)^{k-1} \text{[10]},$$

where $k' = \frac{k}{k-1}$, and repeat the argument with the obvious changes. The argument of § 2 shows that S_n may be nearly as large as

$$n (\log n)^{\frac{1}{2}k};$$

but the gap between $(\log n)^{\frac{1}{2}k}$ and $(\log n)^{k-1}$ remains open.

On the other hand Theorem 3, together with Hölder's inequality, shows that

$$S_n = o(n (\log n)^{\frac{1}{2}k})$$

when $k \leqq 2$; and this, after § 2, is the best possible result.

(2) It is natural to ask whether the Fourier series of an integrable $f(\theta)$ is strongly summable for almost all θ. Our theorems do not settle this question, though they may suggest that the answer is negative. The series is not necessarily strongly summable in the „Lebesgue set“ of $f(\theta)$, but it may conceivably be so in some other „full“ set of θ. Thus, after Kolmogoroff, Seliverstoff, and Plessner,

$$s_n = o(\sqrt{\log n}),$$

when $f(\theta)$ is L^2, for almost all θ; but this is not necessarily true in the Lebesgue set or even at a point of continuity.

This is no doubt the most interesting question still left open.

(3) We have asked ourselves whether

$$\int_0^t |\varphi(x, u)| \, (1 + \log^+ |\varphi(x, u)|) \, du = o(t)$$

is a sufficient condition for strong summability [11]), but without result.

(4) We might define a weaker type of „strong summability“; we might define it, for example,

$$\sum_1^n \frac{|s_\nu - s|}{\nu} = o(\log n) \tag{4.1.1}$$

[10]) This is effectively one of the Hausdorff inequaliies.

[11]) Compare the results of Paley referred to in [1]).

or

$$\sum_{2}^{n} \frac{|s_\nu - s|}{\nu \log \nu} = o(\log \log n),$$

or by similar equations with $|s_\nu - s|^k$. The most interesting of these notions is (4.1.1), which may be called „strong logarithmic summability"; and we may ask whether this property is a consequence of the conditions of Theorem 3. The answer is again negative, as may be shown by an appropriate modification of the argument of § 2.

There is therefore some interest in our final theorem, which follows.

Theorem 4. If

$$f(z) = \sum c_n z^n$$

is a power series of the complex class L, and

(4.1.2)
$$\int_0^\theta |f(e^{it})|\, dt = o(|\theta|),$$

then

(4.1.3)
$$\sum_1^n \frac{|s_\nu|}{\nu} = o(\log n).$$

We say that

$$g(z) = \sum b_n z^n$$

belongs to L if

$$\mu(r, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(r e^{i\theta})|\, d\theta$$

is bounded for $r < 1$. It is known [12]) that then

(4.1.4)
$$\sum_0^\infty \frac{|b_n|}{n+1} r^n \leqq A\, \mu(r, g).$$

Suppose now that $f(z)$ satisfies the conditions of Theorem 4, and apply (4.1.4) to

$$g(z) = \frac{f(z)}{1-z} = \sum s_n z^n$$

[12]) Hardy and Littlewood (**5**, 208, Theorem 16).

on the circle $r = 1 - \delta = 1 - \frac{1}{n}$. Then

$$\sum_{0}^{\infty} \frac{|s_n|}{n+1} r^n \leqq A \int_{-\pi}^{\pi} \left| \frac{f(z)}{1-z} \right| d\theta.$$

An argument like that of § 3.8 shows that the last integral is $o\left(\log \frac{1}{\delta}\right)$, and (4.1.3) follows.

Here then there is a difference between Fourier series and power series: it is the only one which we have found connected with this particular problem.

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COMMENTS

p. 176. Simpler proofs of Theorem 3 have been given by T. Kawata, *Proc. Imp. Acad. Tokyo*, 15 (1939), 243–6, and O. Szász, *Trans. Amer. Math. Soc.* 48 (1940), 117–25. F. T. Wang, *Duke Math. J.* 12 (1945), 77–87, has shown that the hypothesis (3.1.1) in Theorem 3 can be replaced by (1.1.6) and (1.1.7) with $r = 1$.

p. 182. The results of Lemmas 9 and 10 give the estimates

$$|u(r,\theta+t)| \leqslant AC\frac{|1-re^{it}|}{1-r},$$

$$|f'(re^{i\theta+it})| \leqslant AC\frac{|1-re^{it}|}{(1-r)^2},$$

where A is an absolute constant, and

$$C = \sup_{0<|t|\leqslant\pi}\left\{\frac{1}{t}\int_0^t |u(\theta+w)|\,dw\right\}.$$

Combined with the maximal theorem 'Real Max' (1930, 1), these estimates have proved useful in the theory associated with the Littlewood–Paley g functions (for a generalization of Lemma 9, see T. M. Flett, *Proc. London Math. Soc.* (3), 7 (1957), 113–41, Lemma 4).

p. 187. The question (2) whether the Fourier series of an integrable f is strongly summable for almost all θ was answered affirmatively, by Marcinkiewicz for index 2, and by Zygmund for general index (see Z II, p. 184).

The question (3) whether the condition

$$\int_0^t |\phi(x,u)|(1+\log^+|\phi(x,u)|)\,du = o(t)$$

implies strong summability was answered affirmatively by F. T. Wang, loc. cit.

p. 188, *Theorem* 4. Zygmund, *Bull. de l'Acad. Polonaise*, 1924, 243–50, had shown that if u belongs to the real class L and $\int_0^\theta u(t)\,dt = o(|\theta|)$, then

$$\sum_1^n s_\nu/\nu = o(\log n).$$

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SOME MORE THEOREMS CONCERNING FOURIER SERIES AND FOURIER POWER SERIES

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1. Introduction

1.1. The principal theorem in this paper is Theorem 10: if $u(\theta)$ is periodic, with period 2π, and integrable, $s_n(x)$ is the partial sum of the Fourier series of $u(\theta)$, for $\theta = x$, $s(x)$ is arbitrary,

$$\phi(x, \theta) = \tfrac{1}{2}\{u(x + \theta) + u(x - \theta) - 2s(x)\}, \tag{1.1.1}$$

and

$$k \geqq p > 1, \tag{1.1.2}$$

then

$$\left(\sum_1^\infty \frac{|s_n(x) - s(x)|^k}{n}\right)^{1/k} \leqq K(p, k)\left(\int_0^\pi \frac{|\phi(x, \theta)|^p}{\theta}\, d\theta\right)^{1/p}. \tag{1.1.3}$$

This theorem is, in a sense, a theorem of 'strong summability'.[1] It is known that, if

$$\int_0^\theta |\phi(x, t)|^p\, dt = o(\theta), \tag{1.1.4}$$

for some $p > 1$, then

$$\sum_1^n |s_n(x) - s(x)|^k = o(n), \tag{1.1.5}$$

for every positive k. In Theorem 10 both hypothesis and conclusion are stronger. In fact (1.1.4) is equivalent to

$$\int_\theta^{2\theta} \frac{|\phi(x, t)|^p}{t}\, dt = o(1), \tag{1.1.6}$$

and (1.1.5) to

$$\sum_n^{2n} \frac{|s_\nu(x) - s(x)|^k}{\nu} = o(1); \tag{1.1.7}$$

and (1.1.6) and (1.1.7) are plainly consequences of the convergence of the integral and the series in (1.1.3).

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[1] See Hardy and Littlewood (**2**, **4**, **8**), and Zygmund (**15**), 237–241. The bold-faced numbers refer to the list of references at the end of the paper.

There are other points of difference between Theorem 10 and the theorems about strong summability. Thus (1.1.4) says the less the smaller p, and (1.1.5) says the more the larger k, in each case because of Hölder's inequality. There are no such obvious relations of inclusion between different cases of Theorem 10. The convergence of

$$\sum n^{-1} a_n^r \tag{1.1.8}$$

for given positive a_n and r, does not imply its convergence[2] for any other r; and the integral and series in (1.1.3), for different pairs of values of p and k, are similarly independent, so that no case of the theorem implies any other case in any trivial manner.

Finally, (1.1.4) is satisfied for almost all x, if $u(\theta)$ is L^p, while the integral in (1.1.3) may diverge for almost all x.

1.2. In §§2–3 we prove some 'pure inequalities' which we require later; the theorem essential for our applications is Theorem 3. In §2 we deduce this theorem from a very general theorem (Theorem 1) which we have proved elsewhere;[3] but, in view of the length and difficulty of the proof of Theorem 1, we add a direct proof of Theorem 3 in §3.

In §4 we prove our main theorem for a special class of functions, those which are boundary functions of analytic functions $f(re^{i\theta})$ regular for $r < 1$. The Fourier series of such functions are 'Fourier power series'.[4] In this case we can (as in general we cannot) include the value $p = 1$.

In §§5–6 we complete the proof of Theorem 10, for general $u(\theta)$. In §§7–9 we give a more direct proof of an analogous theorem for Fourier cosine transforms; and we conclude, in §10, with a few miscellaneous comments.

2. Inequalities

2.1. Our argument depends upon a number of special cases of a very general inequality[5] which we proved in **7** and restate here.

We suppose that

$$0 < p \leqq q, \qquad r > 0, \qquad \gamma = \alpha + \beta - 1,$$

$$\frac{1}{\tau} = \frac{1}{p} + \frac{1}{q}, \qquad \alpha_0 = 1 - \frac{1}{q} + \frac{1}{r}, \qquad \beta_0 = 1 - \frac{1}{p} + \frac{1}{r},$$

$$a_0 = 0, \qquad b_0 = 0, \qquad a_n \geqq 0, \qquad b_n \geqq 0,$$

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0,$$

[2] Thus (1.1.8) is convergent, with $a_n = (\log n + 1)^{-\alpha}$, if and only if $r > 1/\alpha$. On the other hand, if

$$a_n = n^{\beta} \quad (n = 2^m), \qquad a_n = 0 \quad (n \neq 2^m),$$

then (1.1.8) is convergent if and only if $r < 1/\beta$.

[3] Hardy and Littlewood (**7**), Theorem 1.

[4] For fuller explanations see Hardy and Littlewood (**5**).

[5] Hardy and Littlewood (**7**), Theorem 1.

and

$$A^p = \sum_1^\infty n^{-1}(n^\alpha a_n)^p, \qquad B^q = \sum_1^\infty n^{-1}(n^\beta b_n)^q, \qquad C^r = \sum_1^\infty n^{-1}(n^\gamma c_n)^r.$$

We allow infinite values of p, q, or r; if, for example, $p = \infty$, then A is to be interpreted as

$$\lim_{N\to\infty} \lim_{p\to\infty} \left(\sum_1^N n^{-1}(n^\alpha a_n)^p\right)^{1/p} = \max\,(n^\alpha a_n). \tag{2.1.1}$$

The fundamental inequality is

$$C \leqq KAB, \tag{I: 2.1.2}$$

where

$$K = K(p, q, r, \alpha, \beta). \tag{2.1.3}$$

We say that a set of conditions is necessary and sufficient for (I) if, when the conditions are satisfied, (I) is true for some K of the type (2.1.3) and all a_n, b_n, and, when they are not satisfied, (I) is false for every such K and some a_n, b_n.

In stating the theorem we distinguish between 'ordinary' and 'exceptional' cases. A case is ordinary when p, q, r are finite and $p \neq 1$, and otherwise exceptional.

THEOREM 1. (1) *It is necessary for the truth of* (I) *that*

$$p \geqq 1. \tag{2.1.4}$$

(2) [*Ordinary cases.*] *It is necessary and sufficient for the truth of* (I), *in an ordinary case, that* (2.1.4) *should be satisfied* (*so that* $p > 1$) *and also one of the four* (*mutually exclusive*) *alternative conditions*

$$\tau \leqq r \leqq p, \qquad \alpha < 1, \qquad \beta < 1; \tag{2a; 2.1.5}$$

$$p < r \leqq q, \qquad \alpha < 1, \qquad \beta \leqq \beta_0; \tag{2b; 2.1.6}$$

$$\tau \geqq 1, \qquad q < r, \qquad \alpha \leqq \alpha_0, \qquad \beta \leqq \beta_0; \tag{2c; 2.1.7}$$

$$\tau < 1, \qquad q < r \leqq \frac{\tau}{1-\tau}, \qquad \alpha \leqq \alpha_0, \qquad \beta \leqq \beta_0. \tag{2d; 2.1.8}$$

(3) [*Exceptional cases.*] *The only case*[6] *in which* $p = \infty$ *and* (I) *is true is the case*

$$p = q = r = \tau = \infty, \qquad \alpha < 1, \qquad \beta < 1.$$

When p is finite, all exceptional cases except four are normal, in that the conditions appropriate to them can be derived from those catalogued under (2) *by substitution of the special values of the parameters p, q, r and interpretation of B and C, if necessary, in accordance with the convention* (2.1.1).[7]

[6] In fact, all cases with $p = \infty$ are normal (in the sense defined below).

[7] The four cases (2a)–(2d) are mutally exclusive even when exceptional values of p, q, r are allowed, so that the definition of 'normal' is unambiguous.

The four abnormal cases are[8]

(3a) $p = q = r = 1$. *In this case the conditions are* $\alpha \leqq 1$, $\beta \leqq 1$ (*instead of* $\alpha < 1$, $\beta < 1$).

(3b) $p = 1 < r < q$. *In this case the conditions are* $\alpha < 1$, $\beta < \beta_0$ (*instead of* $\alpha < 1$, $\beta \leqq \beta_0$).

(3c) $p = 1 < r = q$. *In this case the conditions are* $\alpha \leqq 1$, $\beta \leqq \beta_0$ (*instead of* $\alpha < 1$, $\beta \leqq \beta_0$).

(3d) $p > 1$, $\tau > 1$, $r = \infty$. *In this case the conditions are* $\alpha < \alpha_0$, $\beta < \beta_0$ (*instead of* $\alpha \leqq \alpha_0$, $\beta \leqq \beta_0$).

It may be observed that $\alpha_0 < 1$ if $r > q$ and $\beta_0 < 1$ if $r > p$. Thus $\alpha < 1$ and $\beta < 1$ in all of the cases (2a)–(2d).

The case $q = \infty$

2.2. We now specialize the theorem by supposing that $q = \infty$ (so that $\tau = p$). In this case

$$B = \max (n^{\beta} b_n) ,$$

and $B < \infty$ means $b_n = O(n^{-\beta})$; and there is no real loss of generality in supposing that

$$b_n = n^{-\beta} = n^{\omega-1} ,$$

say.

We shall specialize a little further by supposing that

$$\alpha < 1 , \qquad \beta < 1 , \qquad \omega = 1 - \beta > 0 .$$

Then

$$C_n = \sum_{0 < s < n} (n - s)^{\omega-1} a_s = a_n^{(\omega)} \tag{2.2.1}$$

is effectively the Riesz or Cesàro sum,[9] of order ω, formed from the series $\sum a_n$; and (I) becomes

$$(\textstyle\sum n^{-1}(n^{\gamma} a_n^{(\omega)})^r)^{1/r} \leqq K (\textstyle\sum n^{-1}(n^{\alpha} a_n)^p)^{1/p} , \tag{I$_1$: 2.2.2}$$

with

$$K = K(p, r, \alpha, \omega) . \tag{2.2.3}$$

Making these specializations in Theorem 1, and rearranging the results in a more convenient manner, we obtain

THEOREM 2. *Suppose that*

$$1 \leqq p < \infty , \qquad \alpha < 1 , \qquad \omega > 0 .$$

[8] The order in which these cases are catalogued is not the same as in **7**; and there is no parallelism between them and (2a)–(2d).

[9] In fact C_n is Riesz's mean, with integral n, multiplied by a factor $\Gamma(\omega)$.

Riesz admits non-integral values of n, and this is important for some of the more delicate properties of his sums; but the difference is not significant here.

Then it is necessary and sufficient for the truth of (I_1), *with a* K *of type* (2.2.3), *that one or other of the sets of conditions*

$$\omega < \frac{1}{p}, \qquad p \leqq r \leqq \frac{p}{1-\omega p}, \tag{2.2.4}$$

$$\omega = \frac{1}{p}, \qquad p \leqq r < \infty, \tag{2.2.5}$$

$$\omega > \frac{1}{p}, \qquad p \leqq r \leqq \infty \tag{2.2.6}$$

should be satisfied; except that the last $\leqq$ *in* (2.2.4) *must be changed into* $<$, *and the last* $<$ *in* (2.2.5) *into* $\leqq$, *when* $p = 1$.

Finally, if we specialize still further by supposing

$$\omega = \alpha > 0, \qquad \gamma = 0,$$

(I_1) takes the form

$$\left(\sum n^{-1}(a_n^{(\alpha)})^r\right)^{1/r} \leqq K\left(\sum n^{\alpha p-1} a_n^p\right)^{1/p} \tag{I_2: 2.2.7}$$

with

$$K = K(p, r, \alpha); \tag{2.2.8}$$

and we obtain

THEOREM 3. *If*

$$1 \leqq p < \infty, \qquad 0 < \alpha < 1,$$

then it is necessary and sufficient for the truth of (I_2) *that one or other of the sets of conditions*

$$\begin{gathered}\alpha < \frac{1}{p}, \qquad p \leqq r \leqq \frac{p}{1-\alpha p} \qquad \left(\textit{and } r < \frac{1}{1-\alpha} \textit{ when } p = 1\right); \\ \alpha = \frac{1}{p}, \qquad p \leqq r < \infty; \\ \alpha > \frac{1}{p}, \qquad p \leqq r \leqq \infty\end{gathered} \tag{2.2.9}$$

should be satisfied.

3. Direct proof of Theorem 3

3.1. We have deduced Theorem 3 from Theorem 1, whose proof occupies some thirty pages of **7**; and the deduction by specialization, though straightforward, requires a good deal of attention. We therefore add a direct proof of Theorem 3, or rather of its positive clauses, which give sufficient conditions for the truth of (I_2).

We suppose first that

$$1 < p < r < \infty, \tag{3.1.1}$$

and that

$$r < \frac{p}{1 - \alpha p} \tag{3.1.2}$$

if $\alpha < 1/p$. We write[10]

$$\beta = p\alpha - 1, \qquad \gamma = \left(\frac{1}{p} - \frac{1}{r}\right)\beta, \tag{3.1.3}$$

$$S^p = \sum n^\beta a_n^p, \qquad T^r = \sum n^{-1}(a_n^{(\alpha)})^r, \tag{3.1.4}$$

so that the inequality to be proved is

$$T \leqq KS. \tag{3.1.5}$$

We can choose ρ and σ so that

$$\frac{1}{p} - 1 < \rho - \gamma < \frac{1}{p} - \alpha, \tag{3.1.6}$$

$$\frac{1}{p} - \alpha < \sigma < \frac{1}{r}.\text{[11]} \tag{3.1.7}$$

Then

$$a_n^{(\alpha)} = \sum_{s<n} s^\gamma a_s^{1-\frac{p}{r}} \cdot s^{\rho-\gamma}(n-s)^{\sigma+\alpha-1} \cdot s^{-\rho}(n-s)^{-\sigma} a_s^{\frac{p}{r}} \leqq U^{\frac{1}{p}-\frac{1}{r}} V^{\frac{1}{p'}} W^{\frac{1}{r}}, \tag{3.1.8}$$

where p' is defined as usual by

$$p' = \frac{p}{p-1}, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

and

$$U = \sum_{s<n} s^\beta a_s^p \leqq S^p, \tag{3.1.9}$$

$$V = \sum_{s<n} s^{(\rho-\gamma)p'} (n-s)^{(\sigma+\alpha-1)p'}, \tag{3.1.10}$$

$$W = \sum_{s<n} s^{-\rho r} (n-s)^{-\sigma r} a_s^p. \tag{3.1.11}$$

It follows from (3.1.6) and (3.1.7) that

$$(\rho - \gamma)\, p' > -1, \qquad (\sigma + \alpha - 1)\, p' > -1,$$

[10] There will be no disadvantage in using β and γ in senses different from those of §2.
[11] Observe that this choice of σ presupposes (3.1.2.).

and so

(3.1.12) $$V \leqq K n^{(\rho-\gamma+\sigma+\alpha-1)p'+1},$$

with

(3.1.13) $$K = K(p, r, \alpha, \rho, \sigma).$$

Combining (3.1.9) and (3.1.12) with (3.1.8), we obtain

(3.1.14) $$n^{-1}(a_n^{(\alpha)})^r \leqq K\, S^{r-p}\, n^t W,$$

where

(3.1.15) $$t = r\left(\rho - \gamma + \sigma + \alpha - 1 + \frac{1}{p'}\right) - 1.$$

Hence

(3.1.16) $$\begin{aligned} T^r = \sum_n n^{-1}(a_n^{(\alpha)})^r &\leqq K\, S^{r-p} \sum_n n^t \sum_{s<n} s^{-\rho r}(n-s)^{-\sigma r} a_s^p \\ &= K\, S^{r-p} \sum_s s^{-\rho r} a_s^p \sum_{n>s} n^t (n-s)^{-r\sigma} \end{aligned}$$

But $\sigma r < 1$, by (3.1.7), and

$$t - \sigma r = r\left(\rho - \gamma + \alpha - \frac{1}{p}\right) - 1 < -1,$$

by (3.1.6), so that

(3.1.17) $$\sum_{n>s} n^t (n-s)^{-\sigma r} < K\, s^{t-\sigma r+1},$$

with a K of type (3.1.8). Finally

$$t - \sigma r - \rho r + 1 = -\gamma r + \alpha r - \frac{r}{p} = \beta,$$

by (3.1.15) and (3.1.3); so that (3.1.16) and (3.1.17) give

$$T^r \leqq K\, S^{r-p} \sum_s s^\beta a_s^p = K\, S^r.$$

Here $K = K(p, r, \alpha, \rho, \sigma)$, and $K = K(p, r, \alpha)$ when suitable values, satisfying (3.1.6) and (3.1.7), are given to ρ and σ.

3.2. Suppose next that

(3.2.1) $$p = 1 < r < \frac{1}{1-\alpha}.$$

We choose ρ and σ as in §3.1, so that

$$0 < \rho - \gamma < 1 - \alpha < \sigma < \frac{1}{r},$$

and

$$s^{\rho-\gamma}(n-s)^{\sigma+\alpha-1} < n^{\rho-\gamma+\sigma+\alpha-1}.$$

Thus (3.1.8) may be replaced by

$$a_n^{(\alpha)} \leqq n^{\rho-\gamma+\sigma+\alpha-1}\, U^{1/r'}\, W^{1/r},$$

where U is defined as before (with $p = 1$), and r' like p'. The proof then proceeds as in §3.1.

There remain the marginal cases,

$$r = p; \quad p > 1, \quad \alpha < \frac{1}{p}, \quad r = \frac{p}{1-\alpha p}; \quad p > 1, \quad \alpha > \frac{1}{p}, \quad r = \infty.$$

The first of these, like the case (3.2.1) treated above, may be disposed of by an appropriate simplification of the main argument. But in this case we can go further, and find the best possible K. We therefore postpone this case, and treat it, in Theorem 4 below, as a separate theorem.

3.3. When

$$p > 1, \quad \alpha < \frac{1}{p}, \quad r = \frac{p}{1-\alpha p}, \tag{3.3.1}$$

the proof lies a little deeper.[12] It is impossible to choose σ so as to satisfy (3.1.7), and we must appeal to a theorem which Pólya[13] and we have proved elsewhere.

We write

$$b_n = n^{\alpha-1/p}\, a_n,$$

so that $S^p = \sum b_n^p$. Then

$$a_n^{(\alpha)} = \sum_{s<n} (n-s)^{\alpha-1} s^{\frac{1}{p}-\alpha} b_s \leqq n^{\frac{1}{p}-\alpha} \sum_{s<n} (n-s)^{\alpha-1} b_s = n^{\frac{1}{p}-\alpha} b_n^{(\alpha)}.$$

Hence

$$T^{\frac{p}{1-\alpha p}} = \sum n^{-1}(a_n^{(\alpha)})^{\frac{p}{1-\alpha p}} \leqq \sum (b_n^{(\alpha)})^{\frac{p}{1-\alpha p}} \leqq K\left(\sum b_n^p\right)^{\frac{1}{1-\alpha p}} = KS^{\frac{p}{1-\alpha p}},$$

by the theorem referred to.

3.4. There remains the case

$$p > 1, \quad \alpha > \frac{1}{p}, \quad r = \infty. \tag{3.4.1}$$

The theorem then asserts that

$$a_n^{(\alpha)} = \sum_{s<n} (n-s)^{\alpha-1} a_s \leqq KS. \tag{3.4.2}$$

[12] We are in case V of Theorem 1: see **7**, §§7–8.

[13] Hardy, Littlewood and Pólya (**9**, Theorem 5).

Now

$$\sum_{s<n}(n-s)^{\alpha-1}a_s = \sum_{s<n} s^{\alpha-\frac{1}{p}}a_s\cdot s^{\frac{1}{p}-\alpha}(n-s)^{\alpha-1} \tag{3.4.3}$$
$$\leqq S\Big(\sum_{s<n} s^{\frac{p'}{p}-\alpha p'}(n-s)^{p'(\alpha-1)}\Big)^{\frac{1}{p'}}.$$

Since

$$\frac{p'}{p}-\alpha p' = p'-1-\alpha p' = -1+p'(1-\alpha) > -1,$$
$$p'(\alpha-1) = p'\Big(\alpha-\frac{1}{p}\Big)-1 > -1,$$

and the sum of these indices is -1, the second factor on the right hand side of (3.4.3) is bounded; and this proves (3.4.2).

3.5. When $r = p$, we can prove the more precise result which follows.

THEOREM 4. *If $p \geqq 1$, $0 < \alpha < 1$, then*

$$\sum n^{-1}(a_n^{(\alpha)})^p \leqq (\pi \operatorname{cosec} \alpha\pi)^p \sum n^{\alpha p-1}a_n^p. \tag{3.5.1}$$

The constant factor is the best possible.

(1) If $p > 1$, we take

$$\rho = -\frac{\alpha}{p'}, \qquad \sigma = \frac{1-\alpha}{p}. \tag{3.5.2}$$

These values satisfy (3.1.6) and (3.1.7).[14] Then

$$a_n^{(\alpha)} \leqq \Big(\sum_{s<n} s^{\rho p'}(n-s)^{(\sigma+\alpha-1)p'}\Big)^{1/p'}\Big(\sum_{s<n} s^{-\rho p}(n-s)^{-\sigma p}a_s^p\Big)^{1/p} = V^{1/p'}W^{1/p}.$$

Here

$$V = \sum_{s<n} s^{-\alpha}(n-s)^{\alpha-1} < \int_0^n x^{-\alpha}(n-x)^{\alpha-1}\,dx = \pi \operatorname{cosec} \alpha\pi.$$

Hence

$$T^p = \sum n^{-1}(a_n^{(\alpha)})^p \leqq (\pi \operatorname{cosec} \alpha\pi)^{p-1}\sum_n n^{-1}\sum_{s<n} s^{-\rho p}(n-s)^{-\sigma p}a_s^p \tag{3.5.3}$$
$$= (\pi \operatorname{cosec} \alpha\pi)^{p-1}\sum_s s^{-\rho p}a_s^p \sum_{n>s} n^{-1}(n-s)^{-\sigma p}.$$

But

$$\sum_{n>s} n^{-1}(n-s)^{-\sigma p} = \sum_{n>s} n^{-1}(n-s)^{\alpha-1} < \int_s^\infty x^{-1}(x-s)^{\alpha-1}\,dx \tag{3.5.4}$$
$$= \pi \operatorname{cosec} \alpha\pi\cdot s^{\alpha-1}$$

[14] γ is now 0.

and

$$-\rho p + \alpha - 1 = -(p-1)\alpha + \alpha - 1 = \alpha p - 1. \tag{3.5.5}$$

Hence (3.5.1) follows from (3.5.3).

If $p = 1$, $r = 1$, the equations (3.5.2) reduce to $\rho = 0$, $\sigma = 1 - \alpha$, and the conditions (3.1.6) and (3.1.7) are not satisfied. But in this case

$$\sum_n n^{-1} a_n^{(\alpha)} = \sum_n n^{-1} \sum_{s<n} (n-s)^{\alpha-1} a_s = \sum_s a_s \sum_{n>s} n^{-1}(n-s)^{\alpha-1}$$
$$\leqq \pi \operatorname{cosec} \alpha\pi \sum_s s^{\alpha-1} a_s,$$

and (3.5.1) is still correct.

We have now completed the proof of the main clause of Theorem 4 (and so that of Theorem 3). To prove the constant in Theorem 4 the best possible, we take $a_n = n^{-\alpha-\delta}$, where δ is small and positive. Then

$$a_n^{(\alpha)} = \sum_{s<n} (n-s)^{\alpha-1} s^{-\alpha-\delta} \sim \frac{\Gamma(\alpha)\Gamma(1-\alpha-\delta)}{\Gamma(1-\delta)} n^{-\delta};$$

and it follows, by an argument of a familiar type,[15] that the constant cannot be less than

$$\lim_{\delta\to 0} \left(\frac{\Gamma(\alpha)\Gamma(1-\alpha-\delta)}{\Gamma(1-\delta)}\right)^p = (\pi \operatorname{cosec} \alpha\pi)^p.$$

The Riesz mean of a_n is not $a_n^{(\alpha)}$ but $a_n^{(\alpha)}/\Gamma(\alpha)$. If $a_n^{(\alpha)}$ were actually the Riesz mean, the constant would be $(\Gamma(1-\alpha))^p$.

3.6. All these theorems have naturally their analogues for integrals. In particular we require

THEOREM 5. *Suppose that p, r, and α satisfy the conditions of Theorem 3; that $f(x) \geqq 0$; and that*

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y)\, dy \tag{3.6.1}$$

is the Liouville integral of $f(x)$ of order α, with origin 0. Then

$$\left(\int_0^\infty x^{-1}(f_\alpha(x))^r\, dx\right)^{1/r} \leqq K\left(\int_0^\infty x^{\alpha p-1}(f(x))^p\, dx\right)^{1/p}. \tag{3.6.2}$$

When $r = p$ we can take $K = \Gamma(1-\alpha)$, and this is then the best possible value of K.

The proof is the same except for trivial simplifications.

Finally, taking

$$p > 1, \qquad \alpha = 1/p; \qquad p > 1, \qquad \alpha = 1/p',$$

in Theorems 3 and 5, we obtain two theorems which are particularly important for our applications.

[15] See Hardy, Littlewood and Pólya (**10**), 232.

THEOREM 6. *If*

$$p > 1, \qquad p \leqq r < \infty,$$

and $a_n \geqq 0, f(x) \geqq 0,$ *then*

$$(\sum n^{-1}(a_n^{(1/p)})^r)^{1/r} \leqq K(\sum a_n^p)^{1/p}, \tag{3.6.3}$$

$$\left(\int_0^\infty x^{-1}(f_{1/p}(x))^r\,dx\right)^{1/r} \leqq K\left(\int_0^\infty (f(x))^p\,dx\right)^{1/p}. \tag{3.6.4}$$

THEOREM 7. *If*

$$1 < p < 2, \qquad p \leqq r \leqq \frac{p}{2-p},$$

or

$$p = 2, \qquad 2 \leqq r < \infty,$$

or

$$p > 2, \qquad p \leqq r \leqq \infty,$$

and $a_n \geqq 0, f(x) \geqq 0,$ *then*

$$(\sum n^{-1}(a_n^{(1/p')})^r)^{1/r} \leqq K(\sum n^{p-2}a_n^p)^{1/p}, \tag{3.6.5}$$

$$\left(\int_0^\infty x^{-1}(f_{1/p'}(x))^r\,dx\right)^{1/r} \leqq K\left(\int_0^\infty x^{p-2}(f(x))^p\,dx\right)^{1/p}. \tag{3.6.6}$$

In each theorem $K = K(p, r)$.

4. Theorems on power series

4.1. We suppose now that

$$f(z) = \sum_0^\infty c_n z^n \tag{4.1.1}$$

is an analytic function regular for $r = |z| < 1$, that $p \geqq 1$, and that

$$\int_{-\pi}^{\pi} |f(z)|^p\,|1 - z|^{p-1}\,d\theta = \int_{-\pi}^{\pi} |f(re^{i\theta})|^p\,|1 - re^{i\theta}|^{p-1}\,d\theta \tag{4.1.2}$$

is bounded for $r < 1$.

We may always suppose, if we please, that $c_0 = 0$, since the theorems which we prove under this restriction may be extended to the general case by considering $z\,f(z)$ instead of $f(z)$. Series in which n occurs as a denominator are extended over the range 1 to ∞.

If $p = 1, f(z)$ belongs to the (complex) class L, and all the standard relations hold between the function, its boundary values, and its coefficients. In particular,

$$|c_n| \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|\, d\theta$$

and

$$\sum \frac{|c_n|}{n} \leqq \frac{A}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|\, d\theta,$$

where A is a constant.[16] Hence

$$\left(\sum \frac{|c_n|^k}{n}\right)^{1/k} \leqq \frac{A}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|\, d\theta$$

for $1 \leqq k \leqq \infty$.

The situation is not quite so simple when $p > 1$, since $f(z)$ does not usually belong to L. If however $0 < \lambda < 1$, then[17]

$$\int |f|^\lambda\, d\theta = \int |f|^\lambda\, |1 - z|^{\frac{(p-1)\lambda}{p}} \cdot |1 - z|^{-\frac{(p-1)\lambda}{p}}\, d\theta$$

$$\leqq \left(\int |f|^p\, |1 - z|^{p-1}\, d\theta\right)^{\frac{\lambda}{p}} \left(\int |1 - z|^{-\frac{(p-1)\lambda}{p-\lambda}}\, d\theta\right)^{\frac{p-\lambda}{p}},$$

and $(p - 1)\lambda < p - \lambda$, so that the second factor is bounded. Hence $f(z)$ belongs to the class L^λ, and has a boundary function

$$F(\theta) = f(e^{i\theta})$$

of L^λ. Also, since $(1 - z)^{1/p'} f(z)$ is a power series of the class L^p, and $(1 - e^{i\theta})^{1/p'} F(\theta)$ is its boundary function, we have

$$\int |F(\theta)|^p\, |1 - e^{i\theta}|^{p-1}\, d\theta = \lim_{r \to 1} \int |f(z)|^p\, |1 - z|^{p-1}\, d\theta < \infty .$$

On the other hand, if

$$\int |F(\theta)|^p\, |1 - e^{i\theta}|^{p-1}\, d\theta < \infty,$$

then $(1 - e^{i\theta})^{1/p'}F(\theta)$ is L^p, and is the boundary function of a function $(1 - z)^{1/p'}g(z)$ of the complex class L^p; and $g(z)$ must be $f(z)$, since $F(\theta)$ is the boundary function of $f(z)$. Hence

$$\int |f(z)|^p\, |1 - z|^{p-1}\, d\theta$$

is bounded. Finally, since the ratio

$$\int |F(\theta)|^p\, |\theta|^{p-1}\, d\theta \quad : \quad \int |F(\theta)|^p\, |1 - e^{i\theta}|^{p-1}\, d\theta$$

[16] Hardy and Littlewood (**3**), 208.

[17] The range of θ is always supposed to be $(-\pi, \pi)$.

lies between positive bounds depending only on p, our condition on $f(z)$ is equivalent to the condition

$$\int |F(\theta)|^p |\theta|^{p-1} d\theta < \infty . \tag{4.1.3}$$

4.2. Theorem 8. *If*

$$1 \leqq p \leqq k < \infty$$

and the integral (4.1.2) *is bounded, or* $F(\theta)$ *satisfies* (4.1.3), *then*

$$\left(\sum \frac{|c_n|^k}{n}\right)^{1/k} \leqq K\left(\int |F(\theta)|^p |\theta|^{p-1} d\theta\right)^{1/p}, \tag{4.2.1}$$

with $K = K(p, k)$.

(1) We have already disposed of the case $p = 1$; in this case we may include the value $k = \infty$.

(2) We suppose then that $p > 1$. We shall use (besides the theorems of §3) three known theorems concerning Fourier series, expressed by the inequalities

$$\left(\sum |u_n|^{p'}\right)^{1/p'} \leqq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |h|^p d\theta\right)^{1/p} \qquad (1 < p \leqq 2), \tag{4.2.2}$$

$$\sum |n|^{p-2} |u_n|^p \leqq K \int |h|^p d\theta \qquad (1 < p \leqq 2), \tag{4.2.3}$$

$$\sum |u_n|^p \leqq K \int |h|^p |\theta|^{p-2} d\theta \qquad (p \geqq 2). \tag{4.2.4}$$

In these theorems, of which the first is due to Hausdorff[18] and the second and third to ourselves,[19]

$$\sum_{-\infty}^{\infty} u_n e^{ni\theta}$$

is the complex Fourier series of a function $h(\theta)$ for which the integral on the right hand side is finite.

We must distinguish the cases $p \leqq 2$ and $p \geqq 2$.

(3) Suppose first that $1 < p \leqq 2$. It is plain from Hölder's inequality that, if (4.2.1) is true for $k = k_1$ and for $k = k_2$ (and the same p), it is true for $k_1 \leqq k \leqq k_2$. It is therefore sufficient to prove it (a) when

$$p \leqq k \leqq \frac{p}{2 - p} \tag{4.2.5}$$

and (b) when[20]

$$k \geqq p' . \tag{4.2.6}$$

[18] Hausdorff (**11**); Zygmund (**15**), 189–192, 200–202.

[19] Hardy and Littlewood (**3**), Theorems 5 and 3; Zygmund (**15**), 202–215.

[20] The two ranges overlap or abut when $p \geqq \frac{4}{3}$, and then no appeal to Hölder's inequality is necessary.

The function

$$\phi(z) = \sum n^{-1/p} z^n$$

is regular for $r \leqq 1$, $z \neq 1$, and has no zeros, except $z = 0$, in or on the unit circle.[21] Near[22] $z = 1$

$$\phi(z) \sim \frac{\Gamma(1/p')}{(1-z)^{1/p'}},$$

and the ratio

$$| \phi(e^{i\theta}) | : | \theta |^{-1/p'}$$

lies between positive bounds $K(p)$.

If

$$g(z) = \sum b_n z^n = \frac{f(z)}{\phi(z)}, \qquad f(z) = \phi(z)\, g(z),$$

then

$$c_n = \sum_{s<n} (n-s)^{-1/p} b_s = b_n^{(1/p')},$$

in the notation of §2.2. The function $g(z)$ is regular and belongs to L^p, and has a boundary function $g(e^{i\theta}) = G(\theta)$; and the ratio

$$| G(\theta) | : | \theta |^{1/p'} | F(\theta) |$$

lies (for almost all θ) between positive bounds $K(p)$.

We now distinguish cases (a) and (b). In case (a) we use Theorem 7 and the second of the three theorems quoted in (2), viz. (4.2.3). These give

$$\left(\sum \frac{|c_n|^k}{n}\right)^{1/k} = \left(\sum \frac{|b_n^{(1/p')}|^k}{n}\right)^{1/k} \leqq K\left(\sum n^{p-2} |b_n|^p\right)^{1/p}$$

$$\leqq K\left(\int |G|^p\, d\theta\right)^{1/p} \leqq K\left(\int |F|^p |\theta|^{p-1}\, d\theta\right)^{1/p}.$$

In case (b) we use Theorem 6 (with p' in place of p) and the first of the theorems of (2), viz. (4.2.2). We thus obtain

$$\left(\sum \frac{|c_n|^k}{n}\right)^{1/k} = \left(\sum \frac{|b_n^{(1/p')}|^k}{n}\right)^{1/k} \leqq K\left(\sum |b_n|^{p'}\right)^{1/p'}$$

$$\leqq K\left(\int |G|^p\, d\theta\right)^{1/p} \leqq K\left(\int |F|^p |\theta|^{p-1}\, d\theta\right)^{1/p}.$$

[21] This is a case of 'Kakeya's Theorem'.

[22] In fact

$$\phi(z) - \Gamma\left(\frac{1}{p'}\right)\left(\log\frac{1}{z}\right)^{-1/p'}$$

is regular for $z = 1$. See for example Lindelöf (**12**), 138.

Thus the theorem is proved in cases (a) and (b), and so whenever $p \leqq 2$.

(4) When $p \geqq 2$ we use

$$\psi(z) = \sum n^{-1/p'} z^n$$

instead of $\phi(z)$. If

$$f = \psi g , \qquad g = \sum b_n z^n ,$$

then

$$c_n = b_n^{(1/p)} .$$

The function $g(z)$ has a boundary function $g(e^{i\theta}) = G(\theta)$, and the ratio

$$| G(\theta) | : | \theta |^{1/p} | F(\theta) |$$

lies between positive bounds $K(p)$. Hence, using now Theorem 6 and the third of the theorems of (2), viz. (4.2.4), we obtain

$$\left(\sum \frac{| c_n |^k}{n}\right)^{1/k} = \left(\sum \frac{| b_n^{(1/p)} |^k}{n}\right)^{1/k} \leqq (\sum | b_n |^p)^{1/p}$$

$$\leqq K\left(\int | G |^p | \theta |^{p-2} d\theta\right)^{1/p} \leqq K\left(\int | F |^p | \theta |^{p-1} d\theta\right)^{1/p} ,$$

thus completing the proof of Theorem 8.

The result is not true, for any $p > 1$, when $k = \infty$. It would imply that $b_n^{(1/p')}$ is bounded for any $g(z) = \sum b_n z^n$ of the class L^p, and it is not difficult to construct an example to the contrary.

4.3. Theorem 9. *Suppose that $F(\theta)$ is the boundary function of an analytic function $f(z) = \sum c_n z^n$ of the class L; that*

$$s_n(x) = \sum_0^n c_\nu e^{\nu i x} ;$$

that $k \geqq p \geqq 1$; and that

$$| \theta |^{-1} | F(x + \theta) - s(x) |^p$$

is, for a given x and $s(x)$, integrable in θ. Then

$$\left(\sum_1^\infty \frac{| s_n(x) - s(x) |^k}{n}\right)^{1/k} \leqq K\left(\int_{-\pi}^{\pi} \frac{| F(x + \theta) - s(x) |^p}{| \theta |} d\theta\right)^{1/p} ,$$

with $K = K(p, k)$.

We may suppose $x = 0$. We have then only to write

$$f(z) - s(0) = h(z) ,$$

$$g(z) = \frac{h(z)}{1 - z} = \sum (s_n(0) - s(0)) z^n ,$$

and to apply Theorem 8 to $g(z)$.

5. Extension to general Fourier series

5.1. It is natural to expect that, when $p > 1$, there will be a theorem for general Fourier series corresponding to Theorem 9.

Let us suppose that $p > 1$; that $u(\theta)$ is a periodic function of θ of the class L^p; that

$$u(\theta) \sim \sum_{-\infty}^{\infty} c_n e^{nix}$$

(or

$$u(\theta) \sim \tfrac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta))$$

is the Fourier series of $u(\theta)$; that

$$s_n(x) = \sum_{-n}^{n} c_\nu e^{\nu ix}$$

(or

$$s_n(x) = \tfrac{1}{2} a_0 + \sum_{1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)),$$

and that $\phi(x, \theta)$ is defined as in (1.1.1). We shall prove

THEOREM 10. *If $k \geqq p > 1$ and $|\theta|^{-1} |\phi(x, \theta)|^p$ is integrable in θ, for a given x and $s(x)$, then*

$$\left(\sum_{1}^{\infty} \frac{|s_n(x) - s(x)|^k}{n}\right)^{1/k} \leqq K\left(\int_0^\pi \frac{|\phi(x, \theta)|^p}{\theta}\, d\theta\right)^{1/p}, \tag{5.1.1}$$

with $K = K(p, k)$.

5.2. We may make the usual formal simplifications, supposing $x = 0$, $s(x) = 0$, and $u(\theta)$ real and even, so that

$$u(\theta) \sim \tfrac{1}{2}a_0 + \sum a_n \cos n\theta,$$

$$\phi(x, \theta) = u(\theta),$$

$$s_n = s_n(x) = s_n(0) = \tfrac{1}{2}a_0 + \sum_{1}^{n} a_\nu.$$

We have then to prove that

$$\left(\sum_{1}^{\infty} \frac{|s_n|^k}{n}\right)^{1/k} \leqq K\left(\int_0^\pi \frac{|u(\theta)|^p}{\theta}\, d\theta\right)^{1/p}. \tag{5.2.1}$$

The function u has a conjugate v, odd and of L^p, defined by

$$v(\theta) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \cot \frac{1}{2} (\phi - \theta)\, u(\phi)\, d\phi. \tag{5.2.2}$$

The associated harmonic function vanishes at the origin.

Let us assume for a moment that we have proved that

$$\int_0^\pi \frac{|v(\theta)|^p}{\theta}\,d\theta \leqq K(p)\int_0^\pi \frac{|u(\theta)|^p}{\theta}\,d\theta \tag{5.2.3}$$

whenever the integral on the right is finite. Then $F(\theta) = u(\theta) + iv(\theta)$ is the boundary function of an analytic function $f(z) = \sum c_n z^n$ satisfying the conditions of Theorem 9, and $s_n = c_0 + c_1 + \cdots + c_n$. Hence

$$\left(\sum \frac{|s_n|^k}{n}\right)^{1/k} \leqq K\left(\int \frac{|F(\theta)|^p}{|\theta|}\,d\theta\right)^{1/p} \leqq K\left(\int \frac{|u(\theta)|^p}{|\theta|}\,d\theta\right)^{1/p},$$

with $K = K(p, k)$.

The proof of Theorem 10 is thus reduced to the proof of (5.2.3).

6. Theorems on conjugate functions

6.1. It is well known[23] that a function $U(x)$ of $L^p(-\infty, \infty)$, where $p > 1$, possesses a conjugate $V(x)$, defined for almost all x by

$$V(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{U(y)}{y - x}\,dy\,, \tag{6.1.1}$$

which also belongs to $L^p(-\infty, \infty)$. The integral is a Lebesgue integral at infinity and a principal value at $y = x$.

When $U(x)$ is *even* (a hypothesis essential in the sequel), we may also write

$$V(x) = -\frac{1}{\pi}\int_0^{\infty} \frac{2x}{y^2 - x^2}\,U(y)\,dy\,. \tag{6.1.2}$$

This integral exists under wider conditions than that in (6.1.1). Suppose, for example, that $U(x)$ is L^p in every finite positive interval, and that $x^\alpha U$, where

$$\alpha > -1 - \frac{1}{p},$$

is $L^p(0, \infty)$. Then the integral converges as a principal value (for almost all x) across $y = |x|$; and, since

$$\int^{\infty} \frac{|U|}{y^2}\,dy \leqq \left(\int^{\infty} (y^\alpha |U|)^p\,dy\right)^{1/p}\left(\int^{\infty} y^{-(2+\alpha)p'}\,dy\right)^{1/p'}$$

and

$$(2 + \alpha)p' > \left(1 - \frac{1}{p}\right)p' = 1\,,$$

it converges absolutely at infinity. We may therefore define $V(x)$ by (6.1.2).

THEOREM 11. *If*

$$-1 - \frac{1}{p} < \alpha < \frac{1}{p'} = 1 - \frac{1}{p}, \tag{6.1.3}$$

[23] M. Riesz (**13**); Zygmund (**15**), 147–149.

$x^\alpha U(x)$ is $L^p(0, \infty)$, and $V(x)$ is defined by (6.1.2), then $x^\alpha V(x)$ is $L^p(0, \infty)$, and

$$\int_0^\infty (x^\alpha | V |)^p \, dx \leqq K \int_0^\infty (x^\alpha | U |)^p \, dx, \tag{6.1.4}$$

with $K = K(p, \alpha)$.

We denote by $V^*(x)$ the conjugate of the even function $| x |^\alpha U(x)$. Then $V^*(x)$ is L^p and

$$\int_0^\infty | V^* |^p \, dx \leqq K \int_0^\infty (x^\alpha | U |)^p \, dx.$$

It is therefore sufficient to prove that

$$\int_0^\infty | V^* - x^\alpha V |^p \, dx \leqq K \int_0^\infty (x^\alpha | U |)^p \, dx. \tag{6.1.5}$$

Now, when $x > 0$,

$$V^* - x^\alpha V = \frac{2}{\pi} \int_0^\infty M(y, x) y^\alpha U(y) \, dy,$$

where

$$M(y, x) = \frac{x}{y^\alpha} \frac{y^\alpha - x^\alpha}{y^2 - x^2}.$$

This function has a fixed sign, viz. that of α, and is homogeneous of degree -1; and

$$\int_0^\infty | M(y, 1) | \, y^{-1/p} \, dy = \int_0^\infty \left| \frac{y^\alpha - 1}{y^2 - 1} \right| y^{-\alpha - 1/p} \, dy < \infty$$

when α satisfies (6.1.3). Hence[24] (6.1.5) is true under the conditions of the theorem.

In particular, when $\alpha = -1/p$, we obtain

$$\int_0^\infty \frac{| V |^p}{x} \, dx \leqq K(p) \int_0^\infty \frac{| U |^p}{x} \, dx. \tag{6.1.5}$$

6.2. Theorem 12. *If $p > 1$, $u(\theta)$ is periodic and even, $\theta^{-1} | u(\theta) |^p$ is integrable, and $v(\theta)$ is defined by (5.2.2), then*

$$\int_0^\pi \frac{| v(\theta) |^p}{\theta} \, d\theta \leqq K(p) \int_0^\pi \frac{| u(\theta) |^p}{\theta} \, d\theta. \tag{6.2.1}$$

We have

$$\begin{aligned} v(\theta) &= -\frac{1}{2\pi} \int_0^\pi \left(\cot \frac{1}{2} (\phi - \theta) - \cot \frac{1}{2} (\phi + \theta) \right) u(\phi) \, d\phi \\ &= \frac{1}{\pi} \int_0^\pi \frac{\sin \theta}{\cos \phi - \cos \theta} u(\phi) \, d\phi. \end{aligned} \tag{6.2.2}$$

[24] Hardy, Littlewood, and Pólya (**10**), Theorem 319.

If we write

$$x = \tan \tfrac{1}{2}\theta, \qquad y = \tan\tfrac{1}{2}\phi, \qquad u(\phi) = U(y), \qquad v(\theta) = V(x),$$

then (6.2.2) becomes (6.1.2). Also

$$\int_0^\pi \frac{|u(\theta)|^p}{\sin\theta}\,d\theta = \int_0^\infty \frac{|U(x)|^p}{x}\,dx, \qquad \int_0^\pi \frac{|v(\theta)|^p}{\sin\theta}\,d\theta = \int_0^\infty \frac{|V(x)|^p}{x}\,dx,$$

and therefore, by Theorem 11,

$$\int_0^\pi \frac{|v(\theta)|^p}{\sin\theta}\,d\theta \leqq K(p)\int_0^\pi \frac{|u(\theta)|^p}{\sin\theta}\,d\theta.$$

This implies (6.2.1).

In proving this theorem, we have completed the proof of Theorem 10.

6.3. It is to be observed that the truth of Theorems 11 and 12 depends essentially on the hypothesis that $U(x)$ and $u(\theta)$ are even. It is not true, without this restriction, that the integrability of $\theta^{-1} \mid u(\theta) \mid^p$ involves that of $\theta^{-1} \mid v(\theta) \mid^p$. Suppose for example that

$$u(\theta) = \sum \frac{\sin n\theta}{n(\log n)^\beta}, \qquad v(\theta) = -\sum \frac{\cos n\theta}{n(\log n)^\beta},$$

where β is positive. Then $u(\theta)$ behaves like a multiple of $\mid \log\theta \mid^{-\beta}$ for small positive θ, and $\mid \theta \mid^{-1} \mid u \mid^p$ is integrable if (and only if) $\beta p > 1$. On the other hand $v(\theta)$ behaves like a multiple of $\mid \log\theta \mid^{1-\beta}$, if $\beta < 1$, and $v(\theta) - v(0)$ behaves in this way if $\beta > 1$; and neither $\mid \theta \mid^{-1} \mid v \mid^p$ nor $\mid \theta \mid^{-1} \mid v(\theta) - v(0) \mid^p$ is integrable unless $\beta p > p + 1$.

We can show, by an argument like that of §6.1, but based upon the formula (6.1.1) instead of upon (6.1.2), that the conclusion of Theorem 11 is true, for general $U(x)$, when $-1/p < \alpha < 1/p'$; but the value $-1/p$ of α, the critical value for our purpose, is excluded.

7. Fourier transforms

7.1. Our proof of Theorem 10 is comparatively simple (granted the inequalities of §§2–3) but very indirect, and it is natural to ask for a proof independent of the theory of analytic functions. For the sake of variety we give here not this proof but the proof of the analogous theorem for Fourier cosine transforms. To simplify the formulae, we suppose throughout that $k = p$.

We use the notion of a 'limit in mean' or 'strong limit', with index p, of a function $s_a(x)$. This, if it exists, is a function $s(x)$ such that

$$\lim_{a\to\infty} \int_0^X |s_a(x) - s(x)|^p\,dx = 0$$

for every positive and finite X. We write, after Wiener,

$$s(x) = \underset{a\to\infty}{\text{l.i.m.}}\ s_a(x).$$

A limit in mean is, apart from null sets, unique.

THEOREM 13. *Suppose that* $p > 1$,

$$\int_0^\infty \frac{|f(x)|^p}{x}\,dx < \infty, \tag{7.1.1}$$

and

$$s_a(x) = \int_0^a f(t)\,\frac{\sin xt}{t}\,dt. \tag{7.1.2}$$

Then $s_a(x)$ *has a limit in mean* $s(x)$ *when* $a \to \infty$, *and*

$$\int_0^\infty \frac{|s(x)|^p}{x}\,dx \leqq K(p)\int_0^\infty \frac{|f(x)|^p}{x}\,dx. \tag{7.1.3}$$

7.2. There is a simple proof, which we have not succeeded in generalizing, in the case $p = 2$.

Suppose first that $f(x) = 0$ for $x > c$. Then

$$s_a(x) = \int_0^c f(t)\,\frac{\sin xt}{t}\,dt = \int_0^\infty f(t)\,\frac{\sin xt}{t}\,dt = s(x)$$

for $a > c$, so that $s(x)$ is the limit of $s_a(x)$ in the ordinary sense. Also,

$$\begin{aligned}\int_0^\xi \frac{(s(x))^2}{x}\,dx &= \int_0^\xi \frac{dx}{x}\int_0^c f(t)\,\frac{\sin xt}{t}\,dt\int_0^c f(u)\,\frac{\sin xu}{u}\,du\\ &= \int_0^\infty\int_0^\infty \frac{f(t)f(u)}{tu}\,dt\,du\int_0^\xi \frac{\sin xt\sin xu}{x}\,dx.\end{aligned} \tag{7.2.1}$$

The inner integral is

$$\frac{1}{2}\int_0^\xi \frac{1-\cos(t+u)x}{x}\,dx - \frac{1}{2}\int_0^\xi \frac{1-\cos|t-u|\,x}{x}\,dx$$
$$= \frac{1}{2}\int_{|t-u|\xi}^{(t+u)\xi} \frac{1-\cos w}{w}\,dw,$$

and is positive and less than

$$\log\frac{t+u}{|t-u|}.$$

Hence, if we write $g(t) = t^{-\frac{1}{2}}f(t)$, so that $g(t)$ is L^2, we have

$$\int_0^\xi \frac{(s(x))^2}{x}\,dx \leqq \int_0^\infty\int_0^\infty M(t, u)\,g(t)\,g(u)\,dt\,du, \tag{7.2.2}$$

where

$$M(t, u) = \frac{1}{\sqrt{(tu)}}\log\frac{t+u}{|t-u|}. \tag{7.2.3}$$

Finally, since M is homogeneous of degree -1, and

$$m = \int_0^\infty M(t, 1)t^{-\frac{1}{2}}\,dt = \int_0^\infty \frac{1}{t}\log\frac{t+1}{|t-1|}\,dt = \frac{1}{2}\pi^2 < \infty,$$

(7.2.2) implies[25]

$$\int_0^\infty \frac{(s(x))^2}{x}\,dx \leqq m\int_0^\infty (g(t))^2\,dt = m\int_0^\infty \frac{(f(t))^2}{t}\,dt. \tag{7.2.4}$$

Passing to the general case, we observe that

$$s_b(x) - s_a(x) = \int_a^b f(t)\,\frac{\sin xt}{t}\,dt,$$

and so, after (7.2.4),

$$\int_0^\infty \frac{(s_b(x) - s_a(x))^2}{x}\,dx \leqq m\int_a^b \frac{(f(t))^2}{t}\,dt,$$

which tends to 0 when a and b tend to infinity. À fortiori

$$\int_0^X (s_b(x) - s_a(x))^2\,dx \to 0$$

if $0 < X < \infty$. It now follows in the usual manner that $s(x)$ exists for almost all x, and that

$$\int_0^\infty \frac{(s(x))^2}{x}\,dx = \lim_{a\to\infty}\int_0^\infty \frac{(s_a(x))^2}{x}\,dx \leqq m\int_0^\infty \frac{(f(t))^2}{t}\,dt.$$

We observe here, in order to avoid repetition, that the last stage of the argument would run quite similarly for general p. When we have proved the analogue of (7.2.4), with general p and $f = 0$ for $t > c$, the rest of the theorem will follow.

8. Lemmas for the proof of Theorem 13

8.1. Lemma α. *If $f(x) \geqq 0$, $p > 1$, $r > 1$,*

$$f_1(x) = \int_0^x f(t)\,dt, \qquad f_2(x) = \int_x^\infty \frac{f(t)}{t}\,dt,$$

then

$$\int_0^\infty x^{-r}(f_1(x))^p\,dx \leqq K\int_0^\infty x^{-r}(xf(x))^p\,dx, \tag{8.1.1}$$

$$\int_0^\infty x^{r-2}(f_2(x))^p\,dx \leqq K\int_0^\infty x^{r-2}(f(x))^p\,dx, \tag{8.1.2}$$

with $K = K(p, r)$, whenever the integrals on the right are finite.

These are known theorems.[26] The cases we require are $r = p$ and $r = 2$.

[25] Hardy, Littlewood, and Pólya (**10**), Theorem 319.

[26] For (8.1.1) see Hardy, Littlewood, and Pólya (**10**), Theorem 330. The second inequality is not stated explicitly in the book, but will be found in Hardy (**1**).

LEMMA β. *If $1 < p \leqq 2$, and $f(x)$ is $L^p(0, \infty)$, then*

$$F(x) = \int_0^\infty f(t) \frac{\cos}{\sin} xt\, dt = \underset{a\to\infty}{\mathrm{l.i.m.}} \int_0^a f(t) \frac{\cos}{\sin} xt\, dt$$

exists, for almost all x, as a limit in mean with index p', and

$$\int_0^\infty x^{p-2} \mid F(x) \mid^p dx \leqq K(p) \int_0^\infty \mid f(x) \mid^p dx\,. \tag{8.1.3}$$

LEMMA γ. *If $p \geqq 2$, and $x^{(p-2)/p} f(x)$ belongs to $L^p(0, \infty)$, then*

$$F(x) = \int_0^\infty f(t) \frac{\cos}{\sin} xt\, dt = \underset{a\to\infty}{\mathrm{l.i.m.}} \int_0^a f(t) \frac{\cos}{\sin} xt\, dt$$

exists, for almost all x, as a limit in mean with index p, and

$$\int_0^\infty \mid F(x) \mid^p dx \leqq K(p) \int_0^\infty x^{p-2} \mid f(x) \mid^p dx\,. \tag{8.1.4}$$

For these two theorems see Hardy and Littlewood (**3**), Theorems 13 and 14.

8.2. LEMMA δ. *Let*

$$\psi(x) = x^{1/p} \int_0^1 (1 - u)^{-1/p'} \cos xu\, du\,. \tag{8.2.1}$$

Then the result of Lemma β remains true when $\psi(xt)$ is substituted for cos xt *or* sin xt.

It is easily verified by standard methods[27] that $\psi(x)$ is regular for $0 < x < \infty$, that

$$\psi(x) \backsim px^{1/p}$$

for small positive x, and that

$$\psi(x) = \Gamma\left(\frac{1}{p}\right) \cos\left(x - \frac{\pi}{2p}\right) + O\left(\frac{1}{x}\right)$$

for large positive x. Hence

$$\psi(x) = C \cos\left(x - \frac{\pi}{2p}\right) + R(x)\,, \tag{8.2.2}$$

where

$$\mid R(x) \mid < K \ (0 < x \leqq 1)\,, \qquad \mid R(x) \mid < \frac{K}{x}\ (x > 1)\,, \tag{8.2.3}$$

and $C = C(p)$, $K = K(p)$.

[27] The simplest method for finding an asymptotic expansion for $\psi(x)$ is to apply Cauchy's Theorem to

$$\int (1 - u)^{-1/p'} e^{ixu}\, du$$

and the rectangle $(0, 1, 1 + i\infty, i\infty)$.

It is enough to prove the result on the hypothesis that $f(x) = 0$ for $x > c$, when

$$F(x) = \int_0^\infty f(t)\,\psi(xt)\,dt$$

exists, for all x, as a Lebesgue integral; the proof may then be completed as at the end of §7.2.

Now in this case[28]

$$F(x) = C\int_0^\infty f(t)\cos\left(xt - \frac{\pi}{2p}\right)dt + \int_0^{1/x} f(t)\,R(xt)\,dt + \int_{1/x}^\infty f(t)\,R(xt)\,dt$$
$$= F_1(x) + F_2(x) + F_3(x)\,,$$

say; and it is sufficient to show that F_1, F_2, and F_3 satisfy inequalities of the type (8.1.3). This is true of F_1, by Lemma β. Next

$$|F_2| \leqq K\int_0^{1/x} |f(t)|\,dt = Kf_1\left(\frac{1}{x}\right),$$

say; and so

$$\int_0^\infty x^{p-2}\,|F_2|^p\,dx \leqq K\int_0^\infty x^{p-2}\left(f_1\left(\frac{1}{x}\right)\right)^p dx$$
$$= K\int_0^\infty x^{-p}\,(f_1(x))^p\,dx \leqq K\int_0^\infty |f(x)|^p\,dx\,,$$

by (8.1.1), with $r = p$. Finally

$$F_3 \leqq \frac{K}{x}\int_{1/x}^\infty \frac{f(t)}{t}\,dt = \frac{K}{x}f_2\left(\frac{1}{x}\right),$$

say, and

$$\int_0^\infty x^{p-2}\,|F_3|^p\,dx \leqq K\int_0^\infty x^{-2}\left(f_2\left(\frac{1}{x}\right)\right)^p dx$$
$$= K\int_0^\infty (f_2(x))^p\,dx \leqq K\int_0^\infty |f(x)|^p\,dx\,,$$

by (8.1.2), with $r = 2$.

8.3. Lemma ϵ. *Let*

$$\chi(x) = x^{1/p'}\int_0^1 (1-u)^{-1/p}\cos xu\,du\,.$$

Then the result of Lemma γ remains true when $\chi(xt)$ is substituted for cos xt *or* sin xt.

[28] Our argument is suggested by one used by Titchmarsh (**14**) for a different purpose.

Here

$$\chi(x) = C \cos\left(x - \frac{\pi}{2p'}\right) + R(x),$$

where $R(x)$ again satisfies (8.2.3). Arguing as before, we obtain

$$F(x) = F_1(x) + F_2(x) + F_3(x),$$

where

$$\int_0^\infty | F_1(x) |^p \, dx \leqq K(p) \int_0^\infty x^{p-2} | f(x) |^p \, dx$$

and

$$| F_2(x) | \leqq K f_1\left(\frac{1}{x}\right), \qquad | F_3(x) | \leqq \frac{K}{x} f_2\left(\frac{1}{x}\right).$$

We have now

$$\int_0^\infty | F_2(x) |^p \, dx \leqq K \int_0^\infty \left(f_1\left(\frac{1}{x}\right)\right)^p dx$$
$$= K \int_0^\infty x^{-2} (f_1(x))^p \, dx \leqq K \int_0^\infty x^{p-2} | f(x) |^p \, dx$$

by (8.1.1), with $r = 2$; and

$$\int_0^\infty | F_3(x) |^p \, dx \leqq K \int_0^\infty x^{-p} \left(f_2\left(\frac{1}{x}\right)\right)^p dx$$
$$= K \int_0^\infty x^{p-2} (f_2(x))^p \, dx \leqq K \int_0^\infty x^{p-2} | f(x) |^p \, dx,$$

by (8.1.2) with $r = p$. The result follows as before.

9. Proof of Theorem 13

9.1. (1) Suppose that $1 < p \leqq 2$, that $x^{-1/p} f(x)$ is L^p, and, in the first instance, that $f(x) = 0$ for $x > c$.

Let

$$w(x) = \int_0^\infty t^{-1/p} f(t) \psi(xt) \, dt,$$

where ψ is defined as in §8.2. Then

$$\int_0^x (x - y)^{-1/p} w(y) \, dy = \int_0^x (x - y)^{-1/p} \, dy \int_0^\infty t^{-1/p} f(t) \psi(yt) \, dt$$
$$= \int_0^\infty t^{-1/p} f(t) \, dt \int_0^x (x - y)^{-1/p} \psi(yt) \, dy$$

(by absolute convergence). The inner integral is

$$\int_0^x (x-y)^{-1/p}(yt)^{1/p}\,dy \int_0^1 (1-u)^{-1/p'} \cos ytu\,du$$

$$= t^{1/p}\int_0^x (x-y)^{-1/p}\,dy \int_0^y (y-v)^{-1/p'}\cos tv\,dv$$

$$= t^{1/p}\int_0^x \cos tv\,dv \int_v^x (x-y)^{-1/p}(y-v)^{-1/p'}\,dy = \pi \operatorname{cosec}\frac{\pi}{p}\cdot t^{1/p}\frac{\sin xt}{t};$$

so that

$$s(x) = \int_0^\infty f(t)\frac{\sin xt}{t}\,dt = K(p)\int_0^x (x-y)^{-1/p} w(y)\,dy$$

is, apart from a factor K, the $(1/p')$-th integral of $w(x)$.

It follows from Theorem 7 and Lemma δ that

$$\int_0^\infty \frac{|s(x)|^p}{x}\,dx \leqq K\int_0^\infty x^{p-2}|w(x)|^p\,dx \leqq K\int_0^\infty \frac{|f(x)|^p}{x}\,dx.$$

This is the result of Theorem 13, when $f(x)$ is 0 for large x; and the full result follows as in §7.2.

(2) If $p \geqq 2$ we write

$$w(x) = \int_0^\infty t^{-1/p'} f(t)\chi(xt)\,dt,$$

where χ is defined as in §8.3. We again suppose, in the first instance, that $f(x) = 0$ for large x. Then $s(x)$ is substantially the $(1/p)$-th integral of $w(x)$, and

$$\int_0^\infty \frac{|s(x)|^p}{x}\,dx \leqq K\int_0^\infty |w(x)|^p\,dx \leqq K\int_0^\infty x^{p-2}\left(\frac{|f(x)|}{x^{1/p'}}\right)^p dx$$

$$= K\int_0^\infty \frac{|f(x)|^p}{x}\,dx,$$

by Theorem 6 and Lemma ϵ. The proof is then completed as before.

10. Concluding remarks

10.1. We conclude with a few miscellaneous comments.

(1) The result of Theorem 10 becomes false for $p = 1$.

It is plain that $\sum n^{-1}|s_n| < \infty$ implies $\sum n^{-1}|a_n| < \infty$ and so

$$\sum \frac{|s_n - \frac{1}{2}a_n|}{n} < \infty.$$

Also

$$s_n - \frac{1}{2}a_n = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{1}{2}\cot\frac{1}{2}t\, f(t)\sin nt\,dt$$

is the Fourier sine coefficient of the function $\frac{1}{2}\cot\frac{1}{2}t\,f(t)$. Hence, if the result were true for $p = 1$, it would also be true that, if $g(t)$ is odd and integrable, and

$$g(t) \sim \sum b_n \sin nt,$$

then

$$\sum \frac{|b_n|}{n} < \infty.$$

But

$$g(t) = 2\sum \frac{\sin nu}{\log n} \sin nt = h(t-u) - h(t+u),$$

where

$$h(t) = \sum \frac{\cos nt}{\log n},$$

is integrable, for any u; while

$$\sum \frac{|\sin nu|}{n \log n}$$

is generally divergent.

Thus

$$\int_0 \frac{|f(t)|}{t}\,dt < \infty$$

implies $s_n \to 0$ (by Dini's convergence criterion), but not the convergence of $\sum n^{-1} |s_n|$. When $p > 1$ the situation is reversed:

$$\int_0 \frac{|f(t)|^p}{t}\,dt < \infty \tag{10.1.1}$$

implies the convergence of $\sum n^{-1} |s_n|^p$, but does not imply $s_n \to 0$. For (10.1.1) is satisfied whenever

$$f(t) = O\left(\left(\log \frac{1}{t}\right)^{-1}\right),$$

and this is not a sufficient condition for convergence of the Fourier series.[29]

10.2. (2) It is instructive to contrast our results with the much simpler results for the Cesàro mean σ_n of the Fourier series.

If (10.1.1) is satisfied then, à fortiori,

$$\int_0^t |f(u)|^p\,du = o(t)$$

[29] See Hardy and Littlewood (**6**), 47; Zygmund (**15**), 31, 174.

and this is, for $p \geqq 1$, a sufficient condition that $\sigma_n \to 0$. Also (10.1.1) implies

$$\sum \frac{|\sigma_n|^p}{n} < \infty. \tag{10.2.1}$$

When $p > 1$, this is a corollary of Theorem 10; but it is true for $p \geqq 1$, and may be proved much more simply. For

$$|\sigma_n| \leqq A \int_0^\pi |f(t)| \frac{\sin^2 nt}{nt^2}\,dt \leqq An \int_0^{1/n} |f(t)|\,dt + \frac{A}{n} \int_{1/n}^\pi \frac{|f(t)|}{t^2}\,dt, \tag{10.2.2}$$

where the A are constants; and (10.2.1) is an easy deduction.

Consider, for example, the first term

$$An \int_0^{1/n} |f(t)|\,dt = An f_1\left(\frac{1}{n}\right)$$

on the right of (10.2.2). The contribution of this to (10.2.1) does not exceed

$$K \sum n^{p-1} \left(f_1\left(\frac{1}{n}\right)\right)^p \leqq K \sum \int_n^{n+1} x^{p-1} \left(f_1\left(\frac{1}{x}\right)\right)^p dx \leqq K \int_0^\infty x^{p-1} \left(f_1\left(\frac{1}{x}\right)\right)^p dx$$
$$= K \int_0^\infty x^{-p-1} f_1^p(x)\,dx.$$

We now require the inequality

$$\int_0^\infty x^{-p-1}(f_1(x))^p\,dx \leqq K \int_0^\infty x^{-1}(f(x))^p\,dx,$$

which is a case of (8.1.1) when $p > 1$ and may be verified independently when $p = 1$.

The second term in (10.2.2) may be disposed of similarly.

10.3. (3) Theorem 10 has a 'transform,' viz.

THEOREM 14. *If $p > 1$ and*

$$\sum n^{p-1} |b_n|^p < \infty,$$

then there is an odd function $g(x)$ whose Fourier series is

$$\sum b_n \sin nx,$$

and

$$\left(\int_0^\pi \frac{|g(x)|^k}{x}\,dx\right)^{1/k} \leqq K(p, k) \left(\sum n^{p-1} |b_n|^p\right)^{1/p}$$

for $k \geqq p$.

This may be proved independently, or (when $k = p$) deduced from Theorem 13; and there is a simple proof similar to that of §7.2 when $k = p = 2$.

The corresponding theorem for cosine series is false. If $b_1 = 0$, and $b_n = (n \log n)^{-1}$ for $n > 1$, $k = p = 2$, then

$$\sum nb_n^2 = \sum \frac{1}{n(\log n)^2} < \infty;$$

but

$$f(t) = \sum \frac{\cos nt}{n \log n} \sim \int_t^\pi \sum \frac{\sin nu}{\log n}\, du \sim \int_t^\pi \frac{du}{u \log (1/u)} \sim \log \log \frac{1}{t}$$

for small positive t.

(4) Theorem 13 is equivalent to

THEOREM 15. *The bilinear integral form*

$$\int_0^\infty \int_0^\infty \frac{\sin xy}{x^{1/p'} y^{1/p}} a(x)\, b(y)\, dx\, dy$$

is bounded in space $[p, p']$: *i.e.,*

$$\left| \int_0^X \int_0^Y \frac{\sin xy}{x^{1/p'} y^{1/p}} a(x)\, b(y)\, dx\, dy \right| \leqq K(p) \left(\int_0^\infty |a(x)|^p\, dx \right)^{1/p} \left(\int_0^\infty |b(y)|^{p'}\, dy \right)^{1/p'}$$

for all $a(x)$, $b(y)$, X, Y.

The form

$$\sum \sum \frac{\sin mn}{m^{1/p'} n^{1/p}} a_m b_n$$

is not bounded in $[p, p']$, since (*e. g.*)

$$\sum_m \frac{|\sin mn|^{p'}}{m} = \infty$$

for every n.[30]

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COMMENTS

§ 3.6. The cases of Theorem 5 in which $p \neq 1$ and $q \neq \infty$ are contained implicitly in Theorem 7 of 1928, 5 (see the comments on 1928, 5; the proof in 1928, 5 is incomplete, but has been completed by T. M. Flett, and by E. M. Stein and G. Weiss).

§ 4.2. The method used by Hardy and Littlewood to prove Theorem 8 was later extended by H. R. Pitt (see the comments on 1926, 7). Using the method of § 5, Pitt also obtained an extension of Theorem 10.

§ 5.1. Theorem 10 is, strictly speaking, a theorem concerning *cosine* series rather than general Fourier series (i.e. it is a theorem about ϕ rather than f).

§ 6.2. There is an extension of Theorem 12 corresponding precisely to Theorem 11. There is also an analogous result for odd functions (see T. M. Flett, *Proc. London Math. Soc.* (3), 8 (1958), 135–48, and references given there).

§ 10.3. Theorem 14 is a particular case of a general inequality for Fourier sine series obtained by T. M. Flett, loc. cit.

NOTES ON FOURIER SERIES (IV): SUMMABILITY (R_2)

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1. INTRODUCTION

1·1. There are two familiar methods of summation of divergent series usually called the methods $(R, 1)$ and $(R, 2)$†. If $s_n = u_0+u_1+\ldots+u_n$ and, as it will be convenient to suppose throughout, $u_0 = 0$, then

$$\Sigma u_n = s(R,1) \equiv \sum_1^\infty u_n \frac{\sin nh}{nh} \to s‡, \tag{1·1·1}$$

$$\Sigma u_n = s(R,2) \equiv \sum_1^\infty u_n \left(\frac{\sin nh}{nh}\right)^2 \to s, \tag{1·1·2}$$

when $h \to +0$§: the convergence of the series for small positive h is presupposed.

We also suppose throughout that f is L and periodic,

$$f(\theta) \sim \tfrac{1}{2}A_0(\theta) + \sum_1^\infty A_n(\theta) = \tfrac{1}{2}a_0 + \Sigma(a_n \cos n\theta + b_n \sin n\theta), \tag{1·1·3}$$

and $a_0 = 0$, so that

$$\int_{-\pi}^{\pi} f(\theta)\, d\theta = 0. \tag{1·1·4}$$

Then the conditions

$$g_1(t) = \int_0^t g(u)\, du = o(t), \tag{1·1·5}$$ $$g_2(t) = \int_0^t g_1(u)\, du = o(t^2), \tag{1·1·6}$$

where

$$g(t) = g(t,\theta,c) = \phi(t,\theta) - c = \tfrac{1}{2}\{f(\theta+t)+f(\theta-t)-2c\}, \tag{1·1·7}$$

are necessary and sufficient for the summability of the Fourier series (F.s.), for $t = \theta$, to c, $(R, 1)$ and $(R, 2)$ respectively‖.

In this and a later note we consider two roughly parallel definitions, viz.

$$\Sigma a_n = s(R_1) \equiv \frac{2}{\pi}\Sigma s_n \frac{\sin nh}{n} \to s, \tag{1·1·8}$$

$$\Sigma a_n = s(R_2) \equiv \frac{2}{\pi h}\Sigma s_n \left(\frac{\sin nh}{n}\right)^2 \to s. \tag{1·1·9}$$

Here we are concerned with (R_2), which is a 'regular' method, whereas (R_1), like $(R, 1)$, is not. We find necessary and sufficient conditions for summability (R_2) in terms of f

† The method $(R, 1)$ is sometimes called Lebesgue's method: $(R, 2)$ is 'Riemann's method'.
‡ $\equiv$ is the sign of logical equivalence.
§ It is convenient, though in no way essential, to keep h positive.
‖ The notation is that of Hardy and Rogosinski (3), which we refer to as HR, except that we suppress the suffix c in what is there called $g_c(t)$. We use the notation $g_1, f_1, \phi_1, \ldots, g_2, \ldots$ for the successive integrals of $g, f, \phi, \ldots$, from 0, systematically.

1947, 1 (with W. W. Rogosinski) *Proceedings of the Cambridge Philosophical Society*, 43, 10–25.

and, under certain restrictions, also in terms of the conjugate function $\tilde{f}$. Incidentally we give two proofs that (R_2) is 'Fourier-effective', i.e. sums any F.s., to $f(\theta)$, p.p.†. We also show that $(R, 2)$ and (R_2) are 'incomparable' (that neither implies the other) even for F.s.

Not all of this is new, but there is little literature about (R_2). The definition itself stands, in principle, in Riemann, though he naturally did not treat it explicitly as a definition of 'summability'; and he gives what is effectively a proof of its regularity‡. The only writer to consider its application to F.s. in particular, so far as we know, is Wiener: he proves in (8) that it succeeds at a point of continuity or jump, but not that it succeeds p.p. The last result is a corollary of a difficult theorem of Kuttner (4), to the effect that $(R, 1) \longrightarrow (R_2)$§ for any series, from which it follows that a F.s. is summable (R_2), to $f(\theta)$, whenever (1·1·5) is satisfied with $c = f(\theta)$, and so p.p. Our proofs of this are however much simpler. Finally, Marcinkiewicz (5) and Kuttner (4) have given examples of series summable $(R, 2)$, but not (R_2), and conversely; but these series are not F.s.

1·2. We follow generally the notation of HR‖. We denote by $s_n = s_n(\theta) = s_n(\theta, f)$ the partial sum, for $t = \theta$, of the F.s. of $f(t)$. Then

$$s_n(\theta) - c = \frac{2}{\pi}\int_0^\pi g(t)\, D_n(t)\, dt, \quad D_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}, \tag{1·2·1}$$

where g is defined by (1·1·7). It is familiar that, when we are considering a problem of convergence to c, or of summability to c by any regular method, for a particular θ, we may simplify by supposing

$$\text{(A)} \qquad f \textit{ even}, \quad f \sim \Sigma a_n \cos nt \quad (a_0 = 0), \quad \theta = 0, \quad c = 0,$$

and so $g = \phi = f$. We call these assumptions 'assumptions (A)'.

We call the conditions

$$g_1(t) = \int_0^t g(u)\, du = o(t), \quad g_1^*(t) = \int_0^t |\, g(u)\,|\, du = o(t) \tag{1·2·2}$$

l_c and L_c: the first is (1·1·5).

It is plain that $s_n(\theta) \to c\ (R_2)$ is equivalent to

$$\tau(h) = \tau(\theta, h) \to c, \tag{1·2·3}$$

where

$$\tau(h) = \Sigma \alpha_n(h)\, s_n(\theta), \tag{1·2·4}$$

$$\alpha_n(h) = \frac{2}{\pi h}\frac{\sin^2 nh}{n^2} = \frac{1}{\pi h}\frac{1-\cos 2nh}{n^2}, \tag{1·2·5}$$

and the convergence of the series for small $h > 0$ is presupposed. There are various general theorems about $\tau(h)$ in HR, but they are not applicable to this $\tau(h)$, since

$$\Sigma n\, |\, \alpha_n(h)\,| = \infty \quad (0 < h < \pi)$$

and (R_2) is not a 'K-method' in the sense of HR.

† Almost always (*presque partout*): we use this, and certain other abbreviations, as in HR.
‡ For which see HR, 85–6.
§ Here '$\longrightarrow$' is the symbol of logical implication. Kuttner's proof is not written out in detail.
‖ With the exception mentioned in footnote ‖, p. 10.

1·3. In § 2 we state a number of preliminary lemmas, with proofs where necessary. In § 3 we prove some general theorems about methods of summation, supplementary to those of HR, and deduce our first proof that (R_2) is Fourier-effective. In § 4 we obtain a necessary and sufficient condition for summability (R_2), which leads to a second proof of Fourier-effectiveness. We also show that there are F.s. summable $(R, 2)$ but not (R_2). In § 5 we confine ourselves to a slightly restricted class of functions f, and obtain a condition for summability in terms of the conjugate function $\bar{f}$. This leads to an example of a F.s. summable (R_2) but not $(R, 2)$, and we have not carried the analysis of this section farther than is necessary for this purpose.

2. Preliminary lemmas

2·1. Lemma A. *If* $0<t<2\pi$ *then*

$$\left|\sum_{m=1}^{n}\frac{\cos mt}{m}\right| < A\left(1+\log^{+}\frac{1}{t}+\log^{+}\frac{1}{2\pi-t}\right)\dagger. \tag{2·1·1}$$

It is plainly sufficient to prove (2·1·1) for $0<t\leqq\pi$ and without the term involving $2\pi-t$. The inequality is then obvious if $n\leqq t^{-1}$. If $n>t^{-1}$ then

$$\left|\sum_{m=1}^{n}\frac{\cos mt}{m}\right| \leqq \left|\sum_{1\leqq m\leqq t^{-1}}\right| + \left|\sum_{t^{-1}<m\leqq n}\right|.$$

The first term on the right has the upper bound required, and the second does not exceed

$$t\,\underset{M}{\mathrm{Max}}\left|\sum_{t^{-1}\leqq m<M}\cos mt\right| < t\operatorname{cosec}\tfrac{1}{2}t < A.$$

2·2. Lemma B. *If* $f(t)$ is L, $f_1(\pi)=0$, and

$$F(t)=\int_t^{\pi}\frac{f(u)}{u}\,du \quad (0<t\leqq\pi),$$

then (i) $F(t)$ *is* L^t, (ii) $F(t)=o(t^{-1})$, *and*

$$\text{(iii)}\ \ F_1(t)=f_1(t)+tF(t)=t\int_t^{\pi}\frac{f_1(u)}{u^2}\,du.$$

In particular $F_1(\pi)=0$.

First, $\displaystyle\int_0^{\pi}|F(t)|\,dt\leqq\int_0^{\pi}dt\int_t^{\pi}\frac{|f(u)|}{u}\,du=\int_0^{\pi}\frac{|f(u)|}{u}\,du\int_0^{u}dt=\int_0^{\pi}|f(u)|\,du.$

Secondly, $\displaystyle|F(t)|\leqq\left|\int_t^{\delta}\frac{f}{u}\,du\right|+\left|\int_{\delta}^{\pi}\frac{f}{u}\,du\right|=\frac{1}{t}\left|\int_t^{\tau}f\,du\right|+\left|\int_{\delta}^{\pi}\frac{f}{u}\,du\right|,$

where $0<t<\tau<\delta$. The first term here is less than ϵ/t for sufficiently small δ, and the second is bounded when δ is fixed. Hence $F(t)=o(t^{-1})$. It should be observed that in

† Here $\log^{+}x$, as usual, is $\log x$ if $x>1$ and 0 otherwise, and A is a constant. We shall follow the common convention that the value of an A may differ from one occurrence to another. We shall also apply this convention to numbers $A(\alpha)$, $A(\beta)$, ... depending on the parameters α, β, ... of our arguments.

this argument we do not use the full force of the assumption that f is L in $(0,\pi)$; it is sufficient that f should be L in (δ,π) for every positive δ, and that

$$\int_{\to 0}^{\pi} f(t)\,dt = \lim_{\delta\to 0}\int_{\delta}^{\pi} f(t)\,dt$$

should exist†.

Finally $$F_1(t) = \int_0^t du\int_u^\pi \frac{f(v)}{v}\,dv = t\int_t^\pi \frac{f(v)}{v}\,dv + \int_0^t f(u)\,du = tF(t)+f_1(t),$$

and $$\frac{F_1(t)}{t} = F(t)+\frac{f_1(t)}{t} = \frac{f_1(t)}{t}+\int_t^\pi \frac{f(u)}{u}\,du = \int_t^\pi \frac{f_1(u)}{u^2}\,du,$$

in each case by partial integration. In the last transformation we use the assumption $f_1(\pi) = 0$.

2·3. We say that f is Z if

(2·3·1) $$\int_{-\pi}^{\pi} |f(t)|\,(\log^+|f(t)|+1)\,dt<\infty.$$

The importance of this class of functions was first recognized by Zygmund. It is included in L, and includes L^p for any $p>1$. If f is Z, then the conjugate $\bar{f}$ is L‡.

Lemma C. *If f is Z, and $-\pi\leqq x\leqq\pi$, then*

(2·3·2) $$J(x) = \int_{-\pi}^{\pi} |f(t)|\log^+\frac{1}{|t-x|}\,dt$$

is finite, and continuous in x.

We use Young's inequality

(2·3·3) $$uv\leqq u\log u+e^{v-1}$$

for $u>0$ and real v§. We choose n large enough to ensure that $-\pi<x-n^{-1}<x+n^{-1}<\pi$|| and define $l_n t$ by

$$l_n t = \log^+ t = \log t \quad (1\leqq t\leqq n), \qquad 0 \quad (0<t<1 \text{ or } t>n).$$

Then $$J(x) = \int_{-\pi}^{\pi} |f(t)|\,l_n\frac{1}{|t-x|}\,dt + \int_{x-n^{-1}}^{x+n^{-1}} |f(t)|\log\frac{1}{|t-x|}\,dt = J_1(x)+J_2(x),$$

say. We apply (2·3·3), with

$$u = |f(t)|,\quad v = \tfrac{1}{2}\log\frac{1}{|t-x|},$$

to $J_2(x)$. This gives

$$|f|\log\frac{1}{|t-x|}\leqq 2|f|\log|f|+\frac{2}{e}\frac{1}{|t-x|^{\frac{1}{2}}}\leqq 2|f|\log^+|f|+\frac{2}{e}\frac{1}{|t-x|^{\frac{1}{2}}},$$

and so $$J_2(x)\leqq 2\int_{x-n^{-1}}^{x+n^{-1}} |f|\log^+|f|\,dt+\frac{2}{e}\int_{x-n^{-1}}^{x+n^{-1}}\frac{dt}{|t-x|^{\frac{1}{2}}}<\epsilon$$

for $n>n_0(\epsilon)$, uniformly in x. Since $J_1(x)$ is obviously continuous, the conclusion follows.

† Here we use the convenient notation of Titchmarsh (6) for non-absolutely convergent integrals. We call such integrals 'Cauchy integrals'.

‡ See Zygmund (9), 150.

§ Hardy, Littlewood, and Pólya (2), 61, 107; Zygmund (9), 65.

|| We state the argument for $-\pi<x<\pi$: there is an obvious simplification when $x=-\pi$ or π.

2·4. LEMMA D. *If $f(t)$ satisfies l_c for $t = \theta$, then*

$$\int_\delta^\pi \phi(t,\theta)\, D_n(t)\, dt = o(\sqrt{n}) \tag{2·4·1}$$

uniformly for $0 \leqq \delta \leqq \pi.$

This lemma is due to Wang †: it will be convenient to give the proof. We may replace $D_n(t)$ by $(\sin nt)/t$ and, since the integral of this function over any interval is uniformly bounded, we may suppose $c = 0$, so that l_c is $\phi_1(t) = o(t)$.

If $$\chi(t) = \chi(t,\delta) = \phi(t) \quad (\delta \leqq t \leqq \pi), \quad 0 \quad (0 < t < \delta)$$

then $$\chi_1(t) = \phi_1(t) - \phi_1(\delta) \quad (\delta \leqq t \leqq \pi), \quad 0 \quad (0 < t < \delta)$$

and so $\chi_1(t) = o(t)$, uniformly in δ. We write

$$J = \int_\delta^\pi \phi \frac{\sin nt}{t} dt = \int_0^\pi \chi \frac{\sin nt}{t} dt = \int_0^{pn^{-\frac{1}{2}}} + \int_{pn^{-\frac{1}{2}}}^\pi = J_1 + J_2,$$

say. Then $$|J_2| \leqq \frac{\sqrt{n}}{p} \int_0^\pi |\phi|\, dt < \epsilon\sqrt{n}$$

by choice of $p = p(\epsilon)$; and

$$J_1 = \left[\chi_1(t) \frac{\sin nt}{t}\right]_{t=pn^{-\frac{1}{2}}} - \int_0^{pn^{-\frac{1}{2}}} \chi_1(t) \left(\frac{n \cos nt}{t} - \frac{\sin nt}{t^2}\right) dt.$$

The first term here is $o(1)$ when $n \to \infty$, for any fixed p; and the second is

$$\int_0^{pn^{-\frac{1}{2}}} o(t)\, O\left(\frac{n}{t}\right) dt = o(\sqrt{n}).$$

Hence $|J| < 2\epsilon\sqrt{n}$ for $n > n_0(p,\epsilon) = n_0(\epsilon)$.

Actually we shall only use $J = O(\sqrt{n})$.

2·5. LEMMA E. *If $f(t) \sim \Sigma a_n \cos nt$, with $a_0 = 0$, then a necessary and sufficient condition for the convergence of $\Sigma n^{-1} a_n$ is that $\cot \frac{1}{2}t f_1(t)$, or $t^{-1} f_1(t)$, should have a Cauchy integral down to 0; and in this case*

$$\Sigma n^{-1} a_n = \frac{1}{\pi} \int_{\to 0}^\pi \cot \tfrac{1}{2}t f_1(t)\, dt. \tag{2·5·1}$$

This is proved by Hardy and Littlewood (1), 94, but the proof may be simplified. If $a_0 \neq 0$, we must add $a_0 \log 2$ to the left of (2·5·1) ‡.

The series is the conjugate, for $\theta = 0$, of the F.s. of f_1, which is absolutely continuous and *a fortiori* of bounded variation. If follows § that

$$\sum_{m=1}^n \frac{a_m}{m} - \frac{1}{\pi} \int_{\pi/n}^\pi \cot \tfrac{1}{2}t f_1(t)\, dt \to 0.$$

Since $$\int_{\pi/\lambda}^{\pi/n} \cot \tfrac{1}{2}t f_1(t)\, dt = O\left(\frac{n}{n^2}\right) = o(1)$$

for $n < \lambda < n+1$, the lemma follows.

† Wang (7), 95.
‡ See HR, 96 (note on § 4·10).
§ HR, 49 (Theorem 63).

2·6. Finally we collect in a lemma the sums of a number of elementary trigonometrical series which we shall use, leaving the proofs to the reader. All sums are over $(1,\infty)$.

LEMMA F. *If* $0<t\leqq\pi$, $0<h\leqq\pi$, *then*

(2·6·1) $\Sigma n^{-1}\sin nt = \frac{1}{2}(\pi-t)$; (2·6·2) $\Sigma n^{-1}\cos nt = -\log(2\sin\frac{1}{2}t)$;

(2·6·3) $$\Sigma n^{-1}(1-\cos nh)\sin nt = 0,\quad \tfrac{1}{4}\pi,\quad \tfrac{1}{2}\pi,$$

for $h<t$, $h=t$, $h>t$ *respectively*;

(2·6·4) $$\Sigma\frac{1-\cos nh}{n}\cos nt = -\tfrac{1}{2}\log\frac{1-\cos t}{|\cos h-\cos t|};$$

(2·6·5) $$\Sigma n^{-2}(1-\cos nh) = \tfrac{1}{2}\pi h-\tfrac{1}{4}h^2,$$

(2·6·6) $$\Sigma n^{-2}(1-\cos nh)(1-\cos nt) = \tfrac{1}{2}\pi\,\mathrm{Min}\,(h,t).$$

The last two results are true for $0\leqq t\leqq\pi$, $0\leqq h\leqq\pi$. The sums of all the series, for ranges of the variables outside those stated, may be derived from obvious properties of oddness, evenness, and periodicity. We shall usually have $2h$ in the place of h.

3. A GENERAL THEOREM, AND ITS APPLICATION TO (R_2)

3·1. We now consider linear methods of summation defined by transforms

(3·1·1) $$\tau_h = \Sigma\alpha_n(h)\,s_n,$$

where $h>0$, subject to less stringent conditions than those imposed on them in HR.

THEOREM 1. *If T is a linear method with transforms* (3·1·1), *and*

(1) $$\Sigma n^{\frac{1}{2}}\,|\,\alpha_n(h)\,|<\infty\quad (h>0),$$

(2) *$f(t)$ satisfies l_c for $t=\theta$,*

then the transform $$\tau_h(\theta) = \Sigma\alpha_n(h)\,s_n(\theta)$$
converges for $h>0$, and

(3·1·2) $$\tau_h(\theta) = \frac{2}{\pi}\int_{\to 0}^{\pi}\phi(t,\theta)\,K_h(t)\,dt,$$

where

(3·1·3) $$K_h(t) = \Sigma\alpha_n(h)\,D_n(t).$$

That is to say, $\tau_h(\theta)$ may be calculated by formal term-by-term integration, the integral resulting being a Cauchy integral down to 0.

We have $$\tau_h(\theta) = \Sigma\alpha_n(h)\,s_n(\theta) = \frac{2}{\pi}\Sigma\alpha_n(h)\int_0^{\pi}\phi(t,\theta)\,D_n(t)\,dt$$

(if the series is convergent). Since the partial sums of $K_h(t)$ are uniformly bounded in (δ,π), for any $\delta>0$,

$$\frac{2}{\pi}\Sigma\alpha_n(h)\int_{\delta}^{\pi}\phi D_n\,dt = \frac{2}{\pi}\int_{\delta}^{\pi}\phi K_h\,dt$$

(the series being convergent). Also

$$\Sigma\alpha_n(h)\lim_{\delta\to 0}\int_{\delta}^{\pi}\phi D_n\,dt = \lim_{\delta\to 0}\Sigma\alpha_n(h)\int_{\delta}^{\pi}\phi D_n\,dt,$$

since the integral on the right is $O(n^{\frac{1}{2}})$ uniformly in δ, by Lemma D†, and $\Sigma n^{\frac{1}{2}} |\alpha_n(h)| < \infty$. Hence τ_h is convergent and

$$\tau_h = \frac{2}{\pi}\Sigma\alpha_n(h)\int_0^\pi \phi D_n\,dt = \frac{2}{\pi}\lim_{\delta\to 0}\Sigma\alpha_n(h)\int_\delta^\pi \phi D_n\,dt = \frac{2}{\pi}\lim_{\delta\to 0}\int_\delta^\pi \phi K_h\,dt = \frac{2}{\pi}\int_{\to 0}^\pi \phi K_h\,dt.$$

For the sake of completeness we state a similar theorem in which f satisfies L_c and the condition on $\alpha_n(h)$ is weakened. We do not use it and leave the proof to the reader.

THEOREM 2. *If* (1) $\Sigma \log n\,|\alpha_n(h)| < \infty$ *and* (2) f *satisfies* L_c *for* $t = \theta$, *then*

$$\tau_h(\theta) = \frac{2}{\pi}\int_0^\pi \phi(t,\theta)\,K_h(t)\,dt,$$

the integral being a Lebesgue integral.

3·2. THEOREM 3. *If, in addition to the conditions of Theorem* 1, (*a*) T *is regular* (*b*) $K_h(t)$ *is absolutely continuous, except perhaps at* $t = 0$, *and*

$$\int_0^\pi t\,|K_h'(t)|\,dt < A, \tag{3·2·1}$$

where A is independent of h‡, then the F.s. of $f(t)$, for $t = \theta$, is summable (T) *to* c.

It follows that the F.s. is summable (T) p.p. to $f(\theta)$.

The proof follows familiar lines, though a little care is needed because (3·1·2) is a Cauchy integral. We may adopt assumptions (A). Then, by Theorem 1,

$$\tau_h = \frac{2}{\pi}\int_{\to 0}^\pi fK_h\,dt = \frac{2}{\pi}\int_{\to 0}^\delta fK_h\,dt + \frac{2}{\pi}\int_\delta^\pi fK_h\,dt = J_1 + J_2,$$

say. Now

$$K_h(t) = K_h(\pi) - \int_t^\pi K_h'(u)\,du = O(1) + O\left\{\frac{1}{t}\int_t^\pi u\,|K_h'(u)|\,du\right\} = O\left(\frac{1}{t}\right) \tag{3·2·2}$$

for small t, and so

$$J_1 = \frac{2}{\pi}\lim_{\eta\to 0}\int_\eta^\delta = \frac{2}{\pi}\lim_{\eta\to 0}\left\{f_1(\delta)\,K_h(\delta) - f_1(\eta)\,K_h(\eta) - \int_\eta^\delta f_1K_h'\,dt\right\} = \frac{2}{\pi}\left\{f_1(\delta)\,K_h(\delta) - \int_0^\delta f_1K_h'\,dt\right\},$$

the last integral being a Lebesgue integral because $f_1 = o(t)$. The first term here tends to 0 with δ because of (3·2·2), and so

$$|J_1| < \frac{2\epsilon}{\pi}\left(1 + \int_0^\pi t\,|K_h'|\,dt\right) < \frac{2\epsilon}{\pi}(1+A)$$

for sufficiently small δ. Also

$$J_2 = \frac{2}{\pi}\Sigma\alpha_n(h)\int_\delta^\pi fD_n\,dt \to 0$$

for any fixed δ, when $h\to 0$, because $\int_\delta^\pi fD_n\,dt \to 0$ and T is regular. It follows that $\tau_h \to 0$.

We add, without proof, the theorem related to Theorem 2 as Theorem 3 is to Theorem 1.

† We do not need the o of the lemma.

‡ It would be enough that this should be so for small positive h.

THEOREM 4. *If, in addition to the conditions of Theorem 2, (a) T is regular,*

$$(b)\qquad |K_h(t)| \leqq K_h^*(t),$$

where K_h^* is absolutely continuous, except perhaps at the origin, and

$$\int_0^\pi t\,|K_h^{*\prime}(t)|\,dt < A\dagger,$$

then the F.s. is summable (T) to c.

3·3. We now apply Theorem 3 to (R_2).

THEOREM 5. *The kernel $K_h(t)$ of the (R_2) method satisfies* (3·2·1).

Here

$$(3\cdot3\cdot1)\qquad K_h(t) = \Sigma\alpha_n(h)\,D_n(t) = \frac{1}{2\pi h}\Sigma\frac{1-\cos 2nh}{n^2}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}$$

$$= \frac{1}{2\pi h}\Sigma\frac{1-\cos 2nh}{n^2}\cos nt + \frac{1}{2\pi h}\cot\tfrac{1}{2}t\,\Sigma\frac{1-\cos 2nh}{n^2}\sin nt = L_h(t)+M_h(t),$$

say. The series here may be differentiated term-by-term with respect to t‡. Hence, first

$$L_h'(t) = -\frac{1}{2\pi h}\Sigma\frac{1-\cos 2nh}{n}\sin nt,$$

which is 0 if $0<2h<t<\pi$ and $-1/4h$ if $0<t<2h<\pi$, by Lemma F (2·6·3). Hence

$$\int_0^\pi t\,|L_h'(t)|\,dt = \frac{1}{4h}\int_0^{2h} t\,dt = \tfrac{1}{2}h < \tfrac{1}{4}\pi$$

if $h<\frac{1}{2}\pi$§.

It remains to prove that

$$(3\cdot3\cdot2)\qquad \int_0^\pi t\,|M_h'(t)|\,dt < A,$$

or, what is equivalent, that

$$(3\cdot3\cdot3)\qquad \int_0^\pi \sin\tfrac{1}{2}t\left|\frac{d}{dt}\{\cot\tfrac{1}{2}t\,\chi(t)\}\right|dt < Ah,$$

where

$$(3\cdot3\cdot4)\qquad \chi(t) = \Sigma\frac{1-\cos 2nh}{n^2}\sin nt.$$

Now $$\chi'(t) = \Sigma\frac{1-\cos 2nh}{n}\cos nt = -\tfrac{1}{2}\log\frac{1-\cos t}{|\cos 2h-\cos t|},$$

by Lemma F (2·6·4). If $2h<t<\pi$ then

$$1-\cos t > \cos 2h - \cos t = |\cos 2h - \cos t|,$$

and so $\chi'(t)<0$. If $0<t<2h$, then $\chi'(t)<0$ if $1-\cos t>\cos t-\cos 2h$, i.e. $\cos t<\cos^2 h$.

† Note that this implies $K_h^* = O(t^{-1})$, as in the proof of (3·2·2). Thus in HR, 61 (Theorem 72) the first of the conditions (5·6·5) may be omitted.

We take this opportunity of making a few other corrections in HR. In Theorem 59 (p. 45), for 'converges uniformly to $f(\theta)$ in that interval', read 'converges uniformly to $f(\theta)$ in any closed interval strictly interior to that interval'. This mistake was pointed out to us by Prof. Zygmund. On p. 99, line 4 up, for $f\log^+|f|$ read $f(1+\log^+|f|)$. The last sentence on this page should read 'Important consequences are that $s_n\to f(L^p)$ and $\bar{s}_n\to\bar{f}(L^p)$ when f is L^p and $1<p<\infty$; that (2·3·1) is true (without absolute convergence) when f is L^p, $1<p<\infty$, and F is $L^{p'}$; and that $s_n\to f(L)$ and $\bar{s}_n\to\bar{f}(L)$ when $f(1+\log^+|f|)$ is L'.

‡ Because F.s. may be integrated (or on more elementary grounds).

§ See footnote ‡, p. 16.

Thus
$$\chi'(t) > 0 \quad (0 < t < \rho), \quad \chi'(t) < 0 \quad (\rho < t < \pi),$$
where
$$\rho = \operatorname{arc\,cos}(\cos^2 h);$$
and $\chi(0) = \chi(\pi) = 0$. It follows that $\chi(t)$ is positive in $(0, \pi)$ and has one maximum, at $t = \rho$. Hence
$$\int_0^\pi |\chi'(t)|\,dt = \int_0^\rho \chi'(t)\,dt - \int_\rho^\pi \chi'(t)\,dt = 2\chi(\rho).$$
But

(3·3·5)
$$\chi(t) = \Sigma \frac{1-\cos 2nh}{n^2} \sin nt < \Sigma \frac{1-\cos 2nh}{n^2} = \pi h - h^2 < \pi h,$$
by Lemma F (2·6·5); and so

(3·3·6)
$$\int_0^\pi |\chi'(t)|\,dt < Ah.$$

Also
$$\int_0^\pi \cot \tfrac{1}{2}t \; \chi(t)\,dt = \int_0^\pi \cot \tfrac{1}{2}t\, \chi(t)\,dt = \Sigma \frac{1-\cos 2nh}{n^2} \int_0^\pi \cot \tfrac{1}{2}t \sin nt\,dt,$$
and the last integral is π for each n. Hence

(3·3·7)
$$\int_0^\pi \cot \tfrac{1}{2}t\, \chi(t)\,dt = \pi \Sigma \frac{1-\cos 2nh}{n^2} = \pi(\pi h - h^2) < Ah.$$

Finally
$$\int_0^\pi \sin \tfrac{1}{2}t \frac{d}{dt}\{\cot \tfrac{1}{2}t\, \chi(t)\}\,dt \leqq \frac{1}{2}\int_0^\pi \operatorname{cosec} \tfrac{1}{2}t\, \chi(t)\,dt + \int_0^\pi \cot \tfrac{1}{2}t\, |\chi'(t)|\,dt$$
$$\leqq \frac{1}{2}\int_0^\pi \chi(t)\,dt + \frac{1}{2}\int_0^\pi \cot \tfrac{1}{2}t\, \chi(t)\,dt + \int_0^\pi \cos \tfrac{1}{2}t\, |\chi'(t)|\,dt < Ah\dagger,$$
by (3·3·5), (3·3·7), and (3·3·6).

It is plain that (R_2) satisfies the condition of Theorem 1, and Theorem 5 shows that it satisfies (3·2·1). Hence, after Theorem 3, we have

THEOREM 6. *The F.s. of $f(t)$ is summable (R_2) to c, for $t = \theta$, whenever $f(t)$ satisfies l_c for $t = \theta$, and therefore summable to $f(\theta)$ p.p.*

4. Necessary and sufficient conditions for summability (R_2)

4·1. We now consider the problem of necessary and sufficient conditions for summability (R_2). Here we shall not use the general theorems of § 3.

THEOREM 7. *If the series*

(4·1·1)
$$\tau_h(\theta) = \frac{1}{\pi h} \Sigma \frac{1-\cos 2nh}{n^2} s_n(\theta)$$

is convergent for any one h with $0 < h < \pi$, then it is convergent for all such h. For this, any one of the conditions

(1) $\Sigma n^{-2} s_n(\theta)$ *convergent,*

(2) $\Sigma n^{-1} A_n(\theta)$ *convergent,*

(3) $\displaystyle\int_{\to 0}^{\pi} \frac{\phi_1(t)}{t}\,dt$ *convergent,*

is necessary and sufficient. In particular these conditions are satisfied whenever

(4) $\displaystyle\int_{\to 0}^{\pi} \phi(t) \log \frac{1}{t}\,dt$ *is convergent*‡.

† $\operatorname{cosec} \tfrac{1}{2}t - \cot \tfrac{1}{2}t = \tan \tfrac{1}{4}t \leqq 1$ and $\cos \tfrac{1}{2}t \leqq 1$ in $(0, \pi)$.

‡ We may plainly replace ϕ by g, with any c, in (3) and (4).

If f is Z, then this is so for every θ, and in this case the integrals in (3) *and* (4) *are Lebesgue integrals.*

Except in the last clause, we may make assumptions (A)†.

(1) We have identically

$$\sum_{m=1}^{n-1} s_m \frac{\cos 2mh}{m^2} = \sum_{m=1}^{n-1} a_m \sum_{\nu=m}^{n-1} \frac{\cos 2\nu h}{\nu^2} = \sum_{m=1}^{n} a_m \chi(m,h) - s_n \chi(n,h), \tag{4·1·2}$$

where $$\chi(m,h) = \sum_{\nu=m}^{\infty} \frac{\cos 2\nu h}{\nu^2} = O\left(\frac{1}{m^2 h}\right).$$

Since $a_n = o(1)$, $s_n = o(n)$, it follows from (4·1·2) that $\Sigma n^{-2} s_n \cos 2nh$ is convergent for every positive h, and that (4·1·1) is convergent if and only if $\Sigma n^{-2} s_n$ is so.

(2) Since $$\sum_{m=1}^{n-1} \frac{s_m}{m(m+1)} = \sum_{m=1}^{n} \frac{a_m}{m} - \frac{s_n}{n},$$

this is so if and only if $\Sigma n^{-1} a_n$ is convergent.

(3) It now follows from (2) and Lemma E that condition (3) is necessary and sufficient.

(4) If condition (4) is satisfied, and $0 < \eta < t$, then

$$f_1(t) - f_1(\eta) = \int_\eta^t \frac{1}{\log(1/u)} f(u) \log\frac{1}{u}\, du = \frac{1}{\log(1/t)} \int_\tau^t f(u) \log\frac{1}{u}\, du,$$

with $\eta < \tau < t$. This is $o\left\{\left(\log\frac{1}{t}\right)^{-1}\right\}$, uniformly in η; and so $f_1(t) = o\left\{\left(\log\frac{1}{t}\right)^{-1}\right\}$. Hence

$$\int_\epsilon^\pi \frac{f_1(t)}{t}\, dt = f_1(\epsilon) \log\frac{1}{\epsilon} + \int_\epsilon^\pi f(t) \log\frac{1}{t}\, dt \to \int_{\to 0}^\pi f(t) \log\frac{1}{t}\, dt$$

when $\epsilon \to 0$.

As regards the last clause of the theorem, we may suppose $|\theta| \leqq \pi$. Then

$$\int_0^\pi |\phi(t)| \log^* \frac{1}{t}\, dt = \frac{1}{2} \int_0^\pi |f(\theta+t) + f(\theta-t)| \log^+ \frac{1}{t}\, dt$$

$$\leqq \frac{1}{2} \int_{-2\pi}^{2\pi} |f(t)| \left(\log^+ \frac{1}{|t-\theta|} + \log^+ \frac{1}{|t+\theta|}\right) dt < \infty,$$

by Lemma C‡. And

$$\int_0^\pi \frac{|\phi_1(t)|}{t}\, dt \leqq \int_0^\pi \frac{dt}{t} \int_0^t |\phi(u)|\, du \leqq \int_0^\pi |\phi(u)|\, du \int_u^\pi \frac{dt}{t} \leqq \int_0^\pi |\phi(u)| \left(\log \pi + \log^+ \frac{1}{u}\right) du < \infty.$$

4·2. Our next theorem is the main theorem of the paper.

THEOREM 8. *In order that the F.s. of $f(t)$ should be summable (R_2) to c, for $t = \theta$, it is necessary and sufficient that*

$$J(h) = \int_{\to 0}^\pi \frac{g_1(t)}{t^2} \log\left|\frac{t+h}{t-h}\right| dt \tag{4·2·1}$$

should converge, as a Cauchy integral down to 0, *and tend to* 0 *with h.*

† c does not intervene.

‡ Lemma C, as stated, covers only the range $(-\pi, \pi)$, but the rest of the range is easily disposed of. For example

$$\int_\pi^{2\pi} |f(t)| \log^+ \frac{1}{t-\theta}\, dt = \int_{-\pi}^0 |f(t)| \log^+ \frac{1}{t+2\pi-\theta}\, dt \leqq \int_{-\pi}^0 |f(t)| \log^+ \frac{1}{t+\pi}\, dt.$$

We may use assumptions (A), so that $g = f$.

We observe first that
$$\log\left|\frac{t+h}{t-h}\right| = \frac{2t}{h} + O(t^2)$$
for $h > 0$ and small t. Hence $J(h)$ will converge if and only if

$$\int_{\to 0}^{\pi} \frac{f_1(t)}{t}\,dt \tag{4·2·2}$$

converges, a condition which we know, after Theorem 7, to be necessary and sufficient for the convergence of the series (4·1·1). We may therefore assume the convergence of (4·2·2); and it is sufficient to prove that then

$$\Sigma \frac{s_n}{n^2}(1-\cos 2nh) = \frac{2h}{\pi} J(2h) + o(h). \tag{4·2·3}$$

Next,
$$s_n = \frac{2}{\pi}\int_0^\pi f D_n\,dt = \frac{2}{\pi}\int_0^\pi f \frac{\sin nt}{t}\,dt + r_n = s_n^* + r_n,$$
say, where $r_n = o(1)$; and so
$$\Sigma \frac{r_n}{n^2}(1-\cos 2nh) = o\left(\Sigma \frac{1-\cos 2nh}{n^2}\right) = o(h),$$
by Lemma F (2·6·5). We may therefore replace s_n by s_n^* in (4·2·3).

If
$$F(t) = \int_t^\pi \frac{f(u)}{u}\,du,$$
then, by Lemma B, $F(t)$ is L, $F(t) = o(t^{-1})$, and
$$F_1(t) = f_1(t) + tF(t),$$
so that

$$\int_{\to 0}^{\pi} \frac{F_1}{t}\,dt = \int_{\to 0}^{\pi} \frac{f_1}{t}\,dt + \int_0^\pi F\,dt \tag{4·2·4}$$

is convergent. Also

$$F_1(\pi) = f_1(\pi) = 0. \tag{4·2·5}$$

Next,

$$s_n^* = \frac{2}{\pi}\int_0^\pi \frac{f}{t}\sin nt\,di = -\frac{2}{\pi}\int_0^\pi F' \sin nt\,dt = \frac{2n}{\pi}\int_0^\pi F\cos nt\,dt, \tag{4·2·6}$$

integrating by parts and remembering that $F(t) = o(t^{-1})$. Thus
$$F(t) \sim \Sigma \frac{s_n^*}{n}\cos nt,$$
and hence, by Lemma 6,
$$\Sigma \frac{s_n^*}{n^2} = \frac{1}{\pi}\int_{\to 0}^{\pi} \cot \tfrac{1}{2}t\, F_1(t)\,dt.$$
Also

$$F_1(t) = -\int_t^\pi F(u)\,du = \int_t^\pi o\left(\frac{1}{u}\right)du = o\left(\operatorname{iog}\frac{1}{t}\right). \tag{4·2·7}$$

Hence, again integrating by parts,

$$\Sigma \frac{s_n^*}{n^2} = \frac{2}{\pi}\int_{\to 0}^{\pi} \frac{d}{dt}(\log\sin \tfrac{1}{2}t)\, F_1(t)\,dt = -\frac{2}{\pi}\int_{\to 0}^{\pi} \log\sin \tfrac{1}{2}t\, F(t)\,dt. \tag{4·2·8}$$

Next, by (4·2·6), $\Sigma \frac{s_n^*}{n^2} \cos 2nh = \frac{2}{\pi} \Sigma \frac{\cos 2nh}{n} \int_0^\pi F(t) \cos nt\, dt,$

and here we may sum under the integral sign. For (i) $\Sigma n^{-1} \cos 2nh \cos nt$ is boundedly convergent, and F integrable, in $(0, h)$, so that term-by-term integration is permissible over $(0, h)$. And (ii)

$$\left| \sum_1^N \frac{\cos 2nh \cos nt}{n} \right| < A + A \log^+ \frac{1}{|t-2h|} + A \log^+ \frac{1}{t+2h} + A \log^+ \frac{1}{2\pi - t - 2h},$$

by Lemma A, in (h, π)†, and F is continuous in (h, π), so that term-by-term integration is permissible over this interval also. It follows that

$$\Sigma \frac{s_n^*}{n^2} \cos 2nh = \frac{2}{\pi} \int_0^\pi F \Sigma \frac{\cos 2nh \cos nt}{n} dt = -\frac{1}{\pi} \int_0^\pi F \log |4 \sin(\tfrac{1}{2}t + h) \sin(\tfrac{1}{2}t - h)|\, dt.$$

Also $\log \frac{\sin \frac{1}{2}z}{\frac{1}{2}z}$ is an analytic function of z regular for $|z| < 2\pi$, and so

$$\log \left| \frac{\sin(\frac{1}{2}t + h) \sin(\frac{1}{2}t - h)}{(\frac{1}{2}t + h)(\frac{1}{2}t - h)} \right| = \log \left(\frac{\sin \frac{1}{2}t}{\frac{1}{2}t} \right)^2 + O(h^2),$$

uniformly in t; and hence

(4·2·9) $$\Sigma \frac{s_n^*}{n^2} \cos 2nh = -\frac{1}{\pi} \int_0^\pi F \log |t^2 - 4h^2|\, dt - \frac{2}{\pi} \int_0^\pi F \log \frac{\sin \frac{1}{2}t}{\frac{1}{2}t} dt + o(h).$$

Combining (4·2·8) and (4·2·9), we find

(4·2·10) $$\Sigma \frac{s_n^*}{n^2} (1 - \cos 2nh) = P(h) - Q + o(h),$$

where

(4·2·11) $$P(h) = \frac{1}{\pi} \int_0^\pi F(t) \log |t^2 - 4h^2|\, dt, \quad Q = \frac{2}{\pi} \int_{\to 0}^\pi F(t) \log \tfrac{1}{2}t\, dt.$$

Now $$F(t) = -\frac{f_1(t)}{t} + \int_t^\pi \frac{f_1(u)}{u^2} du$$

by Lemma B, and so

(4·2·12) $$P(h) = -P_1(h) + P_2(h),$$

where

(4·2·13) $$P_1(h) = \frac{1}{\pi} \int_{\to 0}^\pi \frac{f_1(t)}{t} \log |t^2 - 4h^2|\, dt,$$

(4·2·14) $$P_2(h) = \frac{1}{\pi} \int_{\to 0}^\pi \log |t^2 - 4h^2|\, dt \int_t^\pi \frac{f_1(u)}{u^2} du.$$

Here $$P_2(h) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_\epsilon^\pi \dots dt \int_t^\pi \dots du = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_\epsilon^\pi \dots du \int_\epsilon^u \dots dt$$

$$= \frac{1}{\pi} \lim_{\epsilon \to 0} \int_\epsilon^\pi \frac{f_1(u)}{u^2} \{\chi(u) - \chi(\epsilon)\}\, du,$$

where $$\chi(u) = \int_0^u \log |t^2 - 4h^2|\, dt = u \log |u^2 - 4h^2| - 2u + 2h \log \left| \frac{u + 2h}{u - 2h} \right|.$$

† $t + 2h < 2\pi$ and $|t - 2h| < \pi$ if $h < \frac{1}{2}\pi$, as we may suppose.

Since $\chi(u) = O(u)$ for fixed h and small u,

$$\chi(\epsilon)\int_\epsilon^\pi \frac{f_1(u)}{u^2}\,du = O(\epsilon)\,o\left(\frac{1}{\epsilon}\right) = o(1)$$

when $\epsilon \to 0$, by Lemma B†. Hence

$$P_2(h) = \frac{1}{\pi}\int_{\to 0}^{\pi} \frac{f_1}{u^2}\chi\,du = P_1(h) - R + S(h), \tag{4·2·15}$$

where $P_1(h)$ is defined by (4·2·13), and

$$R = \frac{2}{\pi}\int_{\to 0}^{\pi} \frac{f_1}{u}\,du, \quad S(h) = \frac{2h}{\pi}\int_{\to 0}^{\pi} \frac{f_1}{u^2}\log\left|\frac{u+2h}{u-2h}\right| du = \frac{2h}{\pi}J(2h). \tag{4·2·16}$$

Collecting our results from (4·2·10), (4·2·12), (4·2·15) and (4·2·16), we find that

$$\Sigma \frac{s_n^*}{n^2}(1-\cos 2nh) = \frac{2h}{\pi}J(2h) - Q - R + o(h). \tag{4·2·17}$$

Finally, integrating by parts and remembering (4·2·11), (4·2·5), (4·2·7), and (4·2·4), we find

$$Q = \frac{2}{\pi}\int_{\to 0}^{\pi} F\log\tfrac{1}{2}t\,dt = \frac{2}{\pi}\int_{\to 0}^{\pi} F\log t\,dt = -\frac{2}{\pi}\int_{\to 0}^{\pi} \frac{F_1}{t}\,dt$$

$$= -\frac{2}{\pi}\int_{\to 0}^{\pi} \frac{f_1}{t}\,dt - \frac{2}{\pi}\int_0^{\pi} F\,dt = -R.$$

Thus (4·2·17) is (4·2·3)‡, and this proves the theorem.

The proof may be simplified appreciably if

$$\int_0^\pi |f(t)|\log^+\frac{1}{|t|}\,dt < \infty,$$

and in particular if f is Z. The term-by-term integrations required may then be justified directly, without the use of Lemma E, and the later transformations also become rather simpler. In this case $J(h)$ is a Lebesgue integral.

Theorem 8 leads at once to an alternative proof of Theorem 6. For if f satisfies l_c, so that $g_1(t) = o(t)$, then

$$J(h) = \int_0^\pi o\left(\frac{1}{t}\right)\log\left|\frac{t+h}{t-h}\right| dt = o\left(\int_0^\infty \frac{1}{u}\log\left|\frac{u+1}{u-1}\right| du\right) = o(1).$$

4·3. We can now prove

THEOREM 9. *There are F.s. summable $(R, 2)$ but not (R_2).*

We take $\quad 0<\alpha<1, \quad 0<k<1, \quad \beta = \alpha^k, \quad \zeta = \alpha^{2-k},$

choosing α small enough to ensure that the intervals $j_n = (\alpha^n - \zeta^n, \alpha^n + \zeta^n)$, for $n = 1, 2, \ldots$, lie in $(0, \pi)$ and do not overlap. We choose a sequence of positive numbers ϵ_n for which $\epsilon_n \to 0$, $n\epsilon_n \to \infty$, and define $f_1(t)$ in j_n by the isosceles triangle whose base is j_n and whose height is $\epsilon_n\beta^n$, and as 0 elsewhere. Then $f_1(t) = O(\epsilon_n\beta^n)$ in j_n, so that

$$f_2(t) = O\Big(\sum_{\alpha^n \le t} \epsilon_n\beta^n\zeta^n\Big) = o\Big(\sum_{\alpha^n \le t} \alpha^{2n}\Big) = o(t^2).$$

Thus the F.s. of $f(t)$, for $t = 0$, is summable $(R, 2)$ to 0.

† See the additional remark in the proof of clause (ii). ‡ With s_n^* for s_n.

On the other hand $f_1 \geqq 0$, and

$$f_1 \geqq \tfrac{1}{2}\epsilon_n \beta^n \quad (\alpha^n - \tfrac{1}{2}\zeta^n \leqq t \leqq \alpha^n).$$

Hence, taking $h = \alpha^n$, we have

$$J(\alpha^n) > \int_{\alpha^n - \frac{1}{2}\zeta^n}^{\alpha^n} \frac{f_1}{t^2} \log \frac{\alpha^n + t}{\alpha^n - t}\, dt > \tfrac{1}{2}\epsilon_n \beta^n \alpha^{-2n} \int_{\alpha^n - \frac{1}{2}\zeta^n}^{\alpha^n} \log \frac{\alpha^n + t}{\alpha^n - t}\, dt;$$

and here
$$\log \frac{\alpha^n + t}{\alpha^n - t} > \log \frac{\alpha^n}{\zeta^n} = n \log \frac{\alpha}{\zeta}.$$

Hence
$$J(\alpha^n) > \tfrac{1}{4} n \epsilon_n \left(\frac{\beta\zeta}{\alpha^2}\right)^n \log \frac{\alpha}{\zeta} = \tfrac{1}{4} n \epsilon_n \log \frac{\alpha}{\zeta} \to \infty,$$

so that the series is not summable (R_2).

We may observe that f is $O\{(\beta/\zeta)^n\}$ in j_n and 0 elsewhere. Hence $\int |f|^p\, dt < \infty$ if

$$\Sigma \zeta^n (\beta/\zeta)^{pn} = \Sigma \alpha^{kpn-(p-1)(2-k)n} < \infty,$$

i.e. if $kp > (p-1)(2-k)$ or $k > (2p-2)/(2p-1)$. We can therefore find a F.s. of a function of the class L^p, for any finite p, satisfying our requirements†.

5. Conditions in terms of $\bar{f}$

5·1. We suppose now that both f and the conjugate function $\bar{f}$ are integrable. It is known that in this case

$$\bar{f}(t) \sim \Sigma B_n(t) = \Sigma(b_n \cos nt - a_n \sin nt),$$

and that the conditions are certainly satisfied if f is Z. We write

$$\bar{\psi}(t) = \bar{f}(\theta + t) - \bar{f}(\theta - t).$$

Theorem 10. *In order that the F.s. of $f(t)$, for $t = 0$, should be summable (R_2), it is necessary and sufficient that*

$$\frac{1}{\pi}\int_{\to 0}^{\pi} \frac{\bar{\psi}_1(t)}{t^2}\, dt$$

should be convergent.

We may make assumptions (A), in which case $\bar{\psi}(t) = 2\bar{f}(t)$.

It is familiar that

$$(5·1·1) \quad s_n = -\frac{1}{\pi}\int_0^{\pi} \bar{f}(t) \cot \tfrac{1}{2}t(1 - \cos nt)\, dt + u_n$$

$$= -\frac{2}{\pi}\int_0^{\pi} \bar{f}(t)\, \frac{1 - \cos nt}{t}\, dt + \frac{2}{\pi}\int_0^{\pi} \bar{f}(t) \left(\frac{1}{t} - \tfrac{1}{2}\cot \tfrac{1}{2}t\right) dt + v_n = -s_n^* + \rho + v_n,$$

say, where u_n and v_n tend to 0‡. Hence

$$(5·1·2) \quad \frac{1}{\pi h} \Sigma \frac{s_n + s_n^*}{n^2}(1 - \cos 2nh) = \frac{1}{\pi h} \Sigma \frac{\rho + v_n}{n^2}(1 - \cos 2nh) \to \rho.$$

† The F.s. of a *continuous* f is (uniformly) summable either $(R, 2)$ or (R_2).

‡ See HR, 47, with $d = 0$. It is to be remembered that the conjugate of $\bar{f}$ is $-f$.

Also
$$\Sigma \frac{s_n^*}{n^2}(1-\cos 2nh) = \frac{2}{\pi}\Sigma \frac{1-\cos 2nh}{n^2}\int_0^\pi \tilde{f}(t)\,\frac{1-\cos nt}{t}\,dt$$
and $\Sigma n^{-2}(1-\cos 2nh)(1-\cos nt)$ converges uniformly, for $0 \leqq t \leqq \pi$, to $\frac{1}{2}\pi\,\mathrm{Min}\,(2h, t)$, by Lemma F (2·6·6). Hence

$$(5{\cdot}1{\cdot}3) \qquad \Sigma \frac{s_n^*}{n^2}(1-\cos 2nh) = \int_0^{2h}\tilde{f}(t)\,dt + 2h\int_{2h}^{\pi}\frac{\tilde{f}(t)}{t}\,dt$$
$$= \tilde{f}_1(2h) - \tilde{f}_1(2h) + 2h\int_{2h}^{\pi}\frac{\tilde{f}_1(t)}{t^2}\,dt = 2h\int_{2h}^{\pi}\frac{\tilde{f}_1(t)}{t^2}\,dt.$$

It follows from (5·1·2) and (5·1·3) that
$$\frac{1}{\pi h}\Sigma \frac{s_n}{n^2}(1-\cos 2nh) \to \rho - \frac{2}{\pi}\int_{\to 0}^{\pi}\frac{\tilde{f}_1(t)}{t^2}\,dt$$
if and only if the last integral is convergent.

5·2. We can now conclude the paper by proving

THEOREM 11. *There are F.s. summable* (R_2) *but not* $(R, 2)$.

We suppose that $f \sim \Sigma a_n \cos nt$, $\tilde{f} \sim -\Sigma a_n \sin nt$. Then
$$a_n = -\frac{2}{\pi}\int_0^\pi \tilde{f}(t)\sin nt\,dt = \frac{2n}{\pi}\int_0^\pi \tilde{f}_1(t)\cos nt\,dt$$
and
$$f_2(h) = \Sigma \frac{a_n}{n^2}(1-\cos nh) = \frac{2}{\pi}\Sigma \frac{1-\cos nh}{n}\int_0^\pi \tilde{f}_1(t)\cos nt\,dt$$
$$= \frac{2}{\pi}\int_0^\pi \tilde{f}_1(t)\,\Sigma \frac{(1-\cos nh)\cos nt}{n}\,dt = \frac{1}{\pi}\int_0^\pi \tilde{f}_1(t)\log\left|\frac{\sin\frac{1}{2}(t-h)\sin\frac{1}{2}(t+h)}{\sin^2\frac{1}{2}t}\right|dt\dagger.$$

Also $\chi(z) = \log \dfrac{\sin\frac{1}{2}z}{\frac{1}{2}z}$ is an analytic function of z regular for $|z| < 2\pi$, and so
$$\log\left|\frac{\sin\frac{1}{2}(t+h)\sin\frac{1}{2}(t-h)}{\frac{1}{2}(t+h)\,\frac{1}{2}(t-h)}\right| - \log\left(\frac{\sin\frac{1}{2}t}{\frac{1}{2}t}\right)^2 = h^2\chi''(t) + o(h^2)\ddagger$$
uniformly in t. It follows that $h^{-2}f_2(h)$ will tend to a limit if and only if
$$\frac{\lambda(h)}{h^2} = \frac{1}{\pi h^2}\int_0^\pi \tilde{f}_1(t)\log\left|\frac{t^2-h^2}{t^2}\right|dt$$
does so. It is therefore sufficient to define $\tilde{f}_1(t)$ so that
$$(5{\cdot}2{\cdot}1) \qquad \int_{\to 0}^{\pi}\frac{\tilde{f}_1(t)}{t^2}\,dt$$
is convergent and $\lambda(h) \neq O(h^2)$. Actually, with our $\tilde{f}$, (5·2·1) will be a Lebesgue integral.

We define $\tilde{f}_1(t)$ as we defined $f_1(t)$ in the proof of Theorem 9, but now choosing ϵ_n so that $\Sigma\epsilon_n < \infty$ and $n\epsilon_n \to \infty$ when $n \to \infty$ through a certain sequence (n_i). First, the integral (5·2·1) converges (as a Lebesgue integral) because
$$\Sigma\epsilon_n\beta^n\zeta^n\alpha^{-2n} = \Sigma\epsilon_n < \infty.$$
Secondly, if we take $h = \alpha^m - 2\zeta^m$, then
$$\lambda(h) = A\Sigma\int_{\alpha^n-\zeta^n}^{\alpha^n+\zeta^n}\tilde{f}_1(t)\log\left|\frac{t^2-h^2}{t^2}\right|dt = A\Sigma w_n$$
$$= A(w_m + \sum_{n<m} w_n + \sum_{n>m} w_n) = A(S_1 + S_2 + S_3),$$

† The term-by-term integration being justified by Lemma A and the continuity of $\tilde{f}_1$.
‡ Compare § 4·2, p. 21.

say. In S_2, $\quad \bar{f}_1 = O(\epsilon_n \beta^n), \quad \log\left|1-\dfrac{h^2}{t^2}\right| = O\left(\dfrac{h^2}{t^2}\right) = O(\alpha^{2m-2n}),$

so that

$$S_2 = O(\sum_{n<m} \epsilon_n \beta^n \zeta^n \alpha^{2m-2n}) = O\{\alpha^{2m}(\epsilon_{m-1}+\epsilon_{m-2}+\ldots)\} = O(\alpha^{2m}) = O(h^2).$$

In S_3, $\quad \log\left|1-\dfrac{h^2}{t^2}\right| = 2\log\dfrac{h}{t} + O\left(\dfrac{t^2}{h^2}\right) = 2\log\dfrac{h}{t} + O(\alpha^{2n-2m}).$

The O-term contributes

$$O\{\alpha^{2m}(\epsilon_{m+1}\alpha^4+\epsilon_{m+2}\alpha^8+\ldots)\} = O(\alpha^{2m}) = O(h^2),$$

and the logarithmic term contributes

$$O\{\sum_{n>m}(n-m)\,\epsilon_n(\beta\zeta)^n\} = O\{\alpha^{2m}(\epsilon_{m+1}\alpha^2+2\epsilon_{m+2}\alpha^4+\ldots)\} = O(\alpha^{2m}) = O(h^2),$$

so that $S_3 = O(h^2)$. It is therefore sufficient to prove that

$$S_1 \neq O(h^2).$$

Now $\quad S_1 = A\displaystyle\int_{\alpha^m-\zeta^m}^{\alpha^m+\zeta^m} \bar{f}_1(t)\log\left|1-\left(\frac{\alpha^m-2\zeta^m}{t}\right)^2\right| dt.$

Here $\bar{f}_1 \geqq 0$, and the logarithm is negative, so that S_1 is negative and numerically greater than the corresponding integral over $(\alpha^m-\frac{1}{2}\zeta^m, \alpha^m+\frac{1}{2}\zeta^m)$. In this interval $\bar{f}_1 > A\epsilon_m\beta^m$, and the logarithm, varying between

$$\log\left\{1-\left(\frac{\alpha^m-2\zeta^m}{\alpha^m-\frac{1}{2}\zeta^m}\right)^2\right\}, \quad \log\left\{1-\left(\frac{\alpha^m-2\zeta^m}{\alpha^m+\frac{1}{2}\zeta^m}\right)^2\right\},$$

is less than $-Am\log(\alpha/\zeta)$. It follows that

$$|S_1| > Am\epsilon_m(\beta\zeta)^m\log\frac{\alpha}{\zeta} > Am\epsilon_m h^2\log\frac{\alpha}{\zeta},$$

which is not $O(h^2)$ when $m = m_i$.

Here again we may choose α, β, ζ so that f and $\bar{f}$ shall belong to any class L^p.

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CORRECTIONS

p. 12, *Lemma* B (i). For L^t read L.

p. 20, (4.2.8). The partial integration is not legitimate. A corrected argument is given in 1949, 1, p. 173, *footnote*.

NOTES ON FOURIER SERIES (V): SUMMABILITY (R_1)

By G. H. Hardy† and W. W. Rogosinski

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1. Introduction

1·1. There is a familiar method of summation of series usually called the method $(R,1)$, or sometimes Lebesgue's method. If $s_n = u_0+u_1+\ldots+u_n$ and, as it will be convenient to suppose throughout, $u_0 = 0$, then

$$\sum u_n = s \quad (R,1) \equiv \sum_1^\infty u_n \frac{\sin nh}{nh} \to s, \ddagger \tag{1·1·1}$$

when $h \to +0$§; the convergence of the series for small positive h is presupposed. The method is not 'regular'; there are convergent series not summable $(R,1)$.

We also suppose throughout that f is L and of period 2π, that

$$f(\theta) \sim \tfrac{1}{2}A_0(\theta) + \sum_1^\infty A_n(\theta) = \tfrac{1}{2}a_0 + \sum(a_n \cos n\theta + b_n \sin n\theta), \tag{1·1·2}$$

and that $a_0 = 0$. Then the condition

$$g_1(t) = \int_0^t g(u)\,du = o(t), \tag{1·1·3}$$

where
$$g(t) = g(t,\theta,c) = \phi(t,\theta) - c = \tfrac{1}{2}\{f(\theta+t)+f(\theta-t)\} - c \tag{1·1·4}$$

is necessary and sufficient for the summability $(R,1)$ to c, of the Fourier series (F.s.) of f for $t = \theta$‖. Since (1·1·3) is satisfied p.p.¶ with $c = f(\theta)$, the method is 'Fourier effective', i.e. sums any F.s. to $f(\theta)$ p.p.

In this note we consider the method (R_1) defined by

$$\sum u_n = s \quad (R_1) \equiv \frac{2}{\pi} \sum s_n \frac{\sin nh}{n} \to s. \tag{1·1·5}$$

The method is related to $(R,1)$ much as the method called (R_2) of our note (IV) is related to $(R,2)$††. Like $(R,1)$ it is not regular. We find necessary and sufficient conditions for summability (R_1) in terms of f, and also, under certain restrictions, in terms of the conjugate function $\tilde{f}$. We also show that $(R,1)$ and (R_1) are 'incomparable',

† The text of this note, in its present form up to § 3·3 inclusive, was drawn up by Prof. Hardy; his last illness prevented him from completing the draft. There exists material for a last common note on Fourier series which I hope to publish in due course. W.W.R.

‡ $\equiv$ is the sign of logical equivalence.

§ It is convenient, though in no way essential, to keep h positive.

‖ The notation is that of Hardy and Rogosinski (2), which we refer to as HR, except that we suppress the suffix c in what is there called $g_c(t)$. We use the notation $f_1, g_1, \phi_1, \ldots$ for the integrals of $f, g, \phi, \ldots$, from 0, systematically.

¶ Almost always (*presque partout*); we use this, and certain other abbreviations, as in HR.

†† Kuttner (4) has shown that either of $(R,1)$ or (R_1) implies both $(R,2)$ and (R_2). We take this opportunity of correcting a slip in (IV), pointed out to us by Dr Kuttner. The partial integration in

1949, 1 (with W. W. Rogosinski) *Proceedings of the Cambridge Philosophical Society*, 45, 173–85.

i.e. that neither implies the other, even for F.s.† Unlike $(R, 1)$, the method (R_1) is not Fourier effective; in fact, we give an example of a F.s. nowhere summable (R_1). It is, however, Fourier effective for wide classes of functions, for instance, for functions of any Lebesgue class L^p with $p > 1$.

1·2. We follow generally the notation of HR. We denote by $s_n = s_n(\theta) = s_n(\theta, f)$ the nth partial sum, for $t = \theta$, of the F.s. of $f(t)$. Then

$$s_n(\theta) - c = \frac{1}{\pi}\int_0^\pi g(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt, \tag{1·2·1}$$

where g is defined by (1·1·4). When considering summability $(R, 1)$, or (R_1), for a particular θ, we may clearly simplify by supposing

$$f \text{ even}, \qquad f \sim \Sigma a_n \cos nt \quad (a_0 = 0), \qquad \theta = 0, \qquad c = 0, \tag{A}$$

and so $g = \phi = f$. We call these assumptions, as in (IV), 'assumptions (A)'.

It is plain that $s_n(\theta) \to c\ (R_1)$ is equivalent to

$$\tau(h) = \tau(\theta, h) \to c, \tag{1·2·2}$$

where

$$\tau(h) = \Sigma \alpha_n(h)\, s_n(\theta), \quad \alpha_n(h) = \frac{2}{\pi}\frac{\sin nh}{n}, \tag{1·2·3}$$

and the convergence of the series for small $h > 0$ is presupposed. Here

$$\Sigma \alpha_n(h) = 1 - \frac{h}{\pi} \to 1, \tag{1·2·4}$$

if $0 < h < \pi$, $h \to +0$. On the other hand, $\Sigma\,|\,\alpha_n(h)\,|$ is divergent, so that the method (R_1) is not regular.

1·3. In § 2 we state a number of preliminary lemmas, with proofs where necessary; some of these lemmas were proved in (IV). In § 3 we obtain our main result, a necessary and sufficient condition for summability (R_1). We also show that there are F.s. which are nowhere summable (R_1). In § 4 we restrict the class of functions f by assuming that the conjugate function $\tilde{f}$ is L; for this class the method is Fourier effective. We then obtain a condition for summability (R_1) in terms of $\tilde{f}$, and this leads to an example of a F.s. summable (R_1) but not $(R, 1)$.

(4·2·8) is illegitimate. To correct the proof, cancel the last four lines on p. 20. In (4·2·10) replace $-Q$ by Q. The formula for Q, in (4·2·11), should now read

$$\begin{aligned} Q &= \frac{2}{\pi}\int_0^\pi F\log\frac{\sin\frac{1}{2}t}{\frac{1}{2}t}\,dt + \frac{1}{\pi}\int_{\to 0}^{\pi} F_1 \cot\tfrac{1}{2}t\,dt \\ &= \frac{2}{\pi}\left[F_1\log\frac{\sin\frac{1}{2}t}{\frac{1}{2}t}\right]_0^\pi - \frac{1}{\pi}\int_0^\pi F_1\left(\cot\tfrac{1}{2}t - \frac{2}{t}\right)dt + \frac{1}{\pi}\int_{\to 0}^{\pi} F_1\cot\tfrac{1}{2}t\,dt \\ &= \frac{2}{\pi}\int_{\to 0}^{\pi}\frac{F_1}{t}\,dt. \end{aligned}$$

In (4·2·17) replace $-Q$ by Q. Two lines later, read

$$Q = \frac{2}{\pi}\int_{\to 0}^{\pi}\frac{F_1}{t}\,dt = \frac{2}{\pi}\int_{\to 0}^{\pi}\frac{f_1}{t}\,dt + \frac{2}{\pi}\int_0^\pi F\,dt = R.$$

The lemma quoted on p. 20, line 5 from the bottom, should be Lemma E.

† That $(R, 1)$ and (R_1) are incomparable has been shown previously by Kuttner (4), but his examples are not F.s.

2. Preliminary lemmas

2·1. Lemma A. *If* $0<t<2\pi$, *then*

$$\left|\sum_{m=1}^{n}\frac{\cos mt}{m}\right|<A\left(1+\log^+\frac{1}{t}+\log^+\frac{1}{2\pi-t}\right),\dagger \tag{2·1·1}$$

$$\left|\sum_{m=N}^{M}\frac{\cos mt}{m}\right|<\frac{A}{N}\left(\frac{1}{t}+\frac{1}{2\pi-t}\right). \tag{2·1·2}$$

The inequality (2·1·1) is Lemma A of (IV), and (2·1·2) follows by partial summation from

$$\left|\sum_{1}^{n}\cos mt\right|<A\left(\frac{1}{t}+\frac{1}{2\pi-t}\right).$$

2·2. Lemma B. *If* $0<t<\pi$, $0<h<\pi$, $t\neq h$, *then*

$$\left|\sum_{m=1}^{n}\frac{\sin mt\sin mh}{m}\right|<A\left(1+\log^+\frac{1}{|t-h|}+\log^+\frac{1}{2\pi-(t+h)}\right), \tag{2·2·1}$$

$$\left|\sum_{m=N}^{M}\frac{\sin mt\sin mh}{m}\right|<\frac{A}{N}\left(\frac{1}{|t-h|}+\frac{1}{2\pi-(t+h)}\right). \tag{2·2·2}$$

Also

$$\left|\sum_{m=1}^{n}\frac{\sin mt\sin mh}{m}\right|<\frac{At}{h} \tag{2·2·3}$$

for

$$0<t\leqslant\tfrac{1}{2}h<\tfrac{1}{2}\pi.$$

First,

$$\sin mt\sin mh=\tfrac{1}{2}(\cos m\,|t-h|-\cos m(t+h)),$$

so that (2·2·1) and (2·2·2) follow from (2·1·1) and (2·1·2) respectively. To prove (2·2·3), suppose first that $n>t^{-1}$. Then

$$\sum_{m=1}^{n}\frac{\sin mt\sin mh}{m}=\sum_{1\leqslant m\leqslant t^{-1}}+\sum_{t^{-1}<m\leqslant n}=S_1+S_2,$$

say. Now

$$|S_1|=\left|t\sum_{1\leqslant m\leqslant t^{-1}}\sin mh\frac{\sin mt}{mt}\right|<\frac{At}{h}$$

by partial summation, since $(mt)^{-1}\sin mt$ decreases for $0\leqslant mt\leqslant 1$ and

$$\left|\sum_{1}^{n}\sin mh\right|<\frac{A}{h}.$$

Also $t\leqslant\frac{1}{2}h$, and so (2·2·2) gives the same estimate for S_2. This proves (2·2·3) when $n>t^{-1}$, and the proof for $n\leqslant t^{-1}$ is a simpler variant.

2·3. Lemma C. *If* $0<t<\pi$, $0<h<\pi$, $t\neq h$, *then*

$$\left|\sum_{m=1}^{n}\frac{(1-\cos mt)\sin mh}{m}\right|<A, \tag{2·3·1}$$

$$\left|\sum_{m=N}^{M}\frac{(1-\cos mt)\sin mh}{m}\right|<\frac{A}{N}\left(\frac{1}{|t-h|}+\frac{1}{2\pi-(t+h)}\right). \tag{2·3·2}$$

Also

$$\left|\sum_{m=1}^{n}\frac{(1-\cos mt)\sin mh}{m}\right|<\frac{At}{h} \tag{2·3·3}$$

for

$$0<t\leqslant\tfrac{1}{2}h<\tfrac{1}{2}\pi.$$

† We use A, $A(\alpha)$, $A(\beta)$, ... as in (IV).

We have $\sin mh(1-\cos mt) = \sin mh - \tfrac{1}{2}(\sin m(h-t)+\sin m(h+t))$,
so that (2·3·1) follows from the familiar inequality

$$\left|\sum_{m=1}^{n}\frac{\sin mt}{m}\right| < A.$$

The proofs of (2·3·2) and (2·3·3) are similar to those of (2·2·2) and (2·2·3); in proving (2·3·3) we use the fact that $(mt)^{-1}(1-\cos mt)$ increases for $0\leqslant mt\leqslant 1$.

2·4. LEMMA D. *If* $0<t<2\pi$, *then*

$$\sum_{1}^{\infty}\frac{\sin mt}{m} = \tfrac{1}{2}(\pi-t), \tag{2·4·1}$$

$$\sum_{1}^{\infty}\frac{\cos mt}{m} = -\log(2\sin\tfrac{1}{2}t). \tag{2·4·2}$$

If $0<t<\pi$, $0<h<\pi$, *then*

$$\sum_{1}^{\infty}\frac{\sin mt\sin mh}{m} = \tfrac{1}{2}\log\frac{\sin\frac{1}{2}(t+h)}{\sin\frac{1}{2}|t-h|}, \tag{2·4·3}$$

and

$$\sum_{1}^{\infty}\frac{(1-\cos mt)\sin mh}{m} = 0,\ \tfrac{1}{4}\pi,\ \tfrac{1}{2}\pi \tag{2·4·4}$$

for $t<h$, $t=h$, $t>h$, *respectively*.

The formulae (2·4·1) and (2·4·2) are familiar, and (2·4·3), (2·4·4) are corollaries.

2·5. LEMMA E. *If* $0<t<\pi$, $0<h\leqslant\frac{1}{2}\pi$, $t\neq h$, *then*

$$\sum_{1}^{\infty}\frac{\sin mt\sin mh}{m} = \tfrac{1}{2}\log\left|\frac{t+h}{t-h}\right|+\rho(t,h), \tag{2·5·1}$$

where

$$|\rho(t,h)|<Ath. \tag{2·5·2}$$

By (2·4·3)

$$\rho(t,h) = \tfrac{1}{2}\log\frac{\sin\frac{1}{2}(t+h)\,\frac{1}{2}|t-h|}{\frac{1}{2}(t+h)\sin\frac{1}{2}|t-h|}.$$

The function $\chi(z) = \log\dfrac{\sin\frac{1}{2}z}{\frac{1}{2}z}$, where $\chi(0)=0$, is regular for $|z|<2\pi$ and even. Hence $|\chi'(z)|<A|z|$ for $|z|\leqslant\frac{3}{2}\pi$, say. It follows that

$$2|\rho| = |\chi(t+h)-\chi(|t-h|)| = \left|\int_{|t-h|}^{t+h}\chi'(u)\,du\right| < A\int_{|t-h|}^{t+h}u\,du = Ath.$$

2·6. LEMMA F. *If the Cauchy integral*

$$\int_{\to 0}^{\pi} f(t)\log\frac{1}{t}\,dt \tag{2·6·1}$$

converges, then $f_1(t) = o[(\log 1/t)^{-1}]$ *and*

$$\int_{\to 0}^{\pi}\frac{f_1(t)}{t}\,dt \tag{2·6·2}$$

converges. The converse is also true.

If $0<\eta<t<1$, then

$$f_1(t)-f_1(\eta) = \int_{\eta}^{t}\frac{1}{\log 1/u}f(u)\log\frac{1}{u}\,du = \frac{1}{\log 1/t}\int_{\tau}^{t}f(u)\log\frac{1}{u}\,du,$$

with $\eta<\tau<t$. If (2·6·1) converges, this is $o[(\log 1/t)^{-1}]$, uniformly in η; and so $f_1(t)=o[(\log 1/t)^{-1}]$. Also

$$\int_\epsilon^\pi \frac{f_1(t)}{t}\,dt = f_1(\pi)\log\pi + f_1(\epsilon)\log\frac{1}{\epsilon}+\int_\epsilon^\pi f(t)\log\frac{1}{t}\,dt.$$

Making $\epsilon\to 0$, we conclude that (2·6·2) converges†. Conversely, if (2·6·2) converges and $f_1(t)=o[(\log 1/t)^{-1}]$, then (2·6·1) converges.

2·7. We say that f is Z if

$$\int_{-\pi}^{\pi} |f(t)|\,(1+\log^+|f(t)|)\,dt<\infty. \tag{2·7·1}$$

The importance of this class of functions was first recognized by Zygmund. It is included in L, and includes L^p for every $p>1$. If f is Z, then the conjugate $\bar{f}$ is L‡.

Lemma G. *If f is Z, and $-\pi\leqslant h\leqslant\pi$, then*

$$\int_{-\pi}^{\pi}\left|f(t)\log\frac{1}{|t-h|}\right|dt \tag{2·7·2}$$

is finite and continuous in h.

This is equivalent to Lemma C of (IV).

2·8. Lemma H. *If $f(t)\sim\Sigma a_n\cos nt$§, then, in order that the series $\Sigma n^{-1}a_n$ should be convergent, it is necessary and sufficient that the integral* (2·6·2) *should be convergent; and in this case*

$$a_0\log 2+\sum_1^\infty\frac{a_n}{n}=\frac{1}{\pi}\int_{\to 0}^{\pi}\cot\tfrac{1}{2}t\,f_1(t)\,dt. \tag{2·8·1}$$

This is Lemma E of (IV); it was first proved, less simply, by Hardy and Littlewood ((1), 94).

2·9. Lemma I. *If $f(t)\sim\Sigma a_n\cos nt$§, and the integral* (2·6·1) *converges, then*

$$\sum_1^\infty\frac{a_n}{n}=-\frac{2}{\pi}\int_{\to 0}^{\pi}f(t)\log(2\sin\tfrac{1}{2}t)\,dt \tag{2·9·1}$$

(*the result of term-by-term integration*).

By Lemma F, the integral (2·6·2) is convergent, and $f_1(t)=o[(\log 1/t)^{-1}]$. Hence, on integrating by parts,

$$\frac{1}{\pi}\int_{\to 0}^{\pi}\cot\tfrac{1}{2}t\,f_1(t)\,dt=-\frac{2}{\pi}\int_{\to 0}^{\pi}f(t)\log(\sin\tfrac{1}{2}t)\,dt=a_0\log 2-\frac{2}{\pi}\int_{\to 0}^{\pi}f(t)\log(2\sin\tfrac{1}{2}t)\,dt,$$

and (2·9·1) follows from (2·8·1). It should be noted that, by (2·4·2), the right-hand side of (2·9·1) is

$$\frac{2}{\pi}\int_{\to 0}^{\pi}f(t)\sum_1^\infty\frac{\cos nt}{n}\,dt,$$

while the left-hand side is the result of integrating this term by term.

† If $a_0=0$ and $f(t)$ is even, then $f_1(\pi)=0$, and the integral (2·6·2) equals (2·6·1).
‡ Zygmund (5), 150.
§ Here we drop the assumption $a_0=0$.

3. NECESSARY AND SUFFICIENT CONDITIONS FOR SUMMABILITY (R_1)

3·1. We begin with a discussion of the convergence of the series

$$\sum_{1}^{\infty} s_n(\theta)\frac{\sin nh}{n}. \tag{3·1·1}$$

It is convenient to suppose throughout that $0<h<\pi$.

THEOREM 1. *If f is L, then the series* (3·1·1) *converges if, and only if,*

$$\int_{\to 0}^{\pi}\frac{dt}{t}\int_{h-t}^{h+t}\phi(u,\theta)\,du \tag{3·1·2}$$

converges. When θ is fixed, the series converges for almost all h.

We write

$$s_n(\theta)=\frac{1}{\pi}\int_0^{\pi}\phi(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt$$
$$=\frac{1}{\pi}\int_0^{\pi}\phi(t)\cos nt\,dt+\frac{1}{\pi}\int_0^{\pi}\phi(t)\cot\tfrac{1}{2}t\sin nt\,dt=\tfrac{1}{2}A_n(\theta)+s_n^*(\theta),$$

and make assumptions (A), so that

$$s_n=s_n(0)=\tfrac{1}{2}a_n+s_n^*(0)=\tfrac{1}{2}a_n+s_n^*,\quad \phi=f.$$

Then what we have to show is that

$$\tfrac{1}{2}\Sigma n^{-1}a_n\sin nh+\Sigma n^{-1}s_n^*\sin nh \tag{3·1·3}$$

converges if, and only if,

$$\int_{\to 0}^{\pi}\frac{dt}{t}\int_{h-t}^{h+t}f(u)\,du=2\int_{\to 0}^{\pi}\frac{\phi_1(t,h)}{t}dt \tag{3·1·4}$$

converges.

First, $\Sigma n^{-1}a_n\sin nh$ is an integrated F.s., and therefore converges for all h†.

Next, we write

$$s_n^*=\frac{1}{\pi}\left(\int_0^{\frac{1}{2}h}+\int_{\frac{1}{2}h}^{\frac{3}{2}h}+\int_{\frac{3}{2}h}^{\pi}\right)f(t)\cot\tfrac{1}{2}t\sin nt\,dt=s_n^{(1)}+s_n^{(2)}+s_n^{(3)},\ddagger \tag{3·1·5}$$

say, and it is easy to prove $\Sigma n^{-1}(s_n^{(1)}+s_n^{(3)})\sin nh$ convergent. In fact,

$$\sum_1^{\infty}(s_n^{(1)}+s_n^{(3)})\frac{\sin nh}{n}=\frac{1}{\pi}\left(\int_0^{\frac{1}{2}h}+\int_{\frac{3}{2}h}^{\pi}\right)f(t)\cot\tfrac{1}{2}t\sum_1^{\infty}\frac{\sin nt\sin nh}{n}dt,$$

by term-by-term integration, this being permissible over $(0,\frac{1}{2}h)$ by (2·2·3), and over $(\frac{3}{2}h,\pi)$ because the series under the integral is, by (2·2·1), boundedly convergent.

It remains to consider $\Sigma n^{-1}s_n^{(2)}\sin nh$. We write

$$s_n^{(2)}\sin nh=\frac{1}{2\pi}\left\{\int_{\frac{1}{2}h}^{\frac{3}{2}h}f(t)\cot\tfrac{1}{2}t\cos n(t-h)\,dt-\int_{\frac{1}{2}h}^{\frac{3}{2}h}f(t)\cot\tfrac{1}{2}t\cos n(t+h)\,dt\right\}$$
$$=p_n-q_n,$$

say. Now
$$\sum_1^{\infty}\frac{q_n}{n}=\frac{1}{2\pi}\int_{\frac{1}{2}h}^{\frac{3}{2}h}f(t)\cot\tfrac{1}{2}t\sum_1^{\infty}\frac{\cos n(t+h)}{n}dt, \tag{3·1·6}$$

† HR, Theorem 44.

‡ We suppose that $\frac{3}{2}h<\pi$. If not, the second integral is over $(\frac{1}{2}h,\pi)$, and $s_n^{(3)}$ may be omitted. The argument is similar.

the term-by-term integration being again justified by (2·1·1). Also

$$2p_n = \frac{1}{\pi}\int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(t+h)\cot\tfrac{1}{2}(t+h)\cos nt\,dt$$

is the nth Fourier cosine coefficient of the function

$$F(t) = f(t+h)\cot\tfrac{1}{2}(t+h) \quad (|t| \leqslant \tfrac{1}{2}h), \qquad 0 \quad (\tfrac{1}{2}h < |t| < \pi). \tag{3·1·7}$$

Hence, by Lemma H, $\Sigma n^{-1}p_n$, and therefore $\Sigma n^{-1}s_n^{(2)}\sin nh$ and $\Sigma n^{-1}s_n^*\sin nh$ converges if, and only if,

$$\int_{\to 0}^{\frac{1}{2}h}\frac{F_1(t)-F_1(-t)}{t}dt = \int_{\to 0}^{\frac{1}{2}h}\frac{dt}{t}\int_{-t}^{t} f(u+h)\cot\tfrac{1}{2}(u+h)\,du = \int_{\to 0}^{\frac{1}{2}h}\frac{dt}{t}\int_{h-t}^{h+t} f(u)\cot\tfrac{1}{2}u\,du \tag{3·1·8}$$

converges. Finally, $|\cot\frac{1}{2}u - \cot\frac{1}{2}h| < A(h)\,t$ in the inner integral, and we may replace $\cot\frac{1}{2}u$ by $\cot\frac{1}{2}h$. Thus the convergence of (3·1·8) is equivalent to that of (3·1·4), and we have proved the first part of the theorem.

Since (3·1·4) converges, even as a Lebesgue integral, whenever $\phi_1(t,h) = o(t)$, and therefore for almost all h, the second part is a corollary.

3·2. We denote the 'principal value' limit

$$\lim_{\eta\to 0}\left(\int_0^{h-\eta} + \int_{h+\eta}^{\pi}\right)\dots dt$$

by

$$\int_{0(h)}^{\pi}\dots dt.$$

THEOREM 2. *If f is L, and if*†

$$\int_{0(h)}^{\pi}\phi(t,\theta)\log\frac{1}{|t-h|}dt \tag{3·2·1}$$

converges, then $$\sum_1^\infty s_n^*(\theta)\frac{\sin nh}{n} = \frac{1}{2\pi}\int_{0(h)}^{\pi}\phi(t)\cot\tfrac{1}{2}t\log\frac{\sin\frac{1}{2}(t+h)}{\sin\frac{1}{2}|t-h|}dt, \tag{3·2·2}$$

where s_n^* is defined as in (3·1·3).

We make the assumptions (A) and use the notations of § 3·1.

First, we wish to prove that the convergence of (3·2·1), with $\phi = f$, or what is the same, of

$$\int_{\to 0}^{\pi}\phi(t,h)\log\frac{1}{t}dt \tag{3·2·3}$$

implies that of

$$\int_{\to 0}^{\pi}\{F(t)+F(-t)\}\log\frac{1}{t}dt = \int_{\to 0}^{\frac{1}{2}h}[f(h+t)\cot\tfrac{1}{2}(h+t)+f(h-t)\cot\tfrac{1}{2}(h-t)]\log\frac{1}{t}dt. \tag{3·2·4}$$

In fact, the integrand in (3·2·4) differs, for fixed h, from $2\cot\frac{1}{2}h\phi(t,h)\log 1/t$ by less than

$$At\log 1/t(|f(t+h)|+|f(t-h)|),$$

which is L whenever f is.

It follows, by Lemma I, that $\Sigma n^{-1}p_n$ converges and can be evaluated by term-by-

† ϕ may be replaced by g.

term integration. Since such integration is permitted, in any case, for the sums $\Sigma n^{-1}(s_n^{(1)}+s_n^{(2)})$ and $\Sigma n^{-1}q_n$, we find that $\Sigma n^{-1}s_n^*\sin nh$ converges, and that

$$\sum_1^\infty s_n^*\frac{\sin nh}{n}=\frac{1}{\pi}\int_{0(h)}^{\pi}f(t)\cot\tfrac{1}{2}t\sum_1^\infty\frac{\sin nt\sin nh}{n}\,dt.$$

Substituting from (2·4·3) we obtain (3·2·2).

3·3. Our main theorem is

THEOREM 3. *In order that the F.s. of $f(t)$ should be summable (R_1) to c, for $t=\theta$, it is necessary and sufficient that*

$$I(h)=\int_{\to 0}^{\pi}\frac{dt}{t}\int_{|h-t|}^{h+t}g(u)\cot\tfrac{1}{2}u\,du \tag{3·3·1}$$

should converge for small h and tend to 0 with h†.

We make the assumptions (A), so that $g=f$, and use the notations of § 3·1. We also observe that $t=h$ is not a singular point for $I(h)$. In fact, for instance,

$$\int_{\frac{1}{2}h}^{h}\frac{dt}{t}\int_{h-t}^{h}|f(u)|\cot\tfrac{1}{2}u\,du<A\int_0^h dt\int_{h-t}^{h}\frac{|f(u)|}{u}du$$
$$=A\int_0^h\frac{|f(u)|}{u}du\int_{h-u}^{h}dt=A\int_0^h|f(u)|\,du,$$

for fixed h and $A=A(h)$.

Summability (R_1) implies the convergence of $\Sigma n^{-1}s_n\sin nh$ for all small h. For this it is necessary and sufficient, by Theorem 1, that the integral (3·1·4) should converge for these h. This, in turn, is equivalent, after § 3·1, to the convergence of

$$\int_{\to 0}^{\frac{1}{2}h}\frac{dt}{t}\int_{h-t}^{h+t}f(u)\cot\tfrac{1}{2}u\,du$$

or to that of $I(h)$.

Assuming now the existence of $I(h)$, that is, the convergence of $\Sigma n^{-1}s_n\sin nh$, we write $s_n=\frac{1}{2}a_n+s_n^*$. Here $\Sigma n^{-1}a_n\sin nh\to 0$ as $h\to 0$‡. We are therefore left with the discussion of

$$\sum_1^\infty s_n^*\frac{\sin nh}{n}=\frac{1}{\pi}\sum_1^\infty\left\{\int_0^\eta+\int_\eta^\pi\right\}f(t)\cot\tfrac{1}{2}t\frac{\sin nt\sin nh}{n}dt$$
$$=\frac{1}{\pi}\sum_1^\infty(J_n^{(1)}+J_n^{(2)}), \tag{3·3·2}$$

say, where $0<\eta<\frac{1}{2}h$. As in (3·1·5) we see that $\Sigma J_n^{(1)}$ converges and that

$$\sum_1^\infty J_n^{(1)}=\int_0^\eta f(t)\cot\tfrac{1}{2}t\sum_1^\infty\frac{\sin nt\sin nh}{n}dt.$$

† The upper limit may be replaced by any positive δ less than π. Also (3·3·1) is equivalent to

$$\frac{1}{\pi^2}\int_{\to 0}^{\pi}\frac{dt}{t}\int_{|h-t|}^{h+t}\phi(u)\cot\tfrac{1}{2}u\,du\to c.$$

For
$$\int_{\to 0}^{\pi}\frac{dt}{t}\int_{|h-t|}^{h+t}\cot\frac{u}{2}du=2\int_{\to 0}^{\pi}\frac{dt}{t}\int_{|h-t|}^{h+t}\frac{du}{u}+o(1)=2\int_{\to 0}^{\pi}\frac{1}{t}\log\left|\frac{h+t}{h-t}\right|dt+o(1)$$
$$=2\int_{\to 0}^{\infty}\frac{1}{t}\log\left|\frac{1+t}{1-t}\right|dt+o(1)=\pi^2+o(1),$$

when $h\to 0$. Similar remarks apply to Theorems 4 and 5.

‡ HR, Theorem 44.

Also, by (2·2·3), $$\left|\sum_1^\infty J_n^{(1)}\right| < \frac{A}{h}\int_0^\eta |f(t)|\,dt,$$

so that this sum tends to 0 as $\eta\to 0$, the h being fixed. We thus conclude that $\Sigma a_n = 0\ (R_1)$ is equivalent to

$$\lim_{h\to 0}\lim_{\eta\to 0}\sum_1^\infty J_n^{(2)} = 0. \tag{3·3·3}$$

Next $$\sum_1^\infty J_n^{(2)} = \sum_1^\infty \frac{1}{2n}\int_\eta^\pi f(t)\cot\tfrac12 t(\cos n(t-h)-\cos n(t+h))\,dt.$$

We define a periodic function $f^*(t)$ by

$$f^*(t) = \begin{cases} f(t) & \text{in } (\eta,\pi),\\ 0 & \text{elsewhere in } (0,2\pi).\end{cases} \tag{3·3·4}$$

Then

$$\sum_1^\infty J_n^{(2)} = \sum_1^\infty \frac{1}{2n}\int_0^{2\pi}(f^*(t+h)\cot\tfrac12(t+h)-f^*(t-h)\cot\tfrac12(t-h))\cos nt\,dt = \frac{\pi}{2}\sum_1^\infty \frac{A_n}{n}, \tag{3·3·5}$$

where A_n is the nth cosine coefficient of

$$F^*(t) = f^*(t+h)\cot\tfrac12(t+h) - f^*(t-h)\cot\tfrac12(t-h). \tag{3·3·6}$$

Clearly $A_0 = 0$. Since $\Sigma J_n^{(2)}$ is convergent, we conclude, by Lemma H, that (3·3·3) is equivalent to

$$\lim_{h\to 0}\lim_{\eta\to 0}\int_{\to 0}^{\pi}\cot\tfrac12 t\,dt\int_{-t}^{t} F^*(u)\,du = 0. \tag{3·3·7}$$

When h is fixed we may suppose that $\eta<\tfrac12 h$, $\eta<\pi-h$. We then write

$$\int_{\to 0}^{\pi}\cot\tfrac12 t\,dt\int_{-t}^{t} F^*(u)\,du = P-Q, \tag{3·3·8}$$

where

$$\begin{aligned} P &= \int_{\to 0}^{\pi}\cot\tfrac12 t\,dt\int_{-t}^{t} f^*(u+h)\cot\tfrac12(u+h)\,du = \int_{\to 0}^{\pi}\cot\tfrac12 t\,dt\int_{h-t}^{h+t} f^*(u)\cot\tfrac12 u\,du\\ &= \left(\int_{\to 0}^{h-\eta}\cdots\int_{h-t}^{h+t} + \int_{h-\eta}^{\pi-h}\cdots\int_{\eta}^{h+t} + \int_{\pi-h}^{\pi}\cot\tfrac12 t\,dt\int_\eta^\pi\right) f(u)\cot\tfrac12 u\,du\\ &= \int_{\to 0}^{h-\eta}\int_{h-t}^{h+t} - \int_{h-\eta}^{\pi-h}\int_{h+t}^{\pi} + \int_{h-\eta}^{\pi}\int_\eta^\pi = P_1-P_2+P_3\dagger, \end{aligned} \tag{3·3·9}$$

$$\begin{aligned} Q &= \int_{\to 0}^{\pi}\cot\tfrac12 t\,dt\int_{-t}^{t} f^*(u-h)\cot\tfrac12(u-h)\,du = \int_{\to 0}^{\pi}\cot\tfrac12 t\,dt\int_{-t-h}^{t-h} f^*(u)\cot\tfrac12 u\,du\\ &= \left(\int_{\eta+h}^{\pi}\cdots\int_{\eta}^{t-h} + \int_{\pi-h}^{\pi}\cot\tfrac12 t\,dt\int_{-t-h}^{-\pi}\right) f(u)\cot\tfrac12 u\,du\\ &= \int_{\eta+h}^{\pi}\int_\eta^\pi - \int_{\eta+h}^{\pi}\int_{t-h}^{\pi} + \int_{\pi-h}^{\pi}\int_{-t-h}^{-\pi} = Q_1-Q_2+Q_3. \end{aligned} \tag{3·3·10}$$

Now Q_3 is independent of η and tends to 0 as $h\to 0$. Next

$$P_3 - Q_1 = \int_{h-\eta}^{h+\eta}\cot\tfrac12 t\,dt\int_\eta^\pi f(u)\cot\tfrac12 u\,du,$$

† The reader should draw a figure; we integrate over a triangle and omit the parts where $f^*(u)$ vanishes. In all the repeated integrals $f^*(u)$ is replaced by $f(u)$.

so that
$$|P_3-Q_1|<\frac{A\eta}{h}\left\{\int_\eta^{\eta^{\frac{1}{2}}}+\int_{\eta^{\frac{1}{2}}}^{\pi}\right\}|f(u)|\cot\tfrac{1}{2}u\,du$$
$$<\frac{A\eta}{h}\left(\eta^{-1}\int_\eta^{\eta^{\frac{1}{2}}}|f(u)|\,du+\eta^{-\frac{1}{2}}\int_0^\pi|f(u)|\,du\right). \tag{3·3·11}$$

Hence $P_3-Q_1\to 0$ as $\eta\to 0$, h being fixed. Also
$$P_1-P_2+Q_2\to\int_{\to 0}^{h}\int_{h-t}^{h+t}-\int_h^{\pi-h}\int_{h+t}^{\pi}+\int_h^\pi\int_{t-h}^{\pi}$$
$$=\left(\int_{\to 0}^{h}\ldots\int_{h-t}^{h+t}+\int_h^\pi\ldots\int_{t-h}^{t+h}-\int_{\pi-h}^{\pi}\cot\tfrac{1}{2}t\,dt\int_\pi^{h+t}\right)f(u)\cot\tfrac{1}{2}u\,du, \tag{3·3·12}$$

when $\eta\to 0$. The last integral is independent of η and tends to 0 as $h\to 0$. It follows that (3·3·7) is equivalent to
$$\lim_{h\to 0}\int_{\to 0}^{\pi}\cot\tfrac{1}{2}t\,dt\int_{|h-t|}^{h+t}f(u)\cot\tfrac{1}{2}u\,du=0. \tag{3·3·13}$$

Lastly, $\cot\frac{1}{2}t=2t^{-1}+O(t)$ in $(0,\pi)$. Hence, when replacing $\cot\frac{1}{2}t$ by $2t^{-1}$ in (3·3·13), the error is less than
$$A\int_0^\pi t\,dt\int_{|h-t|}^{h+t}\frac{|f(u)|}{u}du<A\int_0^{2\pi}\frac{|f(u)|}{u}du\int_{|h-u|}^{h+u}t\,dt=2Ah\int_0^{2\pi}|f(u)|\,du,$$
which tends to 0 with h. This completes the proof of the theorem.

3·4. The following theorem, though included in Theorem 3, is much simpler in proof.

THEOREM 4. *Suppose that*
$$\int_{0(h)}^{\pi}g(t)\log\frac{1}{|t-h|}dt \tag{3·4·1}$$
exists for all small h. In order that the F.s. of $f(t)$ should be summable (R_1) to c, for $t=\theta$, it is necessary and sufficient that
$$J(h)=\int_{0(h)}^{\pi}g(t)\cot\tfrac{1}{2}t\log\left|\frac{t+h}{t-h}\right|dt \tag{3·4·2}$$
should tend to 0 *with h.*

We make the assumptions (A), so that $g=f$. Summability (R_1) of Σa_n to 0 is equivalent to $\Sigma n^{-1}s_n^*\sin nh\to 0$. By (3·2·2) this is the same as
$$\int_{0(h)}^{\pi}f(t)\cot\tfrac{1}{2}t\log\frac{\sin\frac{1}{2}(t+h)}{\sin\frac{1}{2}|t-h|}dt\to 0.$$
Finally, by (2·4·3) and Lemma E, we may replace the logarithmic factor by
$$\log|(t+h)/(t-h)|.$$

3·5. Usually it is not possible to replace $\cot\frac{1}{2}t$ by $2t^{-1}$ in (3·4·2). If f is Z, this can be done.

THEOREM 5. *Suppose that f is Z. In order that the F.s. of $f(t)$ should be summable (R_1) to c, for $t=\theta$, it is necessary and sufficient that*
$$K(h)=\int_0^\pi\frac{g(t)}{t}\log\left|\frac{t+h}{t-h}\right|dt \tag{3·5·1}$$
should tend to 0 *with h.*

We make the assumptions (A) so that $g=f$. By Lemma G, the integral (3·4·1) is

a Lebesgue integral. Hence Theorem 4 is applicable, where (3·4·2) is now an L-integral. Also, if we replace $\cot \frac{1}{2}t$ by $2t^{-1}$ in this integral, the error is less than

$$A\left(\int_0^{2h}+\int_{2h}^{\pi}\right) t\,|\,f(t)\,|\log\left|\frac{t+h}{t-h}\right| dt$$

$$< A\left(\int_0^{2h} h\,|\,f(t)\,|\left(\log\frac{1}{h}+\log\frac{1}{|\,t-h\,|}\right) dt+\int_{2h}^{\pi} t\,|\,f(t)\,|\,\frac{h}{t}\,dt\right)$$

$$< Ah\log\frac{1}{h} = o(1),$$

since

$$\int_0^{\pi} |\,f(t)\,|\log\frac{1}{|\,t-h\,|}\,dt < A,$$

by Lemma G.

3·6. Any F.s. is summable $(R, 1)$ p.p. This is not true for the method (R_1).

THEOREM 6. *There are F.s. summable* $(R, 1)$ *but not* (R_1); *in fact, there are F.s. which are nowhere summable* (R_1).

We need only prove the second clause. Let (α_ν) $(\nu = 1, 2, \ldots)$, be a sequence of numbers everywhere dense in $\langle -\pi, \pi\rangle$, and let $\Sigma A_\nu < \infty$, where $A_\nu > 0$. We consider the periodic function

$$f(t) = \sum_1^{\infty} A_\nu \phi(t-\alpha_\nu), \tag{3·6·1}$$

where

$$\phi(t) = \begin{cases} [\,|\,t\,|\log^2|\,t\,|\,]^{-1} & \text{for } 0<|\,t\,|\leqslant \frac{1}{2}, \\ 0 & \text{elsewhere in } \langle -\pi, \pi\rangle. \end{cases} \tag{3·6·2}$$

Clearly, f is non-negative and L. We wish to prove that, for every fixed θ, there exists a sequence of numbers $h_\mu = h_\mu(\theta)$, tending to 0, such that $\Sigma n^{-1} s_n(\theta) \sin nh_\mu$ diverges.

According to Theorem 1 we have to show that the integrals

$$\int_{\to 0}^{\pi} \frac{dt}{t}\int_{h_\mu - t}^{h_\mu + t} (f(\theta+u)+f(\theta-u))\,du \tag{3·6·3}$$

diverge. Now, given θ in $(-\pi, \pi\rangle$, we can find a subsequence (α_μ), say, of the α_ν, such that $h_\mu = \theta-\alpha_\mu \to +0$. We may assume that all $h_\mu < \frac{1}{2}$.

The integral (3·6·3) is diminished, first, by omitting the term $f(\theta+u)$, further by replacing $f(\theta-u)$ by its term $A_\mu\phi(\theta-u-\alpha_\mu) = A_\mu\phi(h_\mu-u)$, and finally by restricting the integration to the interval $0 \leqslant t \leqslant h_\mu$. The integral (3·6·3) is thus greater than

$$A_\mu\int_0^{h_\mu}\frac{dt}{t}\int_{h_\mu-t}^{h_\mu+t}\phi(h_\mu-u)\,du = A_\mu\int_0^{h_\mu}\frac{dt}{t}\int_{-t}^{t}\phi(u)\,du$$

$$= 2A_\mu\int_0^{h_\mu}\frac{dt}{t}\int_0^t\frac{du}{u\log^2 u} = 2A_\mu\int_0^{h_\mu}\frac{dt}{t\log 1/t},$$

which integral diverges. This proves the theorem.

4. CONDITIONS IN TERMS OF $\tilde{f}$

4·1. We suppose from now on that both f and its conjugate function $\tilde{f}$ are L. It will be convenient to say that then f belongs to the class $\tilde{L}$ (f is $\tilde{L}$). It is known that, if f is $\tilde{L}$, then

$$\tilde{f}(\theta) \sim \sum_1^{\infty} B_n(\theta) = \sum_1^{\infty}(b_n\cos n\theta - a_n\sin n\theta);$$

and that the class $\tilde{L}$ includes Z, and hence all classes L^p, where $p>1$.

We write $$\tilde{\psi}(t) = \tilde{\psi}(t,\theta) = \tilde{f}(\theta+t) - \tilde{f}(\theta-t).$$
If f is $\tilde{L}$, then $-f$ is the conjugate function of $\tilde{f}$, and
$$f(\theta) = -\frac{1}{2\pi}\int_{\to 0}^{\pi} \tilde{\psi}(t)\cot\tfrac{1}{2}t\,dt \tag{4·1·1}$$
for almost all θ.†

THEOREM 7. *Suppose that f is $\tilde{L}$. Then the series* (3·1·1) *converges for all θ and h.*

In order that the F.s. of $f(t)$ should be summable (R_1), for $t = \theta$, it is necessary and sufficient that
$$\int_{\to 0}^{\pi} \frac{\tilde{\psi}(t)}{t}\,dt \tag{4·1·2}$$
should exist.

The F.s. of f is summable (R_1) p.p. to the sum $f(\theta)$‡.

We make assumptions (A), except that we do not assume that $c = 0$. Then $\tilde{f}$ is odd and $\tilde{\psi} = 2\tilde{f}$.

It is familiar§ that
$$s_n = -\frac{1}{\pi}\int_0^{\pi} \tilde{f}(t)\cot\tfrac{1}{2}t(1-\cos nt)\,dt - \frac{1}{\pi}\int_0^{\pi}\tilde{f}(t)\sin nt\,dt = s_n^* + \tfrac{1}{2}a_n, \tag{4·1·3}$$
where $s_n^* = s_n^*(0)$ is defined as in (3·1·3). Now $\Sigma n^{-1}a_n \sin nh$ converges for all h and tends to 0 with h. Also
$$\sum_1^{\infty} s_n^* \frac{\sin nh}{n} = -\frac{1}{\pi}\int_0^{\pi}\tilde{f}(t)\cot\tfrac{1}{2}t\sum_1^{\infty}\frac{(1-\cos nt)\sin nh}{n}\,dt \tag{4·1·4}$$
converges; the integration by terms is justified, over the range $\langle 0, \frac{1}{2}h\rangle$ by (2·3·3), and over the range $\langle \frac{1}{2}h, \pi\rangle$ by (2·3·1). Hence, by (2·4·4),
$$\frac{2}{\pi}\sum_1^{\infty} s_n^* \frac{\sin nh}{n} = -\frac{1}{\pi}\int_h^{\pi}\tilde{f}(t)\cot\tfrac{1}{2}t\,dt, \tag{4·1·5}$$
so that $\Sigma a_n = c\ (R_1)$ is equivalent to
$$-\frac{1}{\pi}\int_{\to 0}^{\pi}\tilde{f}(t)\cot\tfrac{1}{2}t\,dt = c. \tag{4·1·6}$$
The convergence of this integral is equivalent to that of (4·1·2), where $\tilde{\psi} = 2\tilde{f}$.

Finally, according to (4·1·1), the formula (4·1·6) holds, with $c = f(\theta)$, for almost all θ. This completes the proof of the theorem.

4·2. We conclude the paper by proving the converse of the first clause of Theorem 6.

THEOREM 8. *There are F.s. summable (R_1) but not $(R, 1)$.*

Let $$0<\alpha<1, \quad 0<k<1, \quad \beta = \alpha^{k-1}, \quad \zeta = \alpha^{2-k},$$
where α is so small that the intervals $j_n = (\alpha^n - \zeta^n, \alpha^n + \zeta^n)$, where $n = 1, 2, \ldots$, do not overlap and lie in $(0, \pi)$. We note that $\alpha\beta = \alpha^k$, $\beta\zeta = \alpha$, and that $\alpha^n - \zeta^n > A\alpha^n$.

Next, we choose a sequence of positive numbers ϵ_n such that $\Sigma\epsilon_n < \infty$, and also so that it contains a subsequence (ϵ_ν), say, with $\nu\epsilon_\nu \to \infty$.

† HR, Theorem 89.

‡ Hence the function (3·6·1) is not $\tilde{L}$. It is, however, not difficult to give an example of a function of L^p, where $p > 1$, whose F.s. is summable $(R, 1)$ but not (R_1).

§ HR, p. 47, formula (4·9·1).

Now, let $\tilde{f}(t)$ be an odd function equal to $\epsilon_n \beta^n$ in j_n, and vanishing elsewhere in $\langle 0, \pi\rangle$. First,

$$\int_0^\pi \tilde{f}^2(t)\,dt = 2\Sigma\epsilon_n^2 \beta^{2n}\zeta^n = 2\Sigma\epsilon_n^2\beta^n\alpha^n = 2\Sigma\epsilon_n^2\alpha^{kn} < \infty, \tag{4·2·1}$$

so that $\tilde{f}$, and hence its conjugate function $-f$, is L^2; thus f is $\tilde{L}$. Also

$$\int_0^\pi \frac{\tilde{f}(t)}{t}\,dt < 2\Sigma\epsilon_n\beta^n\zeta^n(\alpha^n - \zeta^n)^{-1} < A\Sigma\epsilon_n\alpha^n\alpha^{-n} < \infty, \tag{4·2·2}$$

so that, by Theorem 7, the F.s. of f is summable (R_1) for $t = 0$†. Let

$$f(\theta) \sim \sum_1^\infty a_n \cos n\theta, \quad \tilde{f}(\theta) \sim -\sum_1^\infty a_n \sin n\theta.$$

If $0 < \theta \leqslant \frac{1}{2}\pi$, then

$$f_1(\theta) = \sum_1^\infty a_n \frac{\sin n\theta}{n} = -\frac{2}{\pi}\int_0^\pi \tilde{f}(t) \sum_1^\infty \frac{\sin nt \sin n\theta}{n}\,dt$$

$$= -\frac{1}{\pi}\int_0^\pi \tilde{f}(t)\log\left|\frac{t+\theta}{t-\theta}\right| dt + \lambda(\theta), \tag{4·2·3}$$

where

$$|\lambda(\theta)| < A\theta, \tag{4·2·4}$$

by Lemma E; the integral in (4·2·3) exists since $\tilde{f}$ if L^2, and the integration by terms is justified by (2·2·1).

In order to show that the F.s. of f is not summable $(R, 1)$ at $t = 0$, it is sufficient to prove that $\theta^{-1}f_1(\theta)$ is not bounded. In view of (4·2·4) we may replace $f_1(\theta)$ by the integral in (4·2·3).

Let $\theta_n = \alpha^n - \zeta^n$, so that $\theta_n \to 0$ as $n \to \infty$. Since $\tilde{f}$ is non-negative,

$$H(\theta) = \int_0^\pi \tilde{f}(t)\log\left|\frac{t+\theta}{t-\theta}\right| dt > \int_{j_n} \tilde{f}(t)\log\left|\frac{t+\theta}{t-\theta}\right| dt,$$

$$H(\theta_n) > \epsilon_n\beta^n \int_{j_n} \log\frac{t+\theta_n}{t-\theta_n}\,dt.$$

Also

$$\log\frac{t+\theta_n}{t-\theta_n} = \log\frac{t+\alpha^n-\zeta^n}{t-\alpha^n+\zeta^n}$$

decreases in j_n, and $\alpha^n > \theta_n$, so that

$$H(\theta_n) > 2\epsilon_n\beta^n\zeta^n \log\left(\frac{\alpha}{\zeta}\right)^n = 2n\epsilon_n\alpha^n\log\frac{\alpha}{\zeta} > An\epsilon_n\theta_n. \tag{4·2·5}$$

Now $\nu\epsilon_\nu \to \infty$ for a subsequence (ϵ_ν). Hence $\theta^{-1}H(\theta)$ is not bounded when $\theta = \theta_\nu$ tends to 0. This completes the proof.

† The integral (4·2·2) is an L-integral. It follows, from (4·1·3), that the F.s. of f actually converges for $t = 0$.

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COMMENT

p. **173,** ***first footnote.*** The 'last common note on Fourier series' mentioned here has not been published.

(*c*) The Young–Hausdorff Inequalities

INTRODUCTION TO PAPERS ON THE YOUNG–HAUSDORFF INEQUALITIES

The Young–Hausdorff inequalities state that (i) if $f \in L^p(-\pi, \pi)$, where $1 \leqslant p \leqslant 2$, and c_n is the nth complex Fourier coefficient of f, then

$$\Big(\sum_{-\infty}^{\infty} |c_n|^{p'}\Big)^{1/p'} \leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^p \, d\theta\right)^{1/p},$$

(ii) if (c_n) is a two-way infinite sequence such that $\sum_{-\infty}^{\infty} |c_n|^p$ is convergent, where $1 \leqslant p \leqslant 2$, then there exists $f \in L^{p'}$ such that c_n is the nth complex Fourier coefficient of f, and

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^{p'} \, d\theta\right)^{1/p'} \leqslant \Big(\sum_{-\infty}^{\infty} |c_n|^p\Big)^{1/p}.$$

In 1926, 7 Hardy and Littlewood gave a new proof of these inequalities, and determined the cases of equality. They obtained also important generalizations, giving sufficient conditions for the convergence of the series $\sum |n|^{-k}|c_n|^s$, and for the existence of the integral $\int |\theta|^{-k}|f(\theta)|^s \, d\theta$. These inequalities have proved a fundamental tool in the finer theory of Fourier series. The proofs follow the interpolation argument of Hausdorff, using 'Hausdorff differentiation', and among the papers of Hardy and Littlewood on Fourier series, the two authors' analytical insight and technical mastery is nowhere more apparent than here. Later, the discovery of the convexity theorems of M. Riesz and Thorin enabled the proofs to be simplified, and the inequalities have been extended and generalized to a considerable degree.

In 1931, 4 Hardy and Littlewood themselves generalized the inequalities of 1926, 7, introducing the decreasing rearrangement (c_n^*) of the sequence (c_n), and the series $\Sigma^* = \sum (|n|+1)^{r-2} c_n^{*r}$. They proved that if $r \geqslant 2$, then the necessary and sufficient condition for the c_n to be the Fourier coefficients of an $f \in L^r$, for every arrangement, and every variation of $\arg c_n$, is that Σ^* is convergent, and that if $1 < r < 2$, then the same holds for some such arrangement and variation. In 1935, 6 they proved that if $r = 1$ and $f \in Z$, then Σ^* is convergent, and used this result to give a new proof of the difficult convergence criterion of 1934, 3.

The two papers 1932, 5 and 6 deal with the convolution of two power series in the case where one satisfies an integrated Lipschitz condition and the other belongs to H^p for some $p < 1$. The results obtained are analogues of M. Riesz's well-known extension of Parseval's theorem.

The final paper of this group, 1944, 1, is concerned with the partial sums of a function of L^2.

Some new properties of Fourier constants.

Von

G. H. Hardy in Oxford und J. E. Littlewood in Cambridge.

1.

Introduction.

1.1. The theory of Fourier constants, as distinct from the theory of the convergence of Fourier series, may be said to date from Riemann. The first theorem of the theory, in fact, is the 'theorem of Riemann-Lebesgue', that the Fourier constants of any integrable[1]) function tend to zero, a theorem proved by Riemann[2]) for bounded functions integrable in accordance with his definition, and by Lebesgue in full generality.

The most important general theorems concerning Fourier constants, after the theorem of Riemann-Lebesgue, are Parseval's Theorem and the Riesz-Fischer theorem. Let us suppose for simplicity of statement that $f(\theta)$ is a real and even function, that the fundamental interval is $(-\pi, \pi)$ and that the mean value of $f(\theta)$ over this interval is zero, so that the Fourier series of $f(\theta)$ is a pure cosine series $\Sigma a_n \cos n\theta$ without constant term. Then the two theorems together assert that Σa_n^2 is convergent if, and only if, the square of $f(\theta)$ is integrable.

A very important generalisation of these theorems has been effected recently by W. H. Young[3]) and Hausdorff[4]). Suppose that $1 < p \leqq 2$, so that

$$p' = \frac{p}{p-1} \leqq 2. \tag{1.11}$$

Then the final theorem of Hausdorff asserts (I) that $\Sigma |a_n|^{p'}$ is convergent whenever $|f|^p$ is integrable, and (II) that $|f|^{p'}$ is integrable whenever

[1]) In the sense of Lebesgue.

[2]) Riemann, 1, 239—241.

[3]) W. H. Young, 2, 3.

[4]) Hausdorff, 1.

1926, 7 (with J. E. Littlewood) *Mathematische Annalen*, 97, 159–209.

$\Sigma |a_n|^p$ is convergent. There is an essential asymmetry about the number 2, and both propositions would become false were the condition $p \leqq 2$ omitted. The theorems were proved by Young for a special sequence of values of p, viz. $2, \frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \ldots$, and by Hausdorff generally; an alternative proof, somewhat simpler than Hausdorff's, was given later by F. Riesz[5]), who also extended the results to general orthogonal series.

We give a new proof in § 3. The proof will be useful as an introduction to the more difficult analysis which follows. It has also two points of independent interest; it enables us to complete the solution of the minimal problem suggested in Hausdorff's analysis, and it involves what would appear to be the least possible amount of 'existence-theory'.

In §§ 4—7 we solve the principal problems of the memoir. The problem which originally suggested these investigations may be stated as follows. Suppose that $r > 1$ and that f and $|f|^r$ are integrable, or, in the language of Riesz, that f belongs to L^r. *Then for what values of s and $\varkappa$ does it follow that the series*

$$\Sigma n^{-\varkappa} |a_n|^s \tag{1.12}$$

is convergent? We suppose naturally that s is positive, and the theorems which we prove provide the complete answer to the question.

1.2. We observe first that we may confine ourselves to the case in which $r = p \leqq 2$; if $r = q > 2$, the answer is almost immediate. If f belongs to L^q, it belongs *a fortiori* to L^2, so that Σa_n^2 is convergent. *A fortiori*, $\Sigma |a_n|^s$ is convergent if $s > 2$; so that (1.12) is convergent if $s \geqq 2$, $\varkappa \geqq 0$. If on the other hand $s < 2$, then

$$\Sigma n^{-\varkappa} |a_n|^s \leqq (\Sigma a_n^2)^{\frac{1}{2}s} \left(\Sigma n^{-\frac{2\varkappa}{2-s}}\right)^{1-\frac{1}{2}s}$$

is convergent for $\varkappa > 1 - \frac{1}{2}s$. And it may be shown by examples that these inequalities for $\varkappa$ cannot be improved upon, that (1.12) is not necessarily convergent, when $s \geqq 2$, for any negative $\varkappa$, nor when

$$s < 2, \quad \varkappa = 1 - \tfrac{1}{2}s\text{[6]}).$$

Thus the hypothesis that a power of $|f|$ higher than the second is integrable adds nothing to the inferences which we can make.

We may suppose therefore that $r = p \leqq 2$. There are then three ranges of values of s which require separate treatment, viz., the ranges

[5]) F. Riesz, 3.

[6]) See § 8 for indications as to the methods for constructing appropriate 'Gegenbeispiele'.

$$s < p, \quad p \leqq s \leqq p', \quad p' < s.$$

Of these, the two outer ranges are easily disposed of.

(I) If $s > p'$, Hausdorff's theorem solves the problem; for $\Sigma |a_n|^s$ is convergent, but (1.12) is not necessarily convergent for any negative value of $\varkappa$[6]).

(II) If $0 < s \leqq p'$ and

$$\varkappa > \frac{p+s-ps}{p},$$

so that $\varkappa p' > p' - s$, then

$$\Sigma n^{-\varkappa} |a_n|^s \leqq (\Sigma |a_n|^{p'})^{\frac{s}{p'}} \left(\Sigma n^{-\frac{\varkappa p'}{p'-s}}\right)^{1-\frac{s}{p'}},$$

which is convergent in virtue of Hausdorff's theorem [7]). It is easy to show that the inequality for $\varkappa$ cannot be replaced by any better inequality $\varkappa > \varkappa_0$. It is also easy to show[8]) that the conclusion is false when

$$\varkappa = \frac{p+s-ps}{p}, \quad s < p.$$

If $s < p$, then, the problem is disposed of.

1.3. The only question which remains is whether (1.12) is necessarily convergent when

$$\varkappa = \frac{p+s-ps}{p}, \quad p \leqq s \leqq p',$$

and it is this that is the really interesting and difficult question. When $s = p'$, $\varkappa = 0$, the proposition reduces to Hausdorff's theorem. If we can prove it when $s = p$, $\varkappa = 2 - p$ (the other extreme case), that is to say if we can prove that

$$\Sigma n^{p-2} |a_n|^p \tag{1.31}$$

is convergent, then we can deduce it in the intermediate cases in an elementary way[9]). It is the convergence of the series (1.31) that is the kernel of one of our four principal theorems (Theorem 5), though the actual theorem is stated in a more precise and general form, as an inequality satisfied by the complex Fourier constants of a complex function.

If we assume the truth of this theorem for the moment, we can state the answer to the question put at the end of § 1.1 as follows. *If f belongs to L^r, then the series $\Sigma n^{-\varkappa} |a_n|^s$ is convergent if*

(I) $\quad r = q > 2, \quad s \geqq 2, \quad \varkappa \geqq 0;$

(II) $\quad r = q > 2, \quad s < 2, \quad \varkappa > 1 - \frac{1}{2}s;$

---

7) Young proves this when s is an odd integer, in particular when $s = 1$.

8) See § 8.

9) See § 8.

(III) $\qquad r = p \leqq 2, \quad s > \frac{p}{p-1}, \quad \varkappa \geqq 0;$

(IV) $\qquad r = p \leqq 2, \quad p \leqq s \leqq \frac{p}{p-1}, \quad \varkappa \geqq \frac{p+s-ps}{p};$

(V) $\qquad r = p \leqq 2, \quad 0 \leqq s < p, \quad \varkappa > \frac{p+s-ps}{p};$[10])

and it is not necessarily convergent in any other case.

1.4. Our proof of Theorem 5 is of the same general character as Hausdorff's original proof of his theorem, but is decidedly more difficult, and it seems to us unlikely that there is any really easy proof. That (1.12) is convergent when

$$r = p \leqq 2, \quad \varkappa = \frac{p+s-ps}{p}, \quad 2 \leqq s \leqq p'$$

may indeed be proved in a different manner, as we show in § 8. We have proved elsewhere[11]) that if f belongs to L^p, $0 < \alpha < \frac{1}{p}$, and f_α is the 'Riemann-Liouville' integral of f of order α, then f_α belongs to $L^{\frac{p}{1-p\alpha}}$. From this we can deduce the convergence of (1.12) when $s = 2$, $\varkappa = \frac{2}{p} - 1$, and the full result follows when we combine this particular case with Hausdorff's theorem. This argument (which in any case rests on two by no means easy theorems) fails completely when $s < 2$. We have in short no doubt that Theorem 5 is a really difficult and interesting theorem.

1.5. Hausdorff's theorem consists essentially of two theorems, the proofs of which go together. So, we find, Theorem 5 is essentially one of a group of *four* theorems, two (Theorems 3 and 5) of the 'Parseval' and two (Theorems 2 and 6) of the 'Riesz-Fischer' type. The proofs of these are also associated in pairs, and there is a close formal correspondence between the proofs of the two pairs, though we are not able in any sense to *deduce* one pair from the other. Two of the theorems involve a series of the type $\sum n^{p-2} |a_n|^p$, and two an integral of the type $\int |f|^p |\theta|^{p-2} d\theta$. These integrals might be replaced by integrals of the type

$$\int |f|^p |\theta - \theta_0|^{p-2} d\theta,$$

with uniformity of the conclusions in θ_0; but there seems to be no particular object in such an extension.

[10]) We include here the trivial case $s = 0$.

[11]) See Hardy and Littlewood, 2. The proof of the actual theorem which we quote (Theorem 3) has not yet been published in full, but it is an easy deduction from Theorem 1, the proof of which is given by Hardy, Littlewood and Pólya, 1.

We find it necessary, with each pair of theorems, to give first a special proof, of one theorem of the pair, valid only for a particular sequence of values of the parameter q. In this we revert to the original line of argument of Young and Hausdorff, which is not necessary either in Riesz's proof of Hausdorff's theorem or in our own given in § 3. The fact that the numbers which we call λ, λ' or μ, μ' are not bounded in p or q seems to make some such reversion inevitable.

The proof of the principal theorems is completed by § 7. In §§ 8—9 we indicate certain extensions in other directions. In particular we show that, if $f(z)=\sum a_n z^n$ is an analytic function regular for $\varrho=|z|<1$, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\varrho e^{i\theta})|^{\lambda}\, d\theta$$

where $0<\lambda\leqq 1$, $\varrho<1$, is bounded, then $\sum n^{\lambda-2}|a_n|^{\lambda}$ is convergent. If λ were greater than 1, this would be a corollary of Theorem 5; but the corollary remains valid when the latter theorem fails. The case $\lambda=1$ is particularly interesting. In this case the theorem is equivalent to the theorem that, if two conjugate series

$$\sum(a_n \cos n\theta + b_n \sin n\theta), \qquad \sum(b_n \cos n\theta - a_n \sin n\theta)$$

are both Fourier series, then

$$\sum \frac{|a_n|+|b_n|}{n}$$

is convergent.

2.

Notation and preliminary lemmas.

2. 1. We shall use the following notation throughout the memoir.

The letters r, p, q denote real numbers,

$$r>1, \qquad 1<p\leqq 2, \qquad q\geqq 2, \tag{2. 11}$$

so that an r may be a p or a q. In any case

$$r'=\frac{r}{r-1}, \qquad \frac{1}{r}+\frac{1}{r'}=1, \tag{2. 12}$$

so that $1<q'\leqq 2$, $p'\geqq 2$: thus p' is a q and q' a p.

The function $f(\theta)$, which is generally complex, will be said to be integrable, or to belong to the class L, if it is measurable and integrable in the sense of Lebesgue. If also $|f|^r$ is integrable, f will be said to belong to the class L^r. The foundations of a systematic theory of these 'Lebesgue classes' were laid by F. Riesz[12]).

[12]) F. Riesz, 1.

We write generally $\bar{z}$ for the conjugate of z, $\bar{f}$ for the function whose values are the conjugates of those of f, and

$$\operatorname{sgn} f = \frac{f}{|f|}, \tag{2.13}$$

unless $f = 0$, when we define $\operatorname{sgn} f$ as zero.

If $f(\theta) = g(\theta)$ for almost all values of θ, we say that f and g are equivalent, and write $f \equiv g$. If $f \equiv 0$ we say that f is null.

We call

$$c_m = c_m(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-mi\theta}\, d\theta \tag{2.14}$$

where m is an integer, the (complex) Fourier coefficient of $f(\theta)$. We write

$$f \sim \sum_{-\infty}^{\infty} c_m e^{mi\theta}, \quad f_n = \sum_{-n}^{n} c_m e^{mi\theta}. \tag{2.15}$$

If $f(\theta)$ belongs to L^r, we write

$$J_r = J_r(f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^r\, d\theta\right)^{\frac{1}{r}}. \tag{2.16}$$

If $\sum |c_m|^r$ is convergent, we write

$$S_r = S_r(f) = \left(\sum_{-\infty}^{\infty} |c_m|^r\right)^{\frac{1}{r}}. \tag{2.17}$$

It should be observed that if J_r exists for one r it exists for any smaller r, while if S_r exists for one r it exists for any larger r.

2.2. We shall make repeated use of two inequalities ((2.22) and (2.23) below) which are corollaries of 'Hölder's inequality' (or the 'generalised inequality of Schwarz'), and of certain related propositions. We summarise these propositions here in the form of lemmas. We state, for the sake of clearness, more than we shall actually require. The propositions are well-known, but it would be difficult to refer to any connected statement of them, and we do not attempt to attribute each proposition to its original author. The J's refer to a positive function $|f|$, which we assume not to be null (so that $J_r > 0$), and the S's to a sequence of positive numbers $|c_m|$, which we assume not to be all zero (so that $S_r > 0$). The genesis of c_m, as the Fourier constant of a function f, is here irrelevant.

Lemma 1. *If* $0 < \alpha \leqq \beta \leqq \gamma$, $\alpha < \gamma$, *and*

$$\vartheta = \frac{\alpha(\gamma - \beta)}{\beta(\gamma - \alpha)}, \tag{2.21}$$

so that $0 \leqq \vartheta \leqq 1$, *and* S_α *exists* (*i.e.* $S_\alpha < \infty$), *then*

$$S_\beta \leqq S_\alpha^\vartheta S_\gamma^{1-\vartheta}, \tag{2.22}$$

and S_β *lies (in the wide sense) between* S_α *and* S_γ.

Lemma 2. *If* $\alpha, \beta, \gamma, \vartheta$ *satisfy the same conditions as in Lemma 1, and* $J_\gamma < \infty$, *then*

$$J_\beta \leqq J_\alpha^\vartheta J_\gamma^{1-\vartheta}, \tag{2.23}$$

and J_β *lies (in the wide sense) between* J_α *and* J_γ.

So far there is complete correspondence between S's and J's. When we pass to two-term relations, there is a divergence.

Lemma 3. *If* $\alpha \leqq \gamma$, *then*

$$S_\alpha \geqq S_\gamma, \tag{2.24}$$

so that S_α *is (in the wide sense) a decreasing function of* α. *If* $S_\alpha < \infty$ *for some (sufficiently large)* α, *then*

$$S_\alpha \to \operatorname{Max} |c_m|$$

when $\alpha \to \infty$; *if* $S_\alpha < \infty$ *for all* α, *then*

$$S_\alpha \to \infty$$

when $\alpha \to 0$, *unless there is only one* m, *say* m_0, *for which* $c_m \neq 0$, *in which case*

$$S_\alpha = |c_{m_0}|$$

for all α.

Lemma 4. *If* $\alpha \leqq \gamma$, *then*

$$J_\alpha \leqq J_\gamma, \tag{2.25}$$

so that J_α *is (in the wide sense) an increasing function of* α. *If* $|f|$ *is continuous, then*

$$J_\alpha \to \operatorname{Max} |f|$$

when $\alpha \to \infty$. *If* $J_\alpha < \infty$ *for some (sufficiently small)* α, *then*

$$J_\alpha \to \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log |f|\, d\theta\right)$$

when $\alpha \to 0$.

The remaining lemmas are concerned with the cases in which the inequalities asserted by Lemmas 1—4 may reduce to equalities.

Lemma 5. *Equality can occur in* (2.22), *when* $\alpha < \beta < \gamma$, *only if* $c_m = 0$ *for all but a finite number of values of* m, *for each of which* $|c_m| = \varkappa > 0$. *In this case there is equality for all values of* α, β, γ.

Equality can occur in (2.24), *when* $\alpha < \gamma$, *only if* $c_m = 0$ *for all values of* m *save one, in which case there is equality for all values of* α *and* γ.

Lemma 6. *Equality can occur* (2.23), *when* $\alpha < \beta < \gamma$, *only if* $f = 0$ *in a certain set* E, *and* $|f| = \varkappa > 0$ *in the complementary set* CE (*so that* $|f|$ *is almost always* 0 *or* $\varkappa$). *In this case there is equality for all values of* α, β, γ.

Equality can occur in (2.25), *when* $\alpha < \gamma$, *only when* $|f| = \varkappa > 0$. *In this case it occurs for all values of* α *and* γ.

Thus equality can occur, in the two-term relations, for the S's only when there is the maximum of condensation in a single term, for the J's only when there is the maximum regularity of distribution. The integrals in fact do not really correspond to *sums*, but to *finite mean values* of the type

$$\left(\frac{1}{2n+1}\sum_{-n}^{n}|c_m|^{\alpha}\right)^{\frac{1}{\alpha}}$$

(which have properties corresponding strictly to those of the J's). In the three-term relations the powers of $2n+1$ cancel, and symmetry is restored.

In the sequel we need only Lemmas 1 and 2, the first clause of Lemma 4, and the first half of Lemma 6. An alternative proof in § 3.6, suggested to us by Mr. C. A. Meredith, uses the whole of Lemma 5 and the first clause of Lemma 6. We have stated the lemmas completely, since a partial statement of the position might be misleading.

We shall denote the proposition $S_\beta = S_\alpha^{\vartheta} S_\gamma^{1-\vartheta}$, asserting equality in (2.22), or the corresponding proposition for J's, by $[\alpha, \beta, \gamma]$.

2.3. If $(\varphi_n(\theta))$ $(n = 1, 2, 3, \ldots)$ is a sequence of functions each of which belongs to L^r, and if

$$\int_{-\pi}^{\pi} |\varphi_m - \varphi_n|^r \, d\theta \to 0$$

when m and n tend to infinity, then the sequence (φ_n) is said to *converge strongly with index r*. In these circumstances there is a function $\varphi(\theta)$ of L^r such that

$$\int_{-\pi}^{\pi} |\varphi_n - \varphi|^r \, d\theta \to 0$$

when $n \to \infty$. Also

$$\int_a^b \varphi_n \, d\theta \to \int_a^b \varphi \, d\theta \quad (-\pi \leqq a \leqq b \leqq \pi)$$

uniformly in a, b;

$$\int_{-\pi}^{\pi} |\varphi_n|^r d\theta \to \int_{-\pi}^{\pi} |\varphi|^r d\theta;$$

and

$$\int_{-\pi}^{\pi} \varphi_n \psi \, d\theta \to \int_{-\pi}^{\pi} \varphi \psi \, d\theta$$

if ψ is any function of $L^{r'}$ or, in particular, any continuous function[13]). In our applications of the last result ψ is the function $e^{-mi\theta}$, when the relation becomes

$$c_m(\varphi) = \lim_{n\to\infty} c_m(\varphi_n).$$

3.

A new proof of Hausdorff's Theorem, with determination of the minimal functions.

3.1. In this section we prove

Theorem 1. *If q and q' are subject to (2.11) and (2.12), then*

(3.11) $$S_q \leqq J_{q'},$$

(3.12) $$J_q \leqq S_{q'},$$

the second inequality being interpreted as implying the existence of an $f(\theta)$ of L^q with Fourier constants c_m.

Equality occurs in (3.11) or (3.12), when $q > 2$, if and only if

(3.13) $$f(\theta) = c\, e^{\mu i\theta},$$

where c is a constant and μ an integer.

There is nothing to prove when $q = 2$, and we shall suppose throughout that $q > 2$.

We define λ and λ' by

(3.141) $$\lambda = \lambda(n) = \lambda_q(n) = \overline{\text{bound}}\, \frac{S_q(f_n)}{J_{q'}(f)},$$

(3.142) $$\lambda' = \lambda'(n) = \lambda'_q(n) = \overline{\text{bound}}\, \frac{J_q(f_n)}{S_{q'}(f_n)},$$

for variation of f and f_n respectively. Here f_n is a polynomial

(3.15) $$f_n = f_n(\theta) = \sum_{-n}^{n} c_m e^{mi\theta}$$

not identically zero, and

(3.16) $$f = f(\theta) \sim \sum_{-\infty}^{\infty} c_m e^{mi\theta}$$

[13]) For proofs of these propositions see F. Riesz, 1.

is any integrable function other than a null function. If f does not belong to $L^{q'}$, $J_{q'}(f)$ will be infinite, and the quotient which appears in the definition of λ is in this case to be interpreted as zero.

The bounds λ and λ' exist, for every q and n. For (I) if we suppose, as we may do on grounds of homogeneity, that $S_q(f_n) = 1$, we have

$$|c_m|^q \geqq \frac{1}{2n+1}$$

for some m, and so

$$(2n+1)^{-\frac{1}{q}} \leqq |c_m| \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|\,d\theta \leqq J_{q'}(f), \quad \frac{S_q(f_n)}{J_{q'}(f)} \leqq (2n+1)^{\frac{1}{q}}.$$

And (II) if we suppose that $J_q(f_n) = 1$, as again we may do, we have

$$1 = J_q(f_n) \leqq \operatorname{Max} |f_n(\theta)| \leqq (2n+1) \operatorname{Max} |c_m|,$$

$$S_{q'}(f_n) \geqq \operatorname{Max} |c_m| \geqq \frac{1}{2n+1}, \quad \frac{J_q(f_n)}{S_{q'}(f_n)} \leqq 2n+1.$$

The trivial example $f = 1$ shows that $\lambda \geqq 1$, $\lambda' \geqq 1$. We shall in fact prove that $\lambda = \lambda' = 1$. We begin by proving an (almost trivial) existence theorem.

Lemma 7. *The bound λ' is attained, for every q and n; there is a polynomial f_n, not null, for which*

(3.17) $$J_q(f_n) = \lambda' S_{q'}(f_n).$$

That λ is also attained will appear incidentally in the sequel.

There is, by the definition of λ', a sequence f_n^ν $(\nu = 1, 2, 3, \ldots)$ for which

$$S_{q'}(f_n^\nu) = 1, \quad \lim J_q(f_n^\nu) = \lambda'.$$

The set of points

$$P^\nu = (c_{-n}^\nu, \ldots, c_n^\nu),$$

in space of $2n+1$ dimensions, is bounded, since $|c_m^\nu| \leqq S_{q'}(f_n^\nu) = 1$. It has therefore at least one limit point, say

$$P = (c_{-n}, \ldots, c_n).$$

We can select a sub-sequence P^N $(N = 1, 2, 3, \ldots)$ such that $P^N \to P$ and $f_n^N \to f_n$ uniformly in θ. Then

$$S_{q'}(f_n) = \lim S_{q'}(f_n^N) = 1,$$

so that f_n is not null, and

$$J_q(f_n) = \lim J_q(f_n^N) = \lambda'.$$

3.2. Lemma 8. *The bounds λ and λ' are equal, for every q and n:*

(3.21) $$\lambda_q(n) = \lambda_q'(n).$$

Suppose that f_n is the polynomial of Lemma 7, and that

(3. 221) $$g(-\theta)=|f_n(\theta)|^{q-1}\ \operatorname{sgn}\overline{f_n(\theta)},$$

(3. 222) $$g(\theta)\sim\sum_{-\infty}^{\infty}b_m e^{mi\theta}.$$

Then

(3. 231) $$J_q^q(f_n)=J_{q'}^{q'}(g)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f_n(\theta)g(-\theta)\,d\theta=\sum_{-n}^{n}b_m c_m$$

(3. 232) $$\leqq\sum_{-n}^{n}|b_m||c_m|\leqq S_q(g_n)S_{q'}(f_n),$$

(3. 233) $$\leqq\lambda J_{q'}(g)S_{q'}(f_n)=\lambda J_q^{q-1}(f_n)S_{q'}(f_n)$$

(3. 234) $$=\lambda J_q^{q-1}(f_n)\cdot\frac{1}{\lambda'}J_q(f_n)=\frac{\lambda}{\lambda'}J_q^q(f_n).$$

Hence

(3. 24) $$\lambda'\leqq\lambda.$$

On the other hand, if

(3. 251) $$\varphi(\theta)\sim\sum_{-\infty}^{\infty}\gamma_m e^{mi\theta}$$

is any integrable function, and

(3. 252) $$h_n(-\theta)=\sum_{-n}^{n}|\gamma_m|^{q-1}\operatorname{sgn}\overline{\gamma_m}\,e^{mi\theta},$$

we have

(3. 261) $$S_q^q(\varphi_n)=\sum_{-n}^{n}|\gamma_m|^q=\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(\theta)h_n(\theta)\,d\theta$$

(3. 262) $$\leqq J_{q'}(\varphi)J_q(h_n)\leqq J_{q'}(\varphi)\cdot\lambda' S_{q'}(h_n)=\lambda' J_{q'}(\varphi)S_q^{q-1}(\varphi_n),$$

or

(3. 263) $$S_q(\varphi_n)\leqq\lambda' J_{q'}(\varphi).$$

Since this is true for all φ, we must have

(3. 27) $$\lambda\leqq\lambda',$$

and therefore, by (3. 24),

(3. 28) $$\lambda=\lambda'.$$

We may now replace λ' everywhere by λ.

The proof of Lemma 8 might be presented in a somewhat simpler form, but the analysis which we have set out will be needed later.

3. 3. Since $\lambda=\lambda'$, the extreme terms of the chain (3. 23) are identical,

and all inequalities which occur in the chain must reduce to equalities. Since (3. 233) is an equality

$$S_q(g_n) = \lambda J_{q'}(g), \tag{3. 31}$$

so that the upper bound λ is attained, for the function g.

Since (3. 232) is an equality, we have

$$\operatorname{sgn} b_m c_m \geqq 0, \quad |b_m|^q = t^q |c_m|^{q'} (|m| \leqq n), \tag{3. 32}$$

where t is independent of m, so that

$$\sum_{-n}^{n} |b_m|^q = t^q \sum_{-n}^{n} |c_m|^{q'}$$

or

$$t = S_q(g_n)(S_{q'}(f_n))^{-q'+1}.$$

But

$$S_{q'}(f_n) = \frac{1}{\lambda} J_q(f_n) = \frac{1}{\lambda} J_{q'}^{q'-1}(g) = \lambda^{-q'} S_q^{q'-1}(g_n),$$

and so

$$t = \lambda^{q'(q'-1)} (S_q(g_n))^{1-(q'-1)^2} = \lambda^{\frac{q}{(q-1)^2}} (S_q(g_n))^{\frac{q(q-2)}{(q-1)^2}}.$$

Accordingly (3. 32) gives

$$c_m = \tau |b_m|^{q-1} \operatorname{sgn} \overline{b_m} \qquad (|m| \leqq n), \tag{3. 33}$$

where

$$\tau = t^{-q+1} = \lambda^{-q'} (S_q(g_n))^{-q'(q-2)}. \tag{3. 34}$$

3. 4. Lemma 9. *If* $q_1 = 2q - 2$, *so that* $q_1 > q > 2$, *then*

$$\lambda_q(n) \leqq \lambda_{q_1}(n). \tag{3.41}$$

We apply Parseval's Theorem to $g(\theta)$. We write

$$q_1 = 2q - 2, \quad q_1' = \frac{q_1}{q_1 - 1} = \frac{2q-2}{2q-3}, \quad Q = (q-1)q_1' = \frac{2(q-1)^2}{2q-3}, \tag{3. 42}$$

so that

$$2 < Q < q < q_1. \tag{3. 43}$$

We have then

$$\begin{aligned} S_2^2(g_n) = \sum_{-n}^{n} |b_m|^2 \leqq \sum_{-\infty}^{\infty} |b_m|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |g|^2 d\theta = J_{q_1}^{q_1}(f_n) \\ \leqq \lambda_{q_1}^{q_1}(n) S_{q_1}^{q_1}(f_n) = \lambda_{q_1}^{q_1}(n) \cdot \tau^{q_1} (S_Q(g_n))^{(q-1)q_1} \\ \leqq \lambda_{q_1}^{q_1}(n) \lambda_q^{-2q}(n) (S_q(g_n))^{-2q(q-2)} (S_Q(g_n))^{(q-1)q_1}, \end{aligned} \tag{3. 44}$$

or

$$\frac{\lambda_q^{2q}(n)}{\lambda_{q_1}^{q_1}(n)} \leqq (S_Q(g_n))^{2(q-1)^2} (S_2(g_n))^{-2} (S_q(g_n))^{-2q(q-2)}. \tag{3. 45}$$

This does not exceed unity, by Lemma 1. For (since $2 < Q < q$) we may take $\alpha = 2$, $\beta = Q$, $\gamma = q$, and then the indices on the right hand side of (3. 45) are $2(q-1)^2$ times those of Lemma 1. It follows that

$$\lambda_q^{2q}(n) \leqq \lambda_{q_1}^{q_1}(n)$$

and therefore, since $2q > 2q - 2 = q_1$ and $\lambda \geqq 1$, that

$$\lambda_q(n) \leqq \lambda_{q_1}(n).$$

3. 5. We can now prove our principal lemma, viz.

Lemma 10. *The bounds λ and λ' are each unity*:

$$(3.\ 51) \qquad \lambda_q(n) = \lambda_q'(n) = 1.$$

Suppose (3. 51) false, for a particular n. Then there is, after Lemma 9, a $\Theta = \Theta_n > 1$ such that $\lambda_q(n) > \Theta$ for arbitrarily large values of q, and therefore, after Lemma 7, an f such that

$$(3.\ 52) \qquad S_q(f_n) > \Theta J_{q'}(f).$$

But, if $f_n = \sum_{-n}^{n} c_m e^{mi\theta}$ and $\gamma = \operatorname{Max}|c_m|$, (3. 52) gives

$$\Theta < \frac{1}{\gamma}\left(\sum_{-n}^{n} \gamma^q\right)^{\frac{1}{q}} = (2n+1)^{\frac{1}{q}};$$

and this is false if q is sufficiently large.

We can now prove the first inequality (3. 11) of Theorem 1. We have in fact

$$(3.\ 53) \qquad \left(\sum_{-n}^{n} |c_m|^q\right)^{\frac{1}{q}} = S_q(f_n) \leqq J_{q'}(f),$$

and (3. 11) follows when $n \to \infty$. The second inequality (3. 12) requires a further lemma.

Lemma 11. *Suppose that $\sum |c_m|^{q'}$ is convergent. Then there is an $f(\theta)$ of L^q whose Fourier coefficients are c_m, and $f_n(\theta)$ converges strongly to $f(\theta)$, with index q.*

If $0 < \nu < n$, then

$$\varphi = \varphi_n = \varphi_{n,\nu} = f_n - f_\nu = \sum_{\nu < |m| \leqq n} c_m e^{mi\theta}$$

is a polynomial, so that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\varphi|^q d\theta = J_q^q(\varphi_n) \leqq S_{q'}^q(\varphi_n) = \left(\sum_{\nu < |m| \leqq n} |c_m|^{q'}\right)^{q-1} < \varepsilon$$

if $\nu > n_0(\varepsilon)$. That is to say, f_n converges strongly to an f of L^q[14]); and

[14]) See § 2. 3.

$c_r(f)$, the r-th Fourier constant of f, is the limit of the r-th Fourier constant of f_n, that is to say c_r.

We can at once deduce (3. 12). For, given

$$S_{q'}^{q'} = \sum_{-\infty}^{\infty} |c_m|^{q'},$$

there is an f, whose Fourier constants are c_m, so that $S_{q'} = S_{q'}(f)$, to which f_n converges strongly; and

(3. 54) $\quad J_q(f) = \lim J_q(f_n) \leqq \underline{\lim}\, S_{q'}(f_n) = \lim S_{q'}(f_n) = S_{q'}(f)$.

At this stage we have completed the proof of Hausdorff's inequalities. It remains to determine the minimal solutions.

3. 6. We return to (3. 44), which we may now write in the form

(3. 61)
$$1 \leqq \left(\sum_{-n}^{n} |b_m|^q\right)^{-2(q-2)} \left(\sum_{-\infty}^{\infty} |b_m|^2\right)^{-2} \left(\sum_{-n}^{n} |b_m|^Q\right)^{2q-3}$$
$$\leqq \left(\sum_{-n}^{n} |b_m|^q\right)^{-2(q-2)} \left(\sum_{-n}^{n} |b_m|^2\right)^{-2} \left(\sum_{-n}^{n} |b_m|^Q\right)^{2q-3} \leqq 1,$$

by Lemma 1. It is plain that all inequalities here must be equalities. Hence, first,

(3. 621) $$b_m = 0 \qquad (|m| > n),$$

so that $g = g_n$. Next, by Lemma 5, we must have

(3. 622) $$b_m = \varkappa j_m e^{i\beta_m} \qquad (|m| \leqq n),$$

where $\varkappa$ is a positive constant, j_m is 0 or 1, and β_m is real, so that

(3. 63) $$g(\theta) = g_n(\theta) = \varkappa \sum_{-n}^{n} j_m e^{i\beta_m} e^{mi\theta}.$$

Recurring now to (3. 33), and observing that $j_m^{q-1} = j_m$, we see that

(3. 64) $$c_n = \tau \varkappa^{q-1} j_m \operatorname{sgn} \bar{b}_m = \tau \varkappa^{q-1} j_m e^{-i\beta_m},$$

(3. 65) $$f_n = \tau \varkappa^{q-1} \sum_{-n}^{n} j_m e^{-i\beta_m} e^{mi\theta};$$

and so, by (3. 221),

(3. 66) $$g(\theta) = |f_n(-\theta)|^{q-2} \overline{f_n(-\theta)} = |f_n(-\theta)|^{q-1} \tau \varkappa^{q-1} \sum_{-n}^{n} j_m e^{i\beta_m} e^{mi\theta}$$
$$= \tau \varkappa^{q-2} |f_n(-\theta)|^{q-2} g(\theta).$$

Since $g(\theta)$ can vanish only at isolated points[15]), $|f_n|$ is constant.

If we write

$$z = e^{i\theta}, \qquad F(z) = z^n f_n(\theta) = \sum_{-n}^{n} c_m z^{m+n},$$

[15]) With $f_n(-\theta)$.

then $F(z)$ is a polynomial whose modulus is constant on the unit circle. Hence $F(z)$ is a monomial[16]) and

$$f_n = c\, e^{\mu i \theta}, \tag{3.67}$$

where c is a constant and μ an integer.

Mr. C. A. Meredith has pointed out to us an alternative proof. It follows from (3.61) that $S_\alpha(g_n)$ satisfies one proposition of the form $[\alpha, \beta, \gamma]$. viz. $[2, Q, q]$, and therefore all such propositions. Since $|c_m| = \tau\, |b_m|^{q-1}$ if $|m| \leqq n$, $S_\alpha(f_n)$ also satisfies all such propositions. But $J_2(f_n) = S_2(f_n)$, by Parseval's Theorem, $J_q(f_n) = S_{q'}(f_n)$, by the minimal property of f_n, and $J_{q_1}(f_n) = S_{q_1'}(f_n)$, by (3.44), this being a chain of equalities. Since $S_\alpha(f_n)$ satisfies $[q_1', q', 2]$, and ϑ has the same value for γ', β', α' as for α, β, γ, it follows that $J_\alpha(f_n)$ satisfies $[2, q, q_1]$. Hence, by Lemma 6, $|f_n|$ is, almost always, a constant $\varkappa$ or zero; and as f_n is a polynomial, and not null, we must have

$$|f_n| = \varkappa.$$

Hence $J_2(f_n) = J_q(f_n)$, and therefore $S_2(f_n) = S_{q'}(f_n)$, so that, by Lemma 5, f_n is a monomial.

3.7. We see thus that any solution f_n of (3.17), where λ is of course now to be replaced by unity, or (what is the same thing) any *polynomial* solution of

$$J_q(f) = S_{q'}(f),$$

where $q > 2$ and the right hand side is supposed finite, is necessarily a monomial. We have now to extend this result to an arbitrary solution of (3.71).

If the Fourier constants of f are c_m, and $S_{q'}(f)$ is finite, then, by Lemma 11, f_n converges strongly to f with index q. Suppose now that f is a solution of (3.71), and write

$$g(-\theta) = |f|^{q-1} \operatorname{sgn} \bar{f}, \qquad g(\theta) \sim \sum_{-\infty}^{\infty} b_m e^{m i \theta}. \tag{3.72}$$

Then g belongs to $L^{q'}$, and

$$J_q^q(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, g(-\theta)\, d\theta.$$

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[16]) If not, it must have zeros $z_n \neq 0$ inside the circle, and $F = z^\nu \omega F^*$, where

$$\omega = \prod \frac{z - z_n}{1 - z\bar{z}_n}$$

and F^* has no zeros inside the circle. Then $|F^*| = |F|$ is constant on the circle, and so is a constant, in which case F is no polynomial.

But

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(f-f_n)g(-\theta)\,d\theta\right| \leqq J_q(f-f_n)\,J_{q'}(g) \to 0,$$

so that[17])

$$(3.36)\qquad J_q^q(f) = \lim \frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(\theta)g(-\theta)\,d\theta = \lim \sum_{-n}^{n} b_m c_m \leqq S_{q'}(f)\,S_q(g)$$

$$\leqq S_{q'}(f)\,J_{q'}(g) = J_q(f)\,J_q^{q-1}(f) = J_q^q(f).$$

All inequalities here are equalities, so that

$$(3.74)\qquad S_q(g) = J_{q'}(g);$$

and we may argue as in § 3.3 with the sum $\Sigma b_m c_m$, n, f_n, g_n being replaced now by ∞, f, g. We thus obtain

$$(3.75)\qquad c_m = \tau\,|\,b_m\,|^{q-1}\operatorname{sgn}\bar{b}_m,$$

where

$$\tau = (S_q(g))^{-q'(q-2)},$$

for all values of m. Further we have, by the argument of § 3.4,

$$\sum_{-\infty}^{\infty}|\,b_m\,|^2 = J_2^2(g) = J_{q_1}^{q_1}(f) \leqq S_{q_1'}^{q_1}(f) = \tau^{q_1}(S_Q(g))^{(q-1)q_1}$$

and

$$1 \leqq (S_Q(g))^{2(q-1)^2}(S_2(g))^{-2}(S_q(g))^{-2q(q-2)} \leqq 1,$$

by Lemma 1. This being an equality, we can apply Lemma 5; and b_m must be zero from a certain value of $|\,m\,|$ onwards. Hence g, and therefore, by (3.75), f is a polynomial, and therefore a monomial.

3.8. Finally, suppose that g is any solution of the other equality (3.74), and that

$$g(\theta) \sim \sum_{-\infty}^{\infty} b_m e^{mi\theta}, \qquad h_n(-\theta) = \sum_{-n}^{n}|\,b_m\,|^{q-1}\operatorname{sgn}\bar{b}_m\,e^{mi\theta}.$$

Then, if $0 < \nu < n$, we have

$$J_q(h_n - h_\nu) \leqq S_{q'}(h_n - h_\nu) = \Big(\sum_{\nu<|m|\leqq n}|\,b_m\,|^q\Big)^{\frac{1}{q'}} < \varepsilon$$

if $\nu > n_0(\varepsilon)$, so that $h_n(-\theta)$ tends to

$$h(-\theta) \sim \sum_{-\infty}^{\infty}|\,b_m\,|^{q-1}\operatorname{sgn}\bar{b}_m\,e^{mi\theta}$$

[17]) We repeat what are in substance the arguments of § 3.2, but with ∞ in place of n, f in place of f_n.

strongly, with index q. It follows that $J_q(h-h_n)\to 0$ and

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} g(h-h_n)\,d\theta\right| \leqq J_q(h-h_n)\,J_{q'}(g)\to 0.$$

Hence

$$S_q^q(g)=\lim\sum_{-n}^{n}|b_m|^q=\lim\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,h_n(\theta)\,d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,h(\theta)\,d\theta$$

$$\leqq J_{q'}(g)\,J_q(h)\leqq J_{q'}(g)\,S_{q'}(h)=S_q(g)\,S_q^{q-1}(g)=S_q^q(g).$$

Here once more all inequalities are equalities, so that

$$J_q(h)=S_{q'}(h).$$

Hence h, and so g, is a monomial. This completes the proof of Theorem 1.

4.

Theorems 2—4: proof of Theorem 3 when q is an even integer.

4.1. We pass to our second group of theorems, which may be stated as follows.

Theorem 2. *If $\sum|c_m|^p$ is convergent (so that $\sum|c_m|^2$ is convergent and there is an $f(\theta)$ whose Fourier coefficients are c_m) then*

$$(4.11)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^p\,|\theta|^{p-2}\,d\theta\leqq (A_1(p))^p\sum_{-\infty}^{\infty}|c_m|^p,$$

where $A_1(p)$ is a function of p only.

Theorem 3. *If $|f|^q\,|\theta|^{q-2}$ is integrable, then*

$$(4.12)\qquad \sum_{-\infty}^{\infty}|c_m|^q\leqq (A_2(q))^q\,\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^q\,|\theta|^{q-2}\,d\theta,$$

where $A_2(q)$ is a function of q only.

We may suppose that $A_1(p)$ and $A_2(q)$ in (4.11) and (4.12) have, for every p and q, their smallest possible values.

Theorem 4. *The numbers A_1 and A_2 satisfy the relations*

$$(4.13)\qquad A_1(q')=A_2(q)<A\,q,$$

where A is a constant; and the inequality is a best possible inequality apart from the constant factor.

We begin by proving Theorem 3 in the special case in which q is an even integer $2\varkappa$. Our proof will also embody a proof that

$$A_2(2\varkappa)<2A\varkappa,$$

i. e. a proof of part of Theorem 4. We state the result in the form of a lemma.

Lemma 12. *If $\varkappa$ is a positive integer then*

$$(4.14) \qquad \sum_{-\infty}^{\infty} |c_m|^{2\varkappa} \leqq (2A\varkappa)^{2\varkappa} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^{2\varkappa} |\theta|^{2\varkappa-2} \, d\theta.$$

4.2. If $f(\theta)$ is any integrable function, and

$$f \sim \sum c_m e^{mi\theta},$$

then

$$(4.21) \qquad c_m^{\varkappa} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-mi\theta} d\theta \right)^{\varkappa}$$

$$= \left(\frac{1}{2\pi} \right)^{\varkappa} \int_{\varkappa} \Pi_{\varkappa} f(x_j) e^{-mi \Sigma_{\varkappa} x_j} \Pi_{\varkappa} dx_j,$$

where

$$\Sigma_{\varkappa} x_j = x_1 + x_2 + \ldots + x_{\varkappa}, \qquad \int_{\varkappa} \ldots \Pi_{\varkappa} dx_j = \int\int\int \ldots dx_1 \, dx_2 \ldots dx_{\varkappa}$$

(so that the suffix to Σ, Π or $\int$ indicates the number of variables involved), and the range of integration in each variable is $(-\pi, \pi)$. We shall use this notation systematically in what follows, omitting the suffix where no confusion can arise.

The integral (4.21) exists as a multiple Lebesgue integral, and the range of integration in any variable may be replaced by any congruent range. Bearing this in mind, and replacing $x_{\varkappa}$ by $t - x_1 - \ldots - x_{\varkappa-1}$, we obtain

$$(4.22) \quad c_m^{\varkappa} = \left(\frac{1}{2\pi} \right)^{\varkappa} \int_{\varkappa-1} \Pi_{\varkappa-1} f(x_j) \, \Pi_{\varkappa-1} dx_j \int f\left(t - \sum_{\varkappa-1} x_j \right) e^{-mit} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) e^{-mit} dt,$$

where

$$(4.23) \qquad F(t) = \left(\frac{1}{2\pi} \right)^{\varkappa-1} \int_{\varkappa-1} \Pi f(x_j) f(t - \Sigma x_j) \, \Pi dx_j.$$

The function $F(t)$ is defined for almost all values of t, is integrable, und has $c_m^{\varkappa}$ as its Fourier coefficient. And if also the square of $F(t)$ is integrable, then

$$\sum_{-\infty}^{\infty} |c_m|^{2\varkappa} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(t)|^2 dt.$$

If then we write

$$(4.24)\qquad 1-\frac{1}{\varkappa}=\alpha,\ |f|\,|\theta|^{\alpha}=\varphi,$$

we have

$$(4.25)\qquad |F(t)|\leqq\Phi(t)=\left(\frac{1}{2\pi}\right)^{\varkappa-1}\int\limits_{\varkappa-1}\prod\frac{\varphi(x_j)}{|x_j|^{\alpha}}\,\frac{\varphi(t-\Sigma x_j)}{|t-\Sigma x_j|^{\alpha}}\prod dx_j.$$

Thus Lemma 12 will be a corollary of

Lemma 13. *If* φ *is a positive function of* $L^{2\varkappa}$ *and* Φ *is defined by* (4.25), *then*

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\Phi^2(t)\,dt\leqq(2A\varkappa)^{2\varkappa}\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\varphi^{2\varkappa}(\theta)\,d\theta.$$

4.3. We shall in fact deduce Lemma 13 from

Lemma 14. *If* $\varphi(x)$ *belongs to* $L^{2\varkappa}$, *in the interval* $(-\infty,\infty)$, *and*

$$(4.31)\qquad \Phi(t)=\int\limits_{\varkappa-1}\prod\frac{\varphi(x_j)}{|x_j|^{\alpha}}\,\frac{\varphi(t-\Sigma x_j)}{|t-\Sigma x_j|^{\alpha}}\prod dx_j,$$

where the integrations are over all real values of the variables, then

$$(4.32)\qquad \int\limits_{-\infty}^{\infty}\Phi^2(t)\,dt\leqq(2A\varkappa)^{2\varkappa}\int\limits_{-\infty}^{\infty}\varphi^{2\varkappa}(x)\,dx.$$

We suppose in what follows that all integrations are over $(-\infty,\infty)$, and we write

$$(4.33)\qquad \int\limits_{-\infty}^{\infty}\varphi^{2\varkappa}(x)\,dx=\Omega.$$

We require some preliminary lemmas concerning the values of certain definite integrals, of a type analogous to that of the classical integrals of Dirichlet.

Lemma 15. *If* $0<a<b<1$ *then*

$$(4.34)\qquad \psi(a,b)=\int\limits_{-\infty}^{\infty}|w|^{a-1}|1+w|^{-b}\,dw=\frac{\chi(a)\,\chi(b-a)}{\chi(b)},$$

where

$$(4.35)\qquad \chi(a)=2\,\Gamma(a)\cos\tfrac{1}{2}a\pi.$$

We have in fact

$$\psi(a,b)=\int\limits_{-\infty}^{-1}+\int\limits_{-1}^{0}+\int\limits_{0}^{\infty}=\int\limits_{0}^{\infty}\frac{x^{-b}\,dx}{(1+x)^{1-a}}+\int\limits_{0}^{1}x^{a-1}(1-x)^{-b}\,dx+\int\limits_{0}^{\infty}\frac{x^{a-1}\,dx}{(1+x)^{b}}.$$

and the result follows from classical formulae.

Lemma 16. *If*

$$(4.36)\qquad 0 < s a < b < 1$$

then

$$(4.37)\quad I_s(a,b) = \int\limits_s \Pi_s |x_j|^{a-1} \Big|1 + \sum_s x_j\Big|^{-b} \Pi_s \, d x_j = \frac{(\chi(a))^s \chi(b-sa)}{\chi(b)}.$$

Lemma 16 reduces to Lemma 15 when $s = 1$. We calculate the integral I_s by successive reduction to a product of simple integrals, the convergence of which will secure that of the multiple integral.

We put

$$x_s = (1 + \sum_{s-1} x_j)\, v,$$

so that v moves over $(-\infty, \infty)$ in the same sense as x_s, or in the opposite sense, according to the sign of $1 + \sum\limits_{s-1} x_j$; and we find without difficulty that

$$I_s(a,b) = \int |v|^{a-1} |1+v|^{-b} \, dv \int\limits_{s-1} \Pi |x_j|^{a-1} \,|\, 1 + \sum x_j|^{a-b} \, \Pi \, d x_j$$
$$= \psi(a,b)\, I_{s-1}(a, b-a).$$

Repeating the argument, and observing that

$$I_1(a, b - sa + a) = \psi(a, b - sa + a),$$

we obtain

$$I_s(a,b) = \prod_{\nu=0}^{s-1} \psi(a, b - \nu a) = \prod_{\nu=0}^{s-1} \frac{\chi(a)\,\chi(b-\nu a - a)}{\chi(b-\nu a)} = \frac{(\chi(a))^s \chi(b - sa)}{\chi(b)}.$$

The conditions for convergence are plainly that

$$0 < a < b - \nu a < 1 \qquad (\nu = 0, 1, 2, \ldots, s-1),$$

or $0 < sa < b < 1$.

4.4. We can now prove Lemma 14. We suppose for the moment that $\varkappa > 2$.

In order that $\Phi(t)$ should belong to L^2 in $(-\infty, \infty)$, and that

$$(4.41)\qquad \int_{-\infty}^{\infty} \Phi^2(t)\, dt \leqq J,$$

it is necessary and sufficient that

$$(4.42)\qquad \int_{-\infty}^{\infty} \Phi(t)\, h(t)\, dt \leqq \sqrt{HJ}$$

for every positive $h(t)$ for which

$$(4.43)\qquad \int_{-\infty}^{\infty} h^2(t)\, dt \leqq H$$

We assume (4.43) and prove (4.42) by subjecting the integral to a series of transformations. The equalities or inequalities which occur in this series of transformations are all to be interpreted as stating that, if the right hand side exists, then the left hand side also exists and is equal to or less than the right hand side.

With this understanding, we have

$$(4.44)\qquad M=\int_{-\infty}^{\infty}\Phi h\,dt=\int h(t)\,dt\int_{\varkappa-1}\prod\frac{\varphi(x_j)}{|x_j|^{\alpha}}\frac{\varphi(t-\Sigma x_j)}{|t-\Sigma x_j|^{\alpha}}\prod dx_j$$

$$=\int_{\varkappa-1}\prod\frac{\varphi(x_j)}{|x_j|^{\alpha}}\prod dx_j\int h(t)\varphi(t-\Sigma x_j)\frac{dt}{|t-\Sigma x_j|^{\alpha}}.$$

We now write

$$t=(1+w)\Sigma x_j,$$

so that w describes $(-\infty,\infty)$ in the same sense as t, or in the opposite sense, according as Σx_j is positive or negative; and we find

$$(4.45)\quad M=\int_{\varkappa-1}\prod\frac{\varphi(x_j)}{|x_j|^{\alpha}}|\Sigma x_j|^{1-\alpha}\prod dx_j\int\varphi(w\Sigma x_j)\,h((1+w)\Sigma x_j)\frac{dw}{|w|^{\alpha}}$$

$$=\int\frac{N(w)}{|w|^{\alpha}}dw,$$

where

$$(4.451)\quad N(w)=\int_{\varkappa-1}\prod\frac{\varphi(x_j)}{|x_j|^{\alpha}}|\Sigma x_j|^{1-\alpha}\varphi(w\Sigma x_j)\,h((1+w)\Sigma x_j)\prod dx_j.$$

We now put

$$x_j=y_j x_{\varkappa-1}\qquad (j=1,2,\ldots,\varkappa-2).$$

Observing that y_j describes $(-\infty,\infty)$ in the same sense as x_j, or in the opposite sense, according as $x_{\varkappa-1}$ is positive or negative, we obtain

$$(4.46)\qquad N(w)=\int\varphi(x_{\varkappa-1})\,dx_{\varkappa-1}\int_{\varkappa-2}\prod|y_j|^{-\alpha}|1+\Sigma y_j|^{1-\alpha}\prod\varphi(y_j x_{\varkappa-1})$$

$$\varphi(w x_{\varkappa-1}(1+\Sigma y_j))\,h((1+w)x_{\varkappa-1}(1+\Sigma y_j))\prod dy_j$$

$$=\int_{\varkappa-2}\prod|y_j|^{-\alpha}|1+\Sigma y_j|^{1-\alpha}P\prod dy_j,$$

where

$$(4.461)\qquad P=\int\varphi(x_{\varkappa-1})\prod\varphi(y_j x_{\varkappa-1})\,\varphi(w x_{\varkappa-1}(1+\Sigma y_j))$$

$$\times h((1+w)x_{\varkappa-1}(1+\Sigma y_j))\,dx_{\varkappa-1},$$

and the Π's and Σ's refer to the $\varkappa - 2$ y's. But

$$P \leqq \left(\int \varphi^{2\varkappa}(x_{\varkappa-1})\,dx_{\varkappa-1}\right)^{\frac{1}{2\varkappa}} \Pi\left(\int \varphi^{2\varkappa}(y_j x_{\varkappa-1})\,dx_{\varkappa-1}\right)^{\frac{1}{2\varkappa}}$$

$$\left(\int \varphi^{2\varkappa}(w x_{\varkappa-1}(1+\Sigma y_j))\,dx_{\varkappa-1}\right)^{\frac{1}{2\varkappa}} \left(\int h^2((1+w)x_{\varkappa-1}(1+\Sigma y_j))\,dx_{\varkappa-1}\right)^{\frac{1}{2}}$$

$$= |w|^{-\frac{1}{2\varkappa}} |1+w|^{-\frac{1}{2}} \Pi |y_j|^{-\frac{1}{2\varkappa}} |1+\Sigma y_j|^{-\frac{1}{2}-\frac{1}{2\varkappa}} \sqrt{H\Omega},$$

by Hölder's inequality. Hence

$$N(w) \leqq \sqrt{H\Omega}\,|w|^{-\frac{1}{2\varkappa}} |1+w|^{-\frac{1}{2}} \int\limits_{\varkappa-2} \Pi |y_j|^{\frac{1}{2\varkappa}-1} |1+\Sigma y_j|^{\frac{1}{2\varkappa}-\frac{1}{2}} \Pi dy_j, \tag{4.47}$$

and so

$$M \leqq \sqrt{H\Omega} \int |w|^{\frac{1}{2\varkappa}-1} |1+w|^{-\frac{1}{2}} dw \tag{4.48}$$

$$\times \int\limits_{\varkappa-2} \Pi |y_j|^{\frac{1}{2\varkappa}-1} |1+\Sigma y_j|^{\frac{1}{2\varkappa}-\frac{1}{2}} \Pi dy_j$$

$$= \sqrt{H\Omega}\,\psi\left(\frac{1}{2\varkappa}, \frac{1}{2}\right)\left(\chi\left(\frac{1}{2\varkappa}\right)\right)^{\varkappa-2}\left(\chi\left(\frac{1}{2}-\frac{1}{2\varkappa}\right)\right)^{-1} = \frac{\left(\chi\left(\frac{1}{2\varkappa}\right)\right)^{\varkappa}}{\chi\left(\frac{1}{2}\right)}\sqrt{H\Omega}$$

by Lemmas 15 and 16. It is to be observed that, in applying Lemma 16, we must take

$$s = \varkappa - 2, \quad a = \frac{1}{2\varkappa}, \quad b = \frac{1}{2} - \frac{1}{2\varkappa},$$

so that the inequalities (4. 36) become

$$0 < \frac{1}{2} - \frac{1}{\varkappa} < \frac{1}{2} - \frac{1}{2\varkappa} < 1,$$

and are satisfied because $\varkappa > 2$. This completes the proof of Lemma 14.

4. 5. We have thus

$$M = \int_{-\infty}^{\infty} \Phi h\,dt \leqq C\sqrt{H\Omega}, \tag{4.51}$$

where

$$C = \frac{\left(\chi\left(\frac{1}{2\varkappa}\right)\right)^{\varkappa}}{\chi\left(\frac{1}{2}\right)}; \tag{4.511}$$

and we have accordingly proved (4. 41), with

$$J = C^2 \Omega.$$

That is to say we have proved that

$$(4.52)\quad \int_{-\infty}^{\infty} \Phi^2\,dt \leqq C^2 \int_{-\infty}^{\infty} \varphi^{2\varkappa}\,dx = \frac{1}{2\pi}\left(2\,\Gamma\left(\frac{1}{2\varkappa}\right)\cos\frac{\pi}{4\varkappa}\right)^{2\varkappa} \int_{-\infty}^{\infty} \varphi^{2\varkappa}\,dx\,,$$

which plainly involves (4. 32).

We have supposed that $\varkappa > 2$, but the argument is naturally valid, with appropriate simplifications, when $\varkappa = 2$. In this case

$$M = \int h(t)\,dt \int \frac{\varphi(x)\,\varphi(t-x)}{\sqrt{|x|\,|t-x|}}\,dx = \int \frac{\varphi(x)}{\sqrt{|x|}}\,dx \int \frac{\varphi(t-x)}{\sqrt{|t-x|}}\,h(t)\,dt = \int \frac{N(w)}{\sqrt{|w|}}\,dw$$

where

$$\begin{aligned} N(w) &= \int \varphi(x)\,\varphi(wx)\,h(x+wx)\,dx \\ &\leqq \left(\int \varphi^4(x)\,dx\right)^{\frac{1}{4}} \left(\int \varphi^4(wx)\,dx\right)^{\frac{1}{4}} \left(\int h^2(x+wx)\,dx\right)^{\frac{1}{2}} \\ &= |w|^{-\frac{1}{4}}\,|1+w|^{-\frac{1}{2}}\sqrt{H\Omega}\,, \end{aligned}$$

$$M \leqq \sqrt{H\Omega} \int_{-\infty}^{\infty} \frac{dw}{|w|^{\frac{3}{4}}\,|1+w|^{\frac{1}{2}}} = \frac{\left(\chi\left(\frac{1}{4}\right)\right)^2}{\chi\left(\frac{1}{2}\right)}\sqrt{H\Omega}\,;$$

and the proof then proceeds as in the general case.

4. 6. We do not propose now to attempt to determine any upper bound for $A_2(q)$ more precise than that stated in Theorem 4. It is however interesting to observe that the constant occurring in (4. 52) is in fact the best possible constant in this particular inequality, so that nothing has been lost in our elaborate transformations. We shall content ourselves here with verifying the truth of our statement when $\varkappa = 2$, $q = 4$; there is no particular difficulty in extending the proof.

We take

$$(4.61)\qquad \varphi(t) = |t|^{-\beta}\ (|t| \geqq 1), \qquad \varphi(t) = 0\ \ (|t| < 1),$$

where $\frac{1}{4} < \beta < \frac{1}{2}$, so that

$$(4.62)\qquad \Omega = \int_{-\infty}^{\infty} \varphi^4\,dt = 2\int_{1}^{\infty} \frac{dt}{t^{4\beta}} = \frac{2}{4\beta - 1}.$$

Then

$$(4.63)\quad \begin{aligned} \Phi(t) &= \int_{-\infty}^{\infty} \frac{\varphi(x)\,\varphi(t-x)}{\sqrt{|x|\,|t-x|}}\,dx \geqq \int_{-\infty}^{\infty} |x|^{-\beta-\frac{1}{2}}\,|t-x|^{-\beta-\frac{1}{2}}\,dx - \sigma(t) \\ &= \omega(t) - \sigma(t) \end{aligned}$$

say, where $\sigma(t)$ is the sum of two integrals, with the same integrand as that of $\omega(t)$, over intervals (sometimes overlapping) of length 2.

Suppose that $|t| \geqq 3$. Then

$$\omega(t) = |t|^{-2\beta}\psi\left(\frac{1}{2}-\beta, \frac{1}{2}+\beta\right), \tag{4. 64}$$

$$\sigma(t) = \int_{-1}^{1}|x|^{-\beta-\frac{1}{2}}|t-x|^{-\beta-\frac{1}{2}}\,dx \tag{4. 65}$$

$$+\int_{t-1}^{t+1}|x|^{-\beta-\frac{1}{2}}|t-x|^{-\beta-\frac{1}{2}}\,dx < C|t|^{-\beta-\frac{1}{2}},$$

where C is independent of t and β. Now

$$\int_{-\infty}^{\infty}\Phi^2\,dt \geqq \int_{|t|\geqq 3}\Phi^2\,dt \geqq \int\omega^2\,dt - 2\int\omega\sigma\,dt - \int\sigma^2\,dt \tag{4. 66}$$

$$\geqq \int\omega^2\,dt - 2\left(\int\omega^2\,dt\right)^{\frac{1}{2}}\left(\int\sigma^2\,dt\right)^{\frac{1}{2}} - \int\sigma^2\,dt,$$

all the integrations being over $|t| \geqq 3$. It follows from (4. 64) and (4. 65) that

$$\int\omega^2\,dt \sim 2\psi^2\left(\frac{1}{4}, \frac{3}{4}\right)\int_{3}^{\infty}t^{-4\beta}\,dt \sim \frac{2}{4\beta-1}\psi^2\left(\frac{1}{4}, \frac{3}{4}\right) \tag{4. 67}$$

when $\beta \to \frac{1}{4}$, while $\int\sigma^2\,dt$ is bounded. And then it follows from (4. 63) and (4. 67) that

$$\int_{-\infty}^{\infty}\Phi^2\,dt > K\int_{-\infty}^{\infty}\varphi^4\,dt,$$

where K is any number less than

$$\psi^2\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{1}{2}\left(\Gamma\left(\frac{1}{4}\right)\right)^4\left(\sqrt{2}+1\right)^2 = \frac{\left(\chi\left(\frac{1}{4}\right)\right)^4}{\left(\chi\left(\frac{1}{2}\right)\right)^2},$$

if β is sufficiently near to $\frac{1}{4}$. Thus the constant of (4. 52) is a best possible constant, at any rate when $\varkappa = 2$.

4. 7. It is easy to deduce Lemma 13 from Lemma 14. Suppose that φ has the period 2π inside, and is zero outside, the interval $(-\varkappa\pi, \varkappa\pi)$, that Φ is now defined as in Lemma 13, and Φ^* as is Φ in Lemma 14. In Φ, the argument of every φ lies in $(-\varkappa\pi, \varkappa\pi)$, and so

$$\Phi \leqq (2\pi)^{-\varkappa+1}\,\Phi^* \leqq \Phi^*,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi^2\,dt \leqq \int_{-\infty}^{\infty}\Phi^{*2}\,dt \leqq (2A\varkappa)^{2\varkappa}\int_{-\infty}^{\infty}\varphi^{2\varkappa}\,dt,$$

$$= (2A\varkappa)^{2\varkappa}\int_{-\varkappa\pi}^{\varkappa\pi}\varphi^{2\varkappa}\,dt \leqq \varkappa(2A\varkappa)^{2\varkappa}\int_{-\pi}^{\pi}\varphi^{2\varkappa}\,dt \leqq (2A\varkappa)^{2\varkappa}\int_{-\pi}^{\pi}\varphi^{2\varkappa}\,dt,$$

with an appropriate change of A.

We have thus proved Theorem 3 when $q = 2\varkappa$, and proved also that $A_2(2\varkappa) < 2A\varkappa$, that is to say part of Theorem 4. We shall require all that we have proved in § 5.

5.

Proof of Theorems 2—4.

5.1. We write

$$(5.11) \qquad I_r = I_r(f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^r |\theta|^{r-2} d\theta\right)^{\frac{1}{r}},$$

and we define λ and λ' by

$$(5.12) \qquad \lambda = \lambda(n) = \lambda_q(n) = \overline{\text{bound}}\, \frac{S_q(f_n)}{I_q(f)},$$

$$(5.13) \qquad \lambda' = \lambda'(n) = \lambda_p'(n) = \overline{\text{bound}}\, \frac{I_p(f_n)}{S_p(f_n)},$$

for variation of f and f_n respectively. We may suppose $q > 2$, $p < 2$. The analysis of § 4 shows that, when $q = 2\varkappa$, $\lambda_q(n)$ exists and is less than Aq for all values of n. We have to extend this result to general values of q, and to prove corresponding results for $\lambda_p'(n)$.

We begin by proving the existence of λ and λ'. Suppose first that $S_q(f_n) = 1$. Then

$$|c_m| \geqq (2n+1)^{-\frac{1}{q}}$$

for some m of $(-n, n)$, and

$$(2n+1)^{-\frac{1}{q}} \leqq \operatorname{Max} |c_m| \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|\, d\theta.$$

But

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|\, |\theta|^{1-\frac{2}{q}} |\theta|^{1-\frac{2}{q'}} d\theta \leqq I_q(f) \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta|^{q'-2} d\theta\right)^{\frac{1}{q'}}$$

$$= \pi^{1-\frac{2}{q'}} (q'-1)^{\frac{1}{q'}} I_q(f) = C_q I_q(f),$$

say; so that

$$I_q(f) \geqq (2n+1)^{-\frac{1}{q}} C_q^{-1}, \qquad \frac{S_q(f_n)}{I_q(f)} \leqq (2n+1)^{\frac{1}{q}} C_q,$$

which proves the existence of λ.

Next, suppose that $I_p(f_n) = 1$. Then

$$(2n+1)\operatorname{Max}|c_m| \geqq \operatorname{Max}|f_n| \geqq \frac{I_p(f_n)}{I_p(1)} = \pi^{1-\frac{2}{p}}(p-1)^{-\frac{1}{p}},$$

$$\frac{I_p(f_n)}{S_p(f_n)} \leqq \frac{1}{\operatorname{Max}|c_m|} \leqq (2n+1)\pi^{1-\frac{2}{p}}(p-1)^{-\frac{1}{p}},$$

which proves the existence of λ'.

We have defined λ_r for $r \geqq 2$ and λ_r' for $r \leqq 2$ only. We now define them for $r < 2$ and $r > 2$ respectively by

(5.14) $$\lambda_{q'}(n) = \lambda_q(n), \quad \lambda'_{p'}(n) = \lambda'_p(n).$$

5. 2. Lemma 17. *The bounds λ_q and λ'_q are equal, for every q and n:*

(5.21) $$\lambda_q(n) = \lambda'_q(n).$$

(I) Suppose that $f \sim \sum c_m e^{mi\theta}$ is any integrable function for which I_q is finite, and let

(5.22) $$h_n(-\theta) = \sum_{-n}^{n} |c_m|^{q-1} \operatorname{sgn} \bar{c}_m e^{mi\theta}.$$

Then

$$S_q^q(f_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) h_n(\theta)\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f|\theta|^{1-\frac{2}{q}} \cdot h_n|\theta|^{1-\frac{2}{q'}}\, d\theta$$

$$\leqq I_q(f) I_{q'}(h_n) \leqq \lambda'_{q'}(n) I_q(f) S_{q'}(h_n) \leqq \lambda'_{q'}(n) I_q(f) S_q^{q-1}(f_n),$$

$$\frac{S_q(f_n)}{I_q(f)} \leqq \lambda'_{q'}(n).$$

Since this is true for all f, we must have

(5.23) $$\lambda_q(n) \leqq \lambda'_{q'}(n) = \lambda'_q(n).$$

(II) Suppose that

$$\varphi_n = \sum_{-n}^{n} \gamma_m e^{mi\theta}$$

is any polynomial, and let

$$\psi(-\theta) = |\varphi_n|^{p-1} \operatorname{sgn} \bar{\varphi}_n |\theta|^{p-2}, \quad \psi(\theta) \sim \sum \beta_m e^{mi\theta} \text{ [18]}.$$

Then

$$I_p^p(\varphi_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(\theta)\psi(-\theta)\, d\theta = \sum_{-n}^{n} \beta_m \gamma_m$$

$$\leqq S_p(\varphi_n) S_{p'}(\psi_n) \leqq \lambda_{p'}(n) S_p(\varphi_n) I_{p'}(\psi) = \lambda_{p'}(n) S_p(\varphi_n) I_p^{p-1}(\varphi_n),$$

$$\frac{I_p(\varphi_n)}{S_p(\varphi_n)} \leqq \lambda_{p'}(n).$$

[18]) Plainly ψ is integrable.

Since this is true for all φ_n, we have $\lambda'_p(n) \leqq \lambda_{p'}(n)$ or

$$\lambda'_q(n) = \lambda'_{q'}(n) \leqq \lambda_q(n). \tag{5.24}$$

Plainly (5.23) and (5.24) contain the lemma.

5.3. Lemma 18. *The bound λ' is attained; there is a polynomial f_n, not null, such that*

$$I_p(f_n) = \lambda'_p(n)\, S_p(f_n). \tag{5.31}$$

The proof is similar to that of Lemma 7, and we need not repeat it.

Lemma 19. *If f_n is the polynomial of Lemma 18, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_n|^{p-1} \operatorname{sgn} \bar{f}_n |\theta|^{p-2} e^{mi\theta}\, d\theta = \lambda_p^p(n)\, |c_m|^{p-1} \operatorname{sgn} \bar{c}_m \tag{5.32}$$

for $|m| \leqq n$.

This lemma embodies the result of the process of differentiation used by Hausdorff and F. Riesz in their proofs of Hausdorff's theorem. We are not able to evade this process here as we were in § 3.

The polynomial f_n gives the maximum of $I_p^p(f_n)$ for given $S_p^p(f_n)$. Let

$$h_n = \sum_{-n}^{n} d_m e^{mi\theta}$$

be any other polynomial of the same degree[19]. Then

$$Q(t) = \frac{I_p^p(f_n + t h_n)}{S_p^p(f_n + t h_n)}$$

attains its maximum, for real t, when $t = 0$, and

$$Q(0) = \lambda_p^p(n), \qquad Q'(0) = 0,$$

so that, for $t = 0$,

$$\frac{(I_p^p)'}{(S_p^p)'} = \frac{I_p^p}{S_p^p} = Q(0) = \lambda_p^p(n). \tag{5.33}$$

A simple calculation shows that

$$\frac{d}{dt} I_p^p(f_n + t h_n) = \Re \frac{p}{2\pi} \int_{-\pi}^{\pi} |f_n|^{p-1} \operatorname{sgn} \bar{f}_n \,.\, h_n |\theta|^{p-2}\, d\theta,$$

$$\frac{d}{dt} S_p^p(f_n + t h_n) = \Re\, p \sum_{-n}^{n} |c_m|^{p-1} \operatorname{sgn} \bar{c}_m \,.\, d_m,$$

---

[19]) We follow Riesz's argument, which is simpler than Hausdorff's.

for $t = 0$, so that (5.33) gives

$$(5.34)\quad \frac{1}{2\pi}\Re\int_{-\pi}^{\pi}|f_n|^{p-1}\operatorname{sgn}\bar{f}_n\cdot h_n|\theta|^{p-2}\,d\theta = \lambda_p^p(n)\Re\sum_{-n}^{n}|c_m|^{p-1}\operatorname{sgn}\bar{c}_m\cdot d_m.$$

Take now in particular

$$h_n = e^{i\alpha+mi\theta}\qquad(|m|\leqq n),$$

where α is real. Then

$$(5.35)\quad \frac{1}{2\pi}\Re\left(e^{i\alpha}\int_{-\pi}^{\pi}|f_n|^{p-1}\operatorname{sgn}\bar{f}_n\cdot|\theta|^{p-2}e^{mi\theta}\,d\theta\right)$$
$$= \lambda_p^p(n)\Re(e^{i\alpha}|c_m|^{p-1}\operatorname{sgn}\bar{c}_m).$$

This being true for all real α, we may omit the $\Re$ and the $e^{i\alpha}$ on each side, so obtaining (5.32).

5.4. Lemma 20. *If*

$$(5.41)\qquad q(p-1)\leqq 2$$

then

$$(5.42)\qquad \lambda_p(n)\leqq(\lambda_{q(p-1)}(n))^{\frac{p-1}{p}}(\lambda_q(n))^{\frac{1}{p}}.$$

This lemma contains the kernel of our proof. It follows from Lemma 19 that, if f_n is the polynomial of Lemma 18 and

$$(5.43)\qquad g(-\theta) = |f_n|^{p-1}\operatorname{sgn}\bar{f}_n|\theta|^{p-2},$$

$$(5.44)\qquad g(\theta)\sim\sum b_m e^{mi\theta},$$

then

$$(5.45)\qquad b_m = \lambda_p^p(n)|c_m|^{p-1}\operatorname{sgn}\bar{c}_m\qquad(|m|\leqq n).$$

Using the inequality $S_q(g_n)\leqq\lambda_q(n)I_q(g)$, and observing that

$$q(p-2)+q-2 = q(p-1)-2,$$

we obtain

$$\lambda_p^{pq}(n)S_{q(p-1)}^{q(p-1)}(f_n)\leqq\lambda_q^q(n)I_{q(p-1)}^{q(p-1)}(f_n),$$

or

$$\lambda_p^p(n)\leqq\lambda_q(n)\left(\frac{I_{q(p-1)}(f_n)}{S_{q(p-1)}(f_n)}\right)^{p-1}\leqq\lambda_q(n)\lambda_{q(p-1)}^{p-1}(n),$$

which is (5.42).

5.5. Lemma 21. *If* $2<\alpha<\beta$ *and*

$$(5.51)\qquad \lambda_\alpha(n)\leqq l,\quad \lambda_\beta(n)\leqq l,$$

then there is a γ *such that* $\alpha<\gamma<\beta$ *and*

$$(5.52)\qquad \lambda_\gamma(n)\leqq l.$$

We define q and p by

$$q = \beta, \quad (p-1)q = \alpha',$$

so that

$$p = 1 + \frac{\alpha'}{\beta},$$

and take

$$\gamma = p'.$$

Then $q > 2$, $(p-1)q < 2$, and so $p < 2$. Also γ will lie between α and β if p lies between β' and α'. But

$$p = 1 + \frac{\alpha'}{\beta} < \alpha'$$

if $\alpha' > \beta'$, which is true; and

$$\beta' < p = 1 + \frac{\alpha'}{\beta}$$

if $\beta' = (\beta' - 1)\beta < \alpha'$, which is also true. Hence all the conditions of Lemmas 20 and 21 are satisfied and

$$\lambda_\gamma^p(n) = \lambda_p^p(n) \leqq \lambda_{\alpha'}^{p-1}(n)\lambda_\beta(n) = \lambda_{\alpha'}^{p-1}(n)\lambda_\beta(n) \leqq l^p.$$

Lemma 22. *If $q_N \to q$ when $N \to \infty$, then*

(5.53) $$\lambda_q(n) \leqq \varliminf_{N\to\infty} \lambda_{q_N}(n).$$

We have, by the definition of λ', for any fixed f_n,

$$I_{q'_N}(f_n) \leqq \lambda_{q'_N}(n)\, S_{q'_N}(f_n)$$

and so

$$\frac{I_{q'}(f_n)}{S_{q'}(f_n)} = \lim_{N\to\infty} \frac{I_{q'_N}(f_n)}{S_{q'_N}(f_n)} \leqq \varliminf_{N\to\infty} \lambda_{q'_N}(n),$$

$$\lambda_q(n) = \lambda_{q'}(n) \leqq \varliminf_{N\to\infty} \lambda_{q'_N}(n) = \varliminf_{N\to\infty} \lambda_{q_N}(n).$$

5.6. Lemma 23. *There is a constant A such that*

(5.61) $$\lambda_q(n) = \lambda'_q(n) < Aq$$

when $q = 2\varkappa$ and $\varkappa$ is an integer.

This is included in Lemma 12, since

$$S_{2\varkappa}^{2\varkappa}(f_n) \leqq \sum_{-\infty}^{\infty} |c_m|^{2\varkappa} \leqq (2A\varkappa)^{2\varkappa} I_{2\varkappa}^{2\varkappa}(f).$$

Lemma 24. *The inequality* (5.61) *is true for all q and n.*

Suppose that $2\varkappa \leqq q \leqq 2\varkappa + 2$, and let $\mu_\varkappa$ be the upper bound (for all n) of $\lambda_{2\varkappa}(n)$, or the upper bound of $\lambda_{2\varkappa+2}(n)$, whichever is greater. There is plainly an A such that $\mu_\varkappa < 2A\varkappa$.

We fix n, and consider the set of points, in the interval

$$2\varkappa \leqq q \leqq 2\varkappa + 2,$$

for which $\lambda_q(n) \leqq \mu_\varkappa$. This set is compact, by Lemma 21, and closed, by Lemma 22; and therefore contains all points of the interval. Hence $\lambda_q(n) < Aq$ for all q and n.

5. 7. Theorem 3 is an immediate deduction from Lemma 24. We have in fact

$$\sum_{-n}^{n}{}' |c_m|^q \leqq \lambda_q^q(n) \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^q |\theta|^{q-2} d\theta,$$

and so

$$\sum_{-\infty}^{\infty}{}' |c_m|^q \leqq (A_2(q))^q \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^q |\theta|^{q-2} d\theta,$$

where

$$A_2(q) = \operatorname*{Max}_{(n)} \lambda_q(n) < Aq.$$

We have also proved the inequality $A_2(q) < Aq$ of Theorem 4.

To prove Theorem 2, we suppose that the c's are given so that $\sum |c_m|^p$ is convergent, and write

$$\varphi_n = f_n |\theta|^{1-\frac{2}{p}}.$$

If then $0 < \nu < n$, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi_n - \varphi_\nu|^p \, d\theta \leqq A_p \sum_{\nu < |m| \leqq n}{}' |c_m|^p,$$

where A_p is the upper bound of

$$(\lambda_p'(n))^p = (\lambda_{p'}'(n))^p < (Ap')^p$$

for variation of n. Hence

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi_n - \varphi_\nu|^p \, d\theta < \varepsilon$$

if $\nu > n_0(\varepsilon)$, and φ_n converges strongly, with index p, to a function

$$\varphi = f |\theta|^{1-\frac{2}{p}}.$$

Since

$$\frac{f_n}{\varphi_n} = |\theta|^{\frac{2}{p}-1} = \frac{f}{\varphi}$$

is bounded and independent of n, f_n converges strongly to f, and $c_m(f) = \lim c_m(f_n) = c_m$, or

$$f \sim \sum c_m e^{mi\theta}.$$

Finally

$$I_p(f)=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi|^p\,d\theta\right)^{\frac{1}{p}}=\lim\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi_n|^p\,d\theta\right)^{\frac{1}{p}}$$

$$\leqq \lim A_1(p)\,S_p(f_n)=A_1(p)\,S_p(f),$$

where

$$A_1(p)=\operatorname*{Max}_{(n)}\lambda'_p(n)=\operatorname*{Max}_{(n)}\lambda_{p'}(n)=A_2(p')<Ap'.$$

We have thus proved Theorems 2 to 4, except for the last clause of Theorem 4.

If now

$$f=|\theta|^{\frac{2}{q}-1},$$

we have $I_q(f)=1$. But

$$c_0=\frac{1}{\pi}\int_0^{\pi}\theta^{\frac{2}{q}-1}\,d\theta=\frac{1}{2}q\pi^{\frac{2}{q}-1}>\frac{q}{2\pi},\qquad \lambda_q(n)\geqq\frac{S_q(f_n)}{I_q(f)}\geqq c_0>\frac{q}{2\pi},$$

$$A_2(q)>\frac{q}{2\pi}.$$

This completes the proof of Theorem 4.

6.

Theorems 5—7: proof of Theorem 6 when q is an even integer.

6.1. Our third group of theorems may be derived from the second group by a formal interchange of $\int$ and $\sum$, $f(\theta)$ and c_m, $|\theta|$ and $|m|+1$.

Theorem 5. *If $f(\theta)$ belongs to L^p, then*

$$(6.11)\qquad \sum_{-\infty}^{\infty}|c_m|^p\,(|m|+1)^{p-2}\leqq(A_3(p))^p\,\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^p\,d\theta.$$

Theorem 6. *If $\sum|c_m|^q(|m|+1)^{q-2}$ is convergent, then there is an $f(\theta)$ of L^q whose Fourier coefficients are c_m, and*

$$(6.12)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^q\,d\theta\leqq(A_4(q))^q\sum_{-\infty}^{\infty}|c_m|^q(|m|+1)^{q-2}.$$

Theorem 7. *If $A_3(p)$ and $A_4(q)$, in Theorems 5 and 6, are defined as having their least possible values, then*

$$(6.13)\qquad A_3(q')=A_4(q)<Aq,$$

where A is a constant; and this is a best possible inequality apart from the constant factor A.

We begin by proving Theorem 6 (with part of Theorem 7) in the special case in which q is an even integer $2\varkappa$. The method which we use may be adapted to give a proof of the corresponding case of Theorem 4, but the method of § 4 is somewhat simpler and has in any case an independent interest.

6.2. We shall show first that it is sufficient to prove the theorem *with the additional hypothesis that f is a polynomial.*

Suppose in fact that we have proved (6.12) when f is a polynomial, and that the series

$$\sum |c_m|^q (|m|+1)^{q-2}$$

is convergent. If now $0 < \nu < n$ and

$$f_n = \sum_{-n}^{n} c_m e^{mi\theta},$$

we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_n - f_\nu|^q \, d\theta \leqq (A_4(q))^q \sum_{\nu < |m| \leqq n} |c_m|^q (|m|+1)^{q-2} < \varepsilon$$

if $\nu > n_0(\varepsilon)$. Hence[20] f_n converges strongly to an f of L^q, whose Fourier coefficients are c_m, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^q \, d\theta = \lim \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_n|^q \, d\theta \leqq (A_4(q))^q \sum_{-\infty}^{\infty} |c_m|^q (|m|+1)^{q-2}.$$

6.3. We show next that Theorem 6 (with $q = 2\varkappa$) and that part of Theorem 7 which asserts that $A_4(2\varkappa) < 2A\varkappa$, are corollaries of a proposition concerning series of positive terms: viz:

Lemma 25. *Suppose that $\gamma_m \geqq 0$ and that $\sum \gamma_m^{2\varkappa}$ is convergent, and that D_ν is defined by*

$$D_\nu = \sum \prod_{j=1}^{\varkappa} \left(\gamma_{m_j} (|m_j|+1)^{\frac{1}{\varkappa}-1}\right), \tag{6.31}$$

where the m's assume all integral values such that

$$\sum m_j = m_1 + m_2 + \dots + m_\varkappa = \nu. \tag{6.32}$$

Then D_ν and $\sum D_\nu^2$ are convergent, and

$$\sum_{-\infty}^{\infty} D_\nu^2 \leqq (2A\varkappa)^{2\varkappa} \sum_{-\infty}^{\infty} \gamma_m^{2\varkappa}. \tag{6.33}$$

---

[20]) Cf. § 3.5.

The lemma, if true generally, is true *a fortiori* if $\gamma_m = 0$ when $|m| \geqq n$, and this special case would be sufficient for our purpose.

Suppose now (as after § 6.2 we may) that f is a polynomial f_n, so that $c_m = 0$ if $|m| > n$. Then

$$f^{\varkappa} = \left(\sum_{-n}^{n} c_m e^{mi\theta}\right)^{\varkappa} = \sum C_\nu e^{\nu i\theta},$$

where

$$C_\nu = \sum c_{m_1} c_{m_2} \ldots c_{m_\varkappa} = \sum \Pi c_{m_j}$$

over

$$|m_j| \leqq n, \quad \sum m_j = \nu.$$

Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^{2\varkappa} d\theta = \sum_{-\infty}^{\infty} |C_\nu|^2. \tag{6.34}$$

If now we write

$$|c_m|(|m|+1)^{-1\frac{1}{\varkappa}} = \gamma_m$$

then

$$|C_\nu| \leqq \sum \Pi |c_{m_j}| = \sum \Pi\left(\gamma_{m_j}(|m_j|+1)^{\frac{1}{\varkappa}-1}\right) = D_\nu. \tag{6.35}$$

From (6.33), (6.34), and (6.35) it follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^{2\varkappa} d\theta \leqq (2A\varkappa)^{2\varkappa} \sum_{-\infty}^{\infty} \gamma_m{}^{2\varkappa} = (2A\varkappa)^{2\varkappa} \sum_{-\infty}^{\infty} |c_m|^{2\varkappa}(|m|+1)^{2\varkappa-2},$$

which proves Theorem 6, with $q = 2\varkappa$ and $A_4(2\varkappa) < 2A\varkappa$.

6.4. We state the proof of Lemma 25 on the hypothesis that $\varkappa > 2$. The proof when $\varkappa = 2$ is naturally simpler (though the same in principle), and the simplifications may be left to the reader.

We base the proof on a series of subsidiary lemmas. All of these lemmas suggest obvious generalisations, and some of these have an independent interest. We state them here in the specialised forms in which they are actually required.

Lemma 26. *If ξ is real and*

$$0 < sa < b < 1 \tag{6.41}$$

then

$$\int_s \Pi_s |x_j|^{a-1} |\xi - \sum_s x_j|^{-b} \, \Pi_s \, dx_j = \frac{(\chi(a))^s \chi(b - sa)}{\chi(b)} |\xi|^{sa-b}. \tag{6.42}$$

The notation is that of Lemma 16, from which Lemma 26 follows on writing

$$x_j = -\xi y_j \qquad (j = 1, 2, \ldots, s).$$

Lemma 27. *Suppose that* $s = \varkappa - 2$ *and that* c *has one or other of the two values.*

(6. 43) $$c = 1 - \frac{1}{2(\varkappa - 1)} + \frac{1}{4\varkappa(\varkappa - 1)},$$

(6. 44) $$c = 1 - \frac{1}{2(\varkappa - 1)}.$$

Then

(6. 45) $$\int_{\varkappa - 2} \Pi |x_j|^{-c} |\xi - \Sigma x_j|^{-c} \, \Pi \, d x_j < (A\varkappa)^{\varkappa} |\xi|^{\varkappa - 2 - (\varkappa - 1)c}.$$

In Lemma 26 take

$$s = \varkappa - 2, \quad a = 1 - c, \quad b = c.$$

It may be verified at once that

$$\frac{1}{2\varkappa} < a < \frac{1}{\varkappa}$$

with either value of c, so that $(\chi(a))^s < (A\varkappa)^{\varkappa}$. Also

$$b - s a = \frac{1}{2} + \frac{1}{4\varkappa}, \qquad b - s a = \frac{1}{2}$$

in the two cases, so that $\chi(b - s a) < A$. Finally

$$1 - \frac{1}{\varkappa} < b < 1 - \frac{1}{2\varkappa}$$

so that

$$\chi(b) = 2\,\Gamma(b) \cos \tfrac{1}{2} \pi b > \frac{A}{\varkappa}.$$

Hence

$$\frac{(\chi(a))^s \chi(b - s a)}{\chi(b)} < (A\varkappa)^{\varkappa},$$

which proves the lemma.

6. 5. We pass now from integrals to series.

Lemma 28. *Suppose that* $j = 1, 2, \ldots, \varkappa - 1$, *that* c *has either of the values* (6. 43) *and* (6. 44), *and that the summation applies to all values of the integers* $m_1, m_2, \ldots, m_{\varkappa - 1}$ *such that*

(6. 51) $$m_1 + m_2 + \ldots + m_{\varkappa - 1} = \nu.$$

Then

(6. 52) $$\Sigma \prod_{j=1}^{\varkappa - 1} (|m_j| + 1)^{-c} < (A\varkappa)^{\varkappa} (|\nu| + 1)^{\varkappa - 2 - (\varkappa - 1)c}.$$

We have

(6. 53) $$(|m_j| + 1)^{-c} \leqq \int_{m_j - \frac{1}{2}}^{m_j + \frac{1}{2}} |x_j|^{-c} \, d x_j$$

for every j and m_j. We use this inequality for $j=1, 2, \ldots, \varkappa-2$. We have also

(6. 541) $$|m_{\varkappa-1}|+1=\Big|\nu-\sum_{\varkappa-2} m_j\Big|+1 \geqq \Big|\nu+\frac{1}{2}-\sum_{\varkappa-2} x_j\Big|-\varkappa+1$$

(since no m_j differs from the associated x_j by more than $\frac{1}{2}$), and

(6. 542) $$|m_{\varkappa-1}|+1 \geqq 1.$$

Multiplying (6. 542) by $\varkappa-1$, and adding the result to (6. 541), we obtain

$$\varkappa(|m_{\varkappa-1}|+1) \geqq \Big|\nu+\frac{1}{2}-\sum_{\varkappa-2} x_j\Big| = \Big|\xi-\sum_{\varkappa-2} x_j\Big|,$$

say, or

(6. 55) $$(|m_{\varkappa-1}|+1)^{-c} \leqq \varkappa^c |\xi-\Sigma x_j|^{-c}.$$

From (6. 53) and (6. 55) we deduce

$$\prod_{\varkappa-1}(|m_j|+1)^{-c} \leqq \varkappa^c \int_{\varkappa-2} \Pi |x_j|^{-c} |\xi-\Sigma x_j|^{-c}\, \Pi d x_j,$$

where the volume of integration is the rectangle

$$m_j-\frac{1}{2} \leqq x_j \leqq m_j+\frac{1}{2} \quad (j=1, 2, \ldots, \varkappa-2).$$

Summing with respect to the m's, we obtain

(6. 56) $$\Sigma \Pi(|m_j|+1)^{-c} \leqq \varkappa^c \int_{\varkappa-2} \Pi |x_j|^{-c} |\xi-\Sigma x_j|^{-c}\, \Pi d x_j,$$

where the integration is now extended over all $(\varkappa-2)$-dimensional space. Applying Lemma 27, and observing that

$$|\xi|=\Big|\nu+\frac{1}{2}\Big| > A(|\nu|+1),$$

we obtain (6. 52).

We also require a lemma simpler than Lemma 28 but not actually included in it.

Lemma 29. *There is a constant A such that*

(6. 57) $$\sum_{-\infty}^{\infty}(|n|+1)^{-\frac{1}{2}}(|n-m|+1)^{-\frac{1}{2}-\frac{1}{4\varkappa}} < A\varkappa(|m|+1)^{-\frac{1}{4\varkappa}}.$$

In fact

$$(|n|+1)^{-\frac{1}{2}} < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |x|^{-\frac{1}{2}}\, d x$$

and

$$|n-m|+1 > A(|m|+1-x).$$

Hence the series (6. 57) is less than

$$A\int_{-\infty}^{\infty}|x|^{-\frac{1}{2}}\left||m|+1-x\right|^{-\frac{1}{2}-\frac{1}{4\varkappa}}dx$$

$$=A\frac{\chi\left(\frac{1}{2}\right)\chi\left(\frac{1}{4\varkappa}\right)}{\chi\left(\frac{1}{2}+\frac{1}{4\varkappa}\right)}(|m|+1)^{-\frac{1}{4\varkappa}}<A\varkappa(|m|+1)^{-\frac{1}{4\varkappa}}.$$

6. 6. Lemma 30. *Suppose that $\delta_n \geqq 0$, $\sum \delta_n^\varkappa$ is convergent, j has the values $1, 2, \ldots, \varkappa$, n runs through all integers, and $m_1, m_2, \ldots, m_\varkappa$ through all integers whose sum is n. Then*

$$(6.\,61)\qquad \sum_n(|n|+1)^{-\frac{1}{2}}\sum_{m_j}\Pi\left(\delta_{m_j}(|m_j|+1)^{\frac{3}{2\varkappa}-1}\right)\leqq(A\varkappa)^\varkappa\sum_n\delta_n^\varkappa.$$

We can write the series (6. 61) in the form

$$(6.\,62)\qquad S=\sum_n\sum_{m_j}U=\sum U=\sum u_1u_2\ldots u_\varkappa,$$

where

$$(6.\,631)\qquad u_j=\delta_{m_j}(|n|+1)^{-\frac{1}{2\varkappa}}(|m_j|+1)^{\frac{1}{4\varkappa^2}}\Pi'(|m_i|+1)^{-\alpha},$$

$$(6.\,632)\qquad \alpha=\frac{1}{\varkappa}-\frac{1}{2\varkappa(\varkappa-1)}+\frac{1}{4\varkappa^2(\varkappa-1)},$$

and i in Π' omits the value j. Since

$$\varkappa u_1u_2\ldots u_\varkappa\leqq u_1^\varkappa+u_2^\varkappa+\ldots+u_\varkappa^\varkappa,$$

it is sufficient to prove that

$$(6.\,64)\qquad S_j=\sum u_j^\varkappa\leqq(A\varkappa)^\varkappa\sum\delta_n^\varkappa.$$

We may write S_j in the form

$$(6.\,65)\qquad S_j=\sum_n(|n|+1)^{-\frac{1}{2}}\sum_{m_j}\delta_{m_j}^\varkappa(|m_j|+1)^{\frac{1}{4\varkappa}}\sum_{m_i}\Pi'(|m_i|+1)^{-\varkappa\alpha},$$

where i runs through the values $1, 2, \ldots, \varkappa$ other than j and

$$\sum m_i=n-m_j.$$

If we write $\varkappa\alpha=c$, then c has the value (6. 43). Hence, by Lemma 28, the innermost sum in (6. 65) is less than

$$(A\varkappa)^\varkappa(|n-m_j|+1)^{\varkappa-2-(\varkappa-1)c}=(A\varkappa)^\varkappa(|n-m_j|+1)^{-\frac{1}{2}-\frac{1}{4\varkappa}}.$$

And hence

$$S_j \leqq (A\varkappa)^{\varkappa} \sum_n (|n|+1)^{-\frac{1}{2}} \sum_{m_j} \delta_{m_j}^{\varkappa} (|m_j|+1)^{\frac{1}{4\varkappa}} (|n-m_j|+1)^{-\frac{1}{2}-\frac{1}{4\varkappa}}$$

$$= (A\varkappa)^{\varkappa} \sum_{m_j} \delta_{m_j}^{\varkappa} (|m_j|+1)^{\frac{1}{4\varkappa}} \sum_n (|n|+1)^{-\frac{1}{2}} (|n-m_j|+1)^{-\frac{1}{2}-\frac{1}{4\varkappa}}$$

$$\leqq (A\varkappa)^{\varkappa} \sum_{m_j} \delta_{m_j}^{\varkappa},$$

by Lemma 29. This proves (6.64) and so (6.61).

6.7. We can now prove Lemma 25. We have

$$(6.71) \qquad D_\nu^2 \leqq \sum \Pi\left(\gamma_{m_j}^2 (|m_j|+1)^{\frac{3}{2\varkappa}-1}\right) \sum \Pi (|m_j|+1)^{\frac{1}{2\varkappa}-1}.$$

If in Lemma 28 we replace $\varkappa$ by $\varkappa+1$, and use the second value (6.44) of c, we obtain

$$\sum \Pi (|m_j|+1)^{\frac{1}{2\varkappa}-1} < (A\varkappa)^{\varkappa} (|\nu|+1)^{-\frac{1}{2}}.$$

Hence

$$(6.72) \qquad D_\nu^2 \leqq (A\varkappa)^{\varkappa} (|\nu|+1)^{-\frac{1}{2}} \sum \Pi\left(\gamma_{m_j}^2 (|m_j|+1)^{\frac{3}{2\varkappa}-1}\right),$$

$$(6.73) \qquad \sum D_\nu^2 \leqq (A\varkappa)^{\varkappa} \sum (|\nu|+1)^{-\frac{1}{2}} \sum_{m_j} \Pi\left(\delta_{m_j} (|m_j|+1)^{\frac{3}{2\varkappa}-1}\right),$$

where $\delta_m = \gamma_m^2$ and the m's assume all integral values whose sum is ν. Hence, by Lemma 30,

$$\sum D_\nu^2 \leqq (2A\varkappa)^{2\varkappa} \sum \delta_\nu^{\varkappa} = (2A\varkappa)^{2\varkappa} \sum \gamma_\nu^{2\varkappa};$$

which is (6.33).

7.

Proof of Theorems 5—7.

7.1. We need only sketch the proof of Theorems 5—7, which from this point is almost a direct parallel to that of Theorems 2—4.

We write

$$(7.11) \qquad T_r(f_n) = \Big(\sum_{-n}^{n} |c_m|^r (|m|+1)^{r-2}\Big)^{\frac{1}{r}},$$

and define μ and μ' by

$$(7.12) \qquad \mu = \mu(n) = \mu_p(n) = \overline{\text{bound}}\, \frac{T_p(f_n)}{J_p(f)},$$

$$(7.13) \qquad \mu' = \mu'(n) = \mu'_q(n) = \overline{\text{bound}}\, \frac{J_q(f_n)}{T_q(f_n)},$$

for variation of f and f_n respectively, and

(7.14) $$\mu_{p'}(n) = \mu_p(n), \quad \mu'_{q'}(n) = \mu'_q(n).$$

Both bounds exist. For (I) if $T_p(f_n) = 1$, we have

$$|c_m|^p \geqq |c_m|^p (|m|+1)^{p-2} \geqq \frac{1}{2n+1}$$

for some m, and so

$$J_p(f) \geqq J_1(f) \geqq |c_m| \geqq (2n+1)^{-\frac{1}{p}},$$

$$\frac{T_p(f_n)}{J_p(f)} \leqq (2n+1)^{\frac{1}{p}}.$$

And (II) if $J_q(f_n) = 1$, we have

$$(2n+1)\,\mathrm{Max}\,|c_m| \geqq \mathrm{Max}\,|f_n| \geqq J_q(f_n) = 1,$$

$$T_q(f_n) \geqq \mathrm{Max}\,|c_m| \geqq \frac{1}{2n+1}, \quad \frac{J_q(f_n)}{T_q(f_n)} \leqq 2n+1.$$

It is easy to prove (as in §§ 3.1 and 5.3)

Lemma 31. *The bound $\mu'_q(n)$ is attained; there is an f_n, not null, for which*

(7.15) $$J_q(f_n) = \mu'_q(n)\, T_q(f_n).$$

7.2. Lemma 32. *The bounds μ_q and μ'_q are equal, for every q and n:*

(7.21) $$\mu_q(n) = \mu'_q(n).$$

Suppose that f is any function of L^p, and let

(7.22) $$h_n(-\theta) = \sum_{-n}^{n} |c_m|^{p-1} \operatorname{sgn} \bar{c}_m (|m|+1)^{p-2} e^{mi\theta}.$$

Then

$$T_p^p(f_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f h_n \, d\theta \leqq J_p(f) J_{p'}(h_n) \leqq \mu'_{p'}(n) J_p(f) T_{p'}(h_n).$$

But

$$T_{p'}^{p'}(h_n) = \sum_{-n}^{n} \left(|c_m|^{p-1}(|m|+1)^{p-2}\right)^{p'} (|m|+1)^{p'-2}$$

$$= \sum_{-n}^{n} |c_m|^p (|m|+1)^{p-2} = T_p^p(f_n),$$

and so

$$T_p^p(f_n) \leqq \mu'_{p'}(n) J_p(f) T_p^{p-1}(f_n), \quad T_p(f_n) \leqq \mu'_{p'}(n) J_p(f),$$

(7.23) $$\mu_{p'}(n) = \mu_p(n) \leqq \mu'_{p'}(n).$$

Next, given any polynomial

$$\varphi_n = \sum_{-n}^{n} \gamma_m e^{mi\theta},$$

let

$$\psi(-\theta) = |\varphi_n|^{q-1} \operatorname{sgn} \bar{\varphi}_n, \quad \psi(\theta) \sim \sum \beta_m e^{mi\theta}.$$

Then

$$J_q^q(\varphi_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_n(\theta) \psi(-\theta)\, d\theta = \sum_{-n}^{n} \beta_m \gamma_m$$

$$\leqq \sum_{-n}^{n} |\gamma_m| (|m|+1)^{1-\frac{2}{q}} \cdot |\beta_m| (|m|+1)^{1-\frac{2}{q'}}$$

$$\leqq T_q(\varphi_n) T_{q'}(\psi_n) \leqq \mu_{q'}(n) T_q(\varphi_n) J_{q'}(\psi) = \mu_{q'}(n) T_q(\varphi_n) J_q^{q-1}(\varphi_n),$$

$$J_q(\varphi_n) \leqq \mu_{q'}(n) T_q(\varphi_n),$$

$$\mu_q'(n) \leqq \mu_{q'}(n) = \mu_q(n). \tag{7.24}$$

The lemma follows from (7.23) and (7.24).

7.3. We next prove, as in § 5.3,

Lemma 33. *If f_n is a polynomial solution of* (7.15), *then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f_n|^{q-1} \operatorname{sgn} \bar{f}_n e^{mi\theta}\, d\theta = \mu_q^q(n) |c_m|^{q-1} \operatorname{sgn} \bar{c}_m (|m|+1)^{q-2} \tag{7.31}$$

for $|m| \leqq n$.

It follows that, if

$$g(-\theta) = |f_n|^{q-1} \operatorname{sgn} \bar{f}_n, \quad g(\theta) \sim \sum b_m e^{mi\theta},$$

then

$$b_m = \mu_q^q(n) |c_m|^{q-1} \operatorname{sgn} \bar{c}_m (|m|+1)^{q-2} \qquad (|m| \leqq n). \tag{7.32}$$

From (7.32), with the help of the inequality

$$\sum_{-n}^{n} |b_m|^p (|m|+1)^{p-2} \leqq \mu_p^p(n) \frac{1}{2\pi} \int_{-\pi}^{\pi} |g|^p\, d\theta,$$

we deduce

$$\mu_q^{pq}(n) \sum_{-n}^{n} |c_m|^{p(q-1)} (|m|+1)^{p(q-1)-2} \leqq \mu_p^p(n) \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_n|^{p(q-1)}\, d\theta;$$

and so

Lemma 34. *If $p(q-1) \geqq 2$ then*

$$\mu_q(n) \leqq (\mu_{p(q-1)}(n))^{\frac{q-1}{q}} (\mu_p(n))^{\frac{1}{q}}.$$

7.4. From Lemma 34 we deduce, much as in § 5.5,

Lemma 35. *If $\beta < \alpha < 2$ and*

$$\mu_\alpha(n) \leqq m, \qquad \mu_\beta(n) \leqq m,$$

then there is a γ such that $\beta < \gamma < \alpha$ and

$$\mu_\gamma(n) \leqq m.$$

The rest of the argument follows very nearly the lines of §§ 5.5 to 5.7, and we need not set it out in detail. There are three points of divergence which may be noted.

(I) In § 5.5 we required an independent proof that $\lambda_{2\varkappa}(n) < 2A\varkappa$, and here one that $\mu_{2\varkappa}(n) < 2A\varkappa$. The definition of $\mu_{q'}(n)$ in (7.13) refers only to polynomials; and our discussion of the case $q = 2\varkappa$ in § 6 might therefore have been limited to polynomials without prejudice to the general case.

(II) The deduction of Theorem 5 from Theorem 6 is a little simpler than that of Theorem 2 from Theorem 3, since we can utilize classical properties of the integrals J.

(III) The proof of the final clause of Theorem 7, on the other hand, is a little more difficult than the corresponding proof in § 5.7.

Take

$$f(\theta) = \sum_{m=1}^{e^q} \frac{e^{mi\theta}}{m}.$$

Then

$$T_q^q = \sum_{-\infty}^{\infty} |c_m|^q (|m|+1)^{q-2} \leqq 2^q \sum_1^{\infty} \frac{1}{(m+1)^2} < A^q.$$

But if $0 < \theta < \frac{1}{4}\pi e^{-q}$, we have

$$|f| \geqq \Re f \geqq \cos\frac{\pi}{4} \sum_1^{e^q} \frac{1}{m} > Aq,$$

$$J_q^q > \frac{1}{2\pi} \int_0^{\frac{1}{4}\pi e^{-q}} |f|^q d\theta > Ae^{-q}(Aq)^q > (Aq)^q > (Aq)^q T_q^q.$$

8.

Further theorems.

8.1. To each of Theorems 2, 3, 5 and 6 corresponds a more general theorem involving an additional parameter.

Theorem 8. *If*

(8.11) $$p \leqq r \leqq p'$$

and

$$\mu = \frac{1}{p} + \frac{1}{r} - 1, \tag{8.12}$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^r |\theta|^{-\mu r}\, d\theta \leqq A(p)\Big(\sum_{-\infty}^{\infty} |c_m|^p\Big)^{\frac{r}{p}}. \tag{8.13}$$

Here (8. 13) is to be understood as asserting that the left hand side is finite whenever the right hand side is finite, and $A(p)$ is a function of p only.

Theorem 9. *If*

$$q' \leqq r \leqq q \tag{8.14}$$

and

$$\mu = \frac{1}{r} + \frac{1}{q} - 1, \tag{8.15}$$

then

$$\sum_{-\infty}^{\infty} |c_m|^q \leqq A(q)\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^r |\theta|^{-\mu r}\, d\theta\Big)^{\frac{q}{r}}. \tag{8.16}$$

Theorem 10. *The hypotheses of Theorem* 8 *also imply*

$$\sum_{-\infty}^{\infty} |c_m|^r (|m|+1)^{-\mu r} \leqq A(p)\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^p\, d\theta\Big)^{\frac{r}{p}}. \tag{8.17}$$

Theorem 11. *The hypotheses of Theorem* 9 *also imply*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^q\, d\theta \leqq A(q)\Big(\sum_{-\infty}^{\infty} |c_m|^r (|m|+1)^{-\mu r}\Big)^{\frac{q}{r}}. \tag{8.18}$$

Each of these theorems contains, as extreme cases, (a) one of Theorems 2, 3, 5, 6, and (b) one part of Hausdorff's Theorem. For example Theorem 10 reduces to Theorem 5 when $r = p$, and to a part of Hausdorff's Theorem when $r = p'$ (so that $\mu = 0$).

There is however an important distinction between Theorems 8 and 10 on the one hand and Theorems 9 and 11 on the other. The first two theorems may be deduced immediately from the special cases already proved. For example, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^r |\theta|^{-\mu r}\, d\theta \leqq \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^p |\theta|^{p-2}\, d\theta\Big)^{\frac{p'-r}{p'-p}} \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^{p'}\, d\theta\Big)^{\frac{r-p}{p'-p}},$$

so that Theorem 8 is a corollary of Theorems 1 and 2; and similarly

Theorem 10 is a corollary of Theorems 1 and 5. We are not able to prove Theorems 9 and 11 in any such simple way, and the only proofs of these theorems which we possess demand a repetition of the proofs of Theorems 3 and 6, complicated by the presence of an additional parameter throughout the analysis.

8.2. In Theorems 8–11, the conditions (8.11) and (8.14) are essential. We have in fact

Theorem 12. *The enunciations of Theorem* 8 *and* 10 *become false when* $r < p$ *or* $r > p'$, *and those of Theorems* 9 *and* 11 *become false when* $r < q'$ *or* $r > q$.

The proof of Theorem 12 demands the construction of appropriate 'Gegenbeispiele', eight in all. We need only explain shortly the principles on which the construction rests.

(I) Suppose that

$$c_m = (|m|+1)^{-\frac{1}{p}} (\log(|m|+1))^{-\alpha} \qquad (\alpha > 0).$$

Then $\sum |c_m|^p$ is convergent if $p\alpha > 1$. On the other hand $f(\theta)$ is regular in $(-\pi, \pi)$ except for $\theta = 0$, where $|f(\theta)|$ has an infinity of the type

$$|\theta|^{-\frac{1}{p'}} \left(\log \frac{1}{|\theta|}\right)^{-\alpha};$$

and the integral in (8.13) is convergent if and only if $r\alpha > 1$. If $r < p$, we can choose α so that $p\alpha > 1 > r\alpha$, and then (8.13) is false. The same type of example shows that $r \geqq p$ is essential in Theorem 10, and $r \leqq q$ in Theorems 9 and 11.

(II) Suppose that

$$c_m = (-1)^m (|m|+1)^{-\alpha} \qquad (0 < \alpha < 1).$$

Then $\sum |c_m|^p$ is again convergent if $p\alpha > 1$. In this case $f(\theta)$ is regular except at the ends of the interval $(-\pi, \pi)$, where $|f(\theta)|$ has an infinity of the type

$$|\theta \pm \pi|^{-(1-\alpha)},$$

and the integral in (8.13) is convergent if and only if $r(1-\alpha) < 1$. If $r > p'$, we can choose α so that $p\alpha > 1$ and $r(1-\alpha) > 1$, and then (8.13) is false. A similar example shows that $r \geqq q'$ is essential in Theorem 9.

(III) A different type of example is required to show that $r \leqq p'$ is necessary in Theorem 10 and $r \geqq q'$ in Theorem 11. Suppose for example that $r > p'$, and take

$$c_m = 0 \quad (|m| \neq a^n), \qquad c_m = (n+1)^{-2} \quad (|m| = a^n)$$

where a is an integer greater than 1, so that

$$f(\theta)=1+2\sum_{1}^{\infty}\frac{\cos a^n\theta}{(n+1)^2}.$$

Then f is continuous; but $\mu r<0$, so that the series (8.17) is divergent.

Similarly $\mu r>0$ in (8.18) when $r<q'$, and we can then choose c_m so that (a) the series is convergent and (b) c_m, since it does not tend to zero, is not a Fourier coefficient.

(IV) There is a fourth class of series which (though not actually necessary for the proof of Theorem 12) is exceedingly useful for similar purposes.

The functions

$$g(z)=\sum_{2}^{\infty}n^{\beta-\frac{1}{2}}e^{\alpha i n\log n}z^n,\qquad h(u)=\sum_{1}^{\infty}a^{\beta n}u^{a^n},$$

where β and $\alpha=\alpha(a)$ are real and $a>1$, mimic one another in the unit circle when appropriate relations are set up between the variables and constants[21]). The same is true of the functions obtained by the insertion of convergence factors $(\log n)^{-\beta}$ and $n^{-\beta}$ respectively. In particular there is a close correspondence between the functions

$$G(z)=\sum_{2}^{\infty}n^{-\frac{1}{2}}(\log n)^{-\beta}e^{\alpha i n\log n}z^n,\qquad H(u)=\sum_{1}^{\infty}n^{-\beta}u^{a^n}.$$

It is plain, for example, that each function belongs to L^2 if and only if $\beta>\frac{1}{2}$. Actually, more than this is then true; it is easily proved that, if $a=2$, $H(u)$, and therefore $G(z)$, belongs to L^4, if $a=3$ to L^6, and generally that, whenever $\beta>\frac{1}{2}$, we can make the functions belong to any Lebesgue class L^r by an appropriate choice of α and a. Now for $G(z)$

$$\sum n^{-1+\frac{1}{2}s}|a_n|^s=\sum n^{-1}(\log n)^{-s\beta}$$

is divergent if $s<2$ and β is sufficiently near to $\frac{1}{2}$. We thus obtain the only one of the 'Gegenbeispiele' alluded to in § 1.2[22]) whose construction presents any serious difficulty.

8.3. We have stated all our theorems so far in terms of 'complex Fourier series'. There are alternative statements in terms of *real Fourier series* and *power series.*

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[21]) See Hardy and Littlewood, 1. A mistake in this note is corrected on p. 87 of Vol. 3 of the same periodical.

[22]) See f. n. [6]), p. 160.

Let us consider in particular Theorem 10, and regard it as a 'convergence theorem' only, without reference to its more precise statement as an inequality. Then there are two alternative propositions which, when we recall the fundamental theorem of M. Riesz concerning conjugate trigonometrical series[23]), are easily proved equivalent to Theorem 10[24]). These are

Theorem 10a. *If $f(\theta)$ is a real function of L^p, and*

$$f \sim \frac{1}{2} a_0 + \sum (a_m \cos m\theta + b_m \sin m\theta),$$

then

$$\sum_1^\infty m^{-\varkappa} (|a_m|^r + |b_m|^r),$$

where

$$p \leqq r \leqq p', \qquad \varkappa = \frac{p+r-pr}{p},$$

is convergent.

Theorem 10b. *If $f(z) = \sum a_n z^n$ is regular for $\varrho = |z| < 1$, and*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\varrho e^{i\theta})|^p \, d\theta$$

is bounded for $\varrho < 1$, then

$$\sum_1^\infty n^{-\varkappa} |a_n|^r$$

is convergent.

We have stated these theorems as convergence theorems, when they are equivalent to the corresponding form of Theorem 10. They may also be stated as inequalities; and then a distinction appears between Theorem 10b and the other theorems. Suppose for example that $r = p$, when Theorem 10b asserts the convergence of $\sum n^{p-2} |a_n|^p$. The corresponding inequality is

$$\sum_1^\infty n^{p-2} |a_n|^p \leqq A(p) \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^p \, d\theta, \tag{8.31}$$

where $f(\theta)$ is the 'boundary function' of $f(\varrho e^{i\theta})$. In (6.11), the factor $(A_3(p))^p$ tends to infinity when $p \to 1$; also Theorem 5 becomes false when $p = 1$. But we cannot say that $A(p) \to \infty$ in (8.31), when $p \to 1$; and, as we shall see in § 9, the result of Theorem 10b, with $r = p$, remains valid when $0 < p \leqq 1$.

8.4. It is also worth remarking that when $r \geqq 2$ (but not otherwise) Theorem 10 may be deduced from results which we have proved before

[23]) If a trigonometrical series is the Fourier series of a function of L^r, so also is the conjugate series. See M. Riesz, 1; Titchmarsh, 3.

[24]) In its less precise form.

in a more elementary way. It is plainly sufficient to prove this when $r=2$; and we may envisage the theorem in its 'real Fourier series' form 10a. We have proved[25]) that, if f belongs to L^p, and f_α, where $0<\alpha<\frac{1}{p}$, is the 'integral of order α' of f, defined in the manner of Liouville, Riemann, or Weyl, then f_α belongs to $L^{\frac{p}{1-p\alpha}}$. In particular, if $\alpha=\frac{1}{p}-\frac{1}{2}$, f_α belongs to L^2. If now we suppose for convenience $a_0=0$, then

$$f_\alpha \sim \sum (A_m \cos m\theta + B_m \sin m\theta),$$

where

$$A_m = m^{-\alpha}\left(a_m \cos\frac{1}{2}\alpha\pi + b_m \sin\frac{1}{2}\alpha\pi\right),$$

$$B_m = m^{-\alpha}\left(b_m \cos\frac{1}{2}\alpha\pi - a_m \sin\frac{1}{2}\alpha\pi\right),$$

$$A_n^2 + B_n^2 = n^{-2\alpha}(a_n^2 + b_n^2).$$

It follows that, if f belongs to L^p, then

$$\sum n^{-\frac{2}{p}+1}(a_n^2+b_n^2)$$

is convergent; and this is Theorem 10a with $r=2$.

8.5. We conclude this section by stating the theorems concerning *Fourier transforms* (in the sense of Plancherel and Titchmarsh[26])) which correspond to Theorems 2, 3, 5 and 6. There are, owing to the greater symmetry of the theory of transforms, two theorems only instead of four[27]).

Theorem 13. *If $f(x)$ belongs to L^p in $(0,\infty)$, and $F(x)$ is the cosine or sine transform*[28]) *of $f(x)$, then*

$$\int_0^\infty |F(x)|^p x^{p-2}\,dx \leqq A(p)\int_0^\infty |f(x)|^p\,dx.$$

Theorem 14. *If $|f(x)|^q x^{q-2}$ is integrable in $(0,\infty)$, then $F(x)$ exists and belongs to L^q in $(0,\infty)$, and*

$$\int_0^\infty |F(x)|^q\,dx \leqq A(q)\int_0^\infty |f(x)|^q x^{q-2}\,dx.$$

We need not give the proofs of these theorems, as the argument by which they are deduced from Theorems 2 and 6 is in all essentials the same as that of Titchmarsh.

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[25]) Hardy and Littlewood, 2.

[26]) Plancherel, 1; Titchmarsh, 1, 2.

[27]) There are naturally also theorems corresponding to the remainder of Theorems 2—12.

[28]) Proved to exist by Titchmarsh, 2.

9.

Extension of Theorem 10b to an exponent less than unity.

9.1. We stated in § 8.3 that Theorem 10b remains true when $0 < p \leqq 1$, and the proof of this is our principal object in this final section.

Lemma 35[29]). *If*

$$k \geqq 1, \quad r > 1, \quad b_n \geqq 0, \quad B_n = b_1 + b_2 + \dots + b_n, \tag{9.11}$$

then

$$\sum_1^\infty n^{-r} B_n^k \leqq K \sum_1^\infty n^{-r} (n b_n)^k, \tag{9.12}$$

where $K = K(k, r)$ is a function of k and r only[30]).

If on the other hand

$$k \leqq 1, \qquad r > 1,$$

then

$$\sum_1^\infty n^{-r} (n b_n)^k < K \sum_1^\infty n^{-r} B_n^k. \tag{9.13}$$

9.2. Lemma 36. *If the conditions* (9.11) *are satisfied, and*

$$\mathfrak{B}_n = \sum_{m=1}^\infty b_m e^{-\frac{m}{n}} \tag{9.21}$$

then

$$\sum_{n=1}^\infty n^{-r} \mathfrak{B}_n^k \leqq K \sum_1^\infty n^{-r} (n b_n)^k. \tag{9.22}$$

(I) If $k = 1$, the left hand side of (9.22) is

$$\sum_{n=1}^\infty n^{-r} \sum_{m=1}^\infty b_m e^{-\frac{m}{n}} = \sum_{m=1}^\infty b_m \sum_{n=1}^\infty n^{-r} e^{-\frac{m}{n}}.$$

The function $x^{-r} e^{-\frac{m}{x}}$ has a maximum, for positive values of x, when $x = \frac{m}{r}$, and is otherwise monotonic; and the maximum is $\left(\frac{r}{m e}\right)^r$. Hence

$$\sum_{n=1}^\infty n^{-r} e^{-\frac{m}{n}} < \int_0^\infty x^{-r} e^{-\frac{m}{x}} dx + K m^{-r} < K m^{1-r},$$

so that

$$\sum_1^\infty n^{-r} \mathfrak{B}_n \leqq K \sum_1^\infty m^{-r} (m b_m).$$

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29) Hardy and Littlewood, 3.

30) We use K in this sense throughout §§ 9.1, 9.2. For the case $r = k > 1$ see Hardy, 1, 2; Landau, 1; Elliott, 1.

(II) If $k > 1$, we write

$$b_n = n^{\frac{r}{k}-1} u_n, \qquad \sum n^{-r}(n b_n)^k = \sum u_n^k. \tag{9.23}$$

Suppose first that $r \leqq k$. Then

$$\mathfrak{B}_n = \sum m^{\frac{r}{k}-1} e^{-\frac{m}{n}} u_m = \sum v_m(n) u_m,$$

where $v_m(n)$ is a decreasing function of m for every value of n. Hence, if the u's are given in everything except arrangement, $\mathfrak{B}_n$ is a maximum, for every n, when the u's are arranged in decreasing order, so that $\sum n^{-r}\mathfrak{B}_n^k$ is also a maximum in this case. But then

$$\mathfrak{B}_n \leqq \sum_{m=1}^{n} b_m + n^{\frac{r}{k}-1} u_n \sum_{m=n+1}^{\infty} e^{-\frac{m}{n}} \leqq B_n + K n^{\frac{r}{k}} u_n,$$

$$\sum n^{-r}\mathfrak{B}_n^k \leqq K \sum n^{-r} B_n^k + K \sum u_n^k \leqq K \sum n^{-r}(n b_n)^k,$$

by (9.12) and (9.23).

Next, suppose $r > k$. Then $v_m(n)$ has a maximum $l n^{\frac{r}{k}-1}$ for $m = hn$, h and l being numbers of the type K. If we write

$$\mathfrak{B}_n^* = \sum_{m \leqq hn} v_{hn}(n) u_m + \sum_{m > hn} v_m(n) u_m = \sum v_m^*(n) u_m, \tag{9.24}$$

then

$$\mathfrak{B}_n \leqq \mathfrak{B}_n^*. \tag{9.25}$$

Also $v_m^*(n)$ is a decreasing function of m for every n, so that we may estimate $\mathfrak{B}_n^*$, as before we estimated $\mathfrak{B}_n$, on the hypothesis that u_m decreases steadily. In this case

$$\mathfrak{B}_n^* \leqq K n^{\frac{r}{k}-1} U_\nu + u_\nu \sum_{m > hn} v_m(n),$$

where

$$\nu = [hn] + 1, \qquad U_\nu = u_1 + u_2 + \cdots + u_\nu,$$

so that, when n increases, ν can dwell on the same value at most K times. Also

$$\sum_{m > hn} v_m(n) = \sum_{m > hn} m^{\frac{r}{k}-1} e^{-\frac{m}{n}} < K n^{\frac{r}{k}-1} \sum_{m > hn} e^{-\frac{m}{n}} < K n^{\frac{r}{k}}.$$

Hence

$$\sum n^{-r}\mathfrak{B}_n^k \leqq \sum n^{-r}\mathfrak{B}_n^{*k} \leqq K \sum \left(\frac{U_\nu}{\nu}\right)^k + K \sum u_\nu^k$$
$$\leqq K \sum \left(\frac{U_n}{n}\right)^k + K \sum u_n^k \leqq K \sum u_n^k = K \sum n^{-r}(n b_n)^k,$$

again by Lemma 35, but in this case with $r = k$.

9.3. Theorem 15. *If*

(9.31) $$0 < \lambda \leqq 2,$$

(9.32) $$f(z) = \sum_0^\infty a_n z^n$$

is regular for $\varrho = |z| < 1$,

(9.33) $$M_\lambda(\varrho, f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\varrho e^{i\theta})|^\lambda d\theta\right)^{\frac{1}{\lambda}} \leqq C$$

for $\varrho < 1$, *and*

(9.34) $$F(x) = \sum_0^\infty |a_n| x^n \qquad (0 < x < 1),$$

then

(9.35) $$\int_0^1 F^\lambda(x)\,dx \leqq KC^\lambda,$$

where K is now a function of λ only.

It should be observed that the theorem is false when $\lambda > 2$, an example to the contrary being given by the function

$$f(z) = \sum n^{-\frac{1}{2}-\delta} e^{a i n \log n} z^n \qquad (\delta > 0,\ \alpha > 0).$$

When $\lambda \leqq 2$, the theorem includes (except for a determination of the value of K) a theorem of Fejér and F. Riesz[31]), in which $|f(x)|$ stands in the place of $F(x)$ in (9.35).

In what follows K is a function of λ only. Suppose first that $\lambda > 1$. Then

$$|a_n| \leqq \varrho^{-n} M_1(\varrho, f) \leqq \varrho^{-n} M_\lambda(\varrho, f) \leqq C\varrho^{-n},$$

so that

$$|a_n| \leqq C, \quad F(e^{-1}) \leqq C \sum e^{-n} \leqq KC.$$

Hence

$$\int_0^1 F^\lambda(x)\,dx \leqq KC^\lambda + \int_{e^{-1}}^1 F^\lambda(x)\,dx = KC^\lambda + \int_1^\infty F^\lambda\left(e^{-\frac{1}{y}}\right) e^{-\frac{1}{y}} \frac{dy}{y^2}$$

$$= KC^\lambda + \sum_{n=1}^\infty \int_n^{n+1} \leqq KC^\lambda + \sum_1^\infty \frac{1}{n^2} F^\lambda\left(e^{-\frac{1}{n+1}}\right).$$

If we observe that $n^2 \geqq \frac{1}{4}(n+1)^2$, and then replace $n+1$ by n, we obtain

[31]) Fejér and F. Riesz, 1. Their theorem is true for all positive λ.

$$\int_0^1 F^\lambda(x)\,dx \leqq KC^\lambda + K\sum_1^\infty \frac{1}{n^2} F^\lambda\left(e^{-\frac{1}{n}}\right) = KC^\lambda + K\sum_{n=1}^\infty \frac{1}{n^2}\Big(\sum_{m=0}^\infty |a_m|\, e^{-\frac{m}{n}}\Big)^\lambda$$

$$\leqq KC^\lambda + K|a_0|^\lambda + K\sum_{n=1}^\infty \frac{1}{n^2}\Big(\sum_{m=1}^\infty |a_m|\, e^{-\frac{m}{n}}\Big)^\lambda$$

$$\leqq KC^\lambda + K\sum_1^\infty n^{\lambda-2}|a_n|^\lambda = KC^\lambda + K\lim_{\varrho\to 1}\sum_1^\infty n^{\lambda-2}|a_n|^\lambda \varrho^{n\lambda}$$

$$\leqq KC^\lambda + K\varliminf_{\varrho\to 1}\frac{1}{2\pi}\int_{-\pi}^{\pi} |\tilde f(\varrho e^{i\theta})|^\lambda\, d\theta \leqq KC^\lambda,$$

by Lemma 36 and Theorem 5.

Next, suppose that $0 < \lambda \leqq 1$. Then we can write $f = \omega g$, where $|\omega| < 1$ for $\varrho < 1$, g has no zeros for $\varrho < 1$, and

$$M_\lambda(\varrho, g) \leqq C.\ ^{32)}$$

Thus

$$f = \omega g = g + (\omega - 1)g = f_1 + f_2,$$

where neither f_1 nor f_2 has zeros for $\varrho < 1$, and

$$M_\lambda(\varrho, f_1) \leqq C,\quad M_\lambda(\varrho, f_2) \leqq 2M_\lambda(\varrho, g) \leqq 2C.$$

Now (9.35) is true if a corresponding inequality holds for *some* majorant of f, and $F_1 + F_2$, where F_1 and F_2 are any majorants of f_1 and f_2, is a majorant of f. It is therefore sufficient to prove the analogues of (9.35) for f_1 and f_2, or to prove (9.35) on the assumption that f has no zeros inside the circle.

If however f has no zeros, we may write $f = \varphi^k$, where k is an integer such that $1 < \lambda k \leqq 2$. If

$$\varphi = \sum b_n z^n,\quad \Phi = \sum |b_n|\, x^n$$

then Φ^k is plainly a majorant of f. Hence

$$\int_0^1 F^\lambda\, dx \leqq \int_0^1 \Phi^{\lambda k}\, dx \leqq K \operatorname{Max}\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Phi(\varrho e^{i\theta})|^{\lambda k}\, d\theta$$

$$= K \operatorname{Max}\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\varrho e^{i\theta})|^\lambda\, d\theta \leqq KC^\lambda$$

by what we have proved already.

32) See F. Riesz, 2.

9.4. **Theorem 16.** *If* $0 < \lambda \leqq 1$ *and*

(9.41) $$M_\lambda(\varrho, f) \leqq C,$$

then

(9.42) $$\sum_0^\infty (n+1)^{\lambda-2} |a_n|^\lambda \leqq K C^\lambda.$$

This theorem is a corollary of Theorem 15 and the second half of Lemma 35. In fact $|a_0| = M_\lambda(0, f) \leqq C$ and so, by (9.13),

(9.43) $$\sum_0^\infty (n+1)^{\lambda-2} |a_n|^\lambda \leqq C^\lambda + K \sum_1^\infty n^{\lambda-2} |a_n|^\lambda \leqq C^\lambda + K \sum_1^\infty n^{-2} \mathfrak{A}_n^\lambda,$$

where $\mathfrak{A}_n = |a_1| + |a_2| + \dots + |a_n|$.

Also $\mathfrak{A}_n \leqq e F\left(e^{-\frac{1}{n}}\right)$, and so

(9.44) $$\sum_1^\infty n^{-2} \mathfrak{A}_n^\lambda \leqq K \sum_1^\infty n^{-2} F^\lambda\left(e^{-\frac{1}{n}}\right) \leqq K \sum_1^\infty (n+1)^{-2} e^{-\frac{1}{n}} F^\lambda\left(e^{-\frac{1}{n}}\right)$$
$$\leqq K \sum_1^\infty \int_n^{n+1} F^\lambda\left(e^{-\frac{1}{y}}\right) e^{-\frac{1}{y}} \frac{dy}{y^2} \leqq K \int_1^\infty F^\lambda\left(e^{-\frac{1}{y}}\right) e^{-\frac{1}{y}} \frac{dy}{y^2}$$
$$\leqq K \int_0^1 F^\lambda(x)\, dx \gtreqless K C^\lambda,$$

by Theorem 15.

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(Eingegangen am 3. 2. 1926.)

CORRECTIONS

p. 159, (1.11). The inequality should be reversed.

p. 164, (2.21). For $\bar{\alpha}$ in the denominator read α.

p. 166, *line* 4. Insert 'in' after 'occur'.

p. 171, *line* 12. For Lemma 7 read Lemmas 7 and 8.

p. 173, § 3.7. The first displayed equation should be labelled (3.71).

p. 188, *line* 3. For 'compact' read 'everywhere dense in the interval'.

p. 191, *displayed equation following* (6.34). The index should be $-1+1/\kappa$.

p. 201, *line* 17. For C read G.

COMMENTS

The results proved in this paper have been generalized in two directions. First, the theorems were generalized by Hardy and Littlewood themselves by the use of rearrangements. For example, corresponding to Theorem 5 they proved:

(i) *Let $f \in L^p(-\pi,\pi)$, where $1 < p \leqslant 2$, let $f(\theta) \sim \sum_{-\infty}^{\infty} c_n e^{n\theta i}$, and let the numbers c_n^* be the numbers $|c_n|$ arranged so that*

$$c_0^* \geqslant c_{-1}^* \geqslant c_1^* \geqslant c_{-2}^* \geqslant c_2^* \geqslant c_{-3}^* \geqslant \dots .$$

Then
$$\sum_{-\infty}^{\infty} (|n|+1)^{p-2} c_n^{*p} \leqslant A(p) \int_{-\pi}^{\pi} |f(\theta)|^p \, d\theta.$$

This is a stronger result than Theorem 5, since

$$\sum_{-\infty}^{\infty} (|n|+1)^{p-2} |c_n|^p \leqslant \sum_{-\infty}^{\infty} (|n|+1)^{p-2} c_n^{*p}.$$

Outlines of two proofs of (i) and related results were given by Hardy and Littlewood in 1931, 4 and 1948, 1, but the complete proofs were not published, since improvements of the main theorems were made by R. E. A. C. Paley (see Z II, p. 120). Paley's results apply to orthogonal series with respect to a uniformly bounded orthonormal sequence of functions, the result corresponding to (i) being:

(ii) *Let* (ϕ_n) *be an orthonormal sequence of functions on an interval* $I \subset \mathbf{R}$, *and suppose that* $|\phi_n(\theta)| \leqslant M$ *for all* $\theta \in I$ *and all* n. *Let also* $f \in L^p(I)$, *where* $1 < p \leqslant 2$, *and let* c_n *be the n-th* ϕ*-coefficient of* f. *Then*

$$\sum_1^\infty n^{p-2}|c_n|^p \leqslant A(p)M^{2-p}\int_I |f(\theta)|^p \, d\theta.$$

This result includes (i), since we can take (ϕ_n) to be a rearrangement of the sequence $(e^{n\theta i})$.

The second type of generalization is in the direction of Theorems 8–11, but deals with power series rather than Fourier series. Partial results were obtained by Hardy and Littlewood in Theorems 15 and 16 and in Theorem 8 of 1936, 2, and their results were completed by H. R. Pitt, *Duke Math. J.* 3 (1937), 747–55. Pitt's main theorem is as follows:

(iii) *Let* $g(z) = \sum_0^\infty c_n z^n$ $(|z| < 1)$, *and let* $0 < p \leqslant q < \infty$, $\gamma \geqslant \max\{1/p, 1/q'\}$. *Then*

$$\left\{\sum_0^\infty (n+1)^{q-1-q\gamma}|c_n|^q\right\}^{1/q} \leqslant A(p,q,\gamma) \lim_{\rho\to 1-} \left\{\int_{-\pi}^{\pi} |g(\rho e^{i\theta})|^p|1-\rho e^{i\theta}|^{-1+p\gamma}\, d\theta\right\}^{1/p}.$$

By combining this with a theorem of Hardy and Littlewood on conjugate functions (1936, 2, § 6), Pitt obtained the following generalization of Theorems 8–11:

(iv) *Suppose that* $1 < p \leqslant q < \infty$, $\max\{1/p, 1/q'\} \leqslant \gamma < 1$, *that*

$$|f(\theta)|^p|\theta|^{-1+p\gamma} \in L(-\pi, \pi)$$

(*so that* $f \in L(-\pi, \pi)$), *and that* $f(\theta) \sim \sum_{-\infty}^{\infty} c_n e^{ni\theta}$. *Then*

$$\left\{\sum_{-\infty}^{\infty} (|n|+1)^{q-1-q\gamma}|c_n|^q\right\}^{1/q} \leqslant A(p,q,\gamma)\left\{\int_{-\pi}^{\pi} |f(\theta)|^p|\theta|^{-1+p\gamma}\, d\theta\right\}^{1/p}.$$

For odd functions there is a further extension (T. M. Flett, *Proc. London Math. Soc.* (3), 8 (1958), 135–48):

(v) *The condition* $\max\{1/p, 1/q'\} \leqslant \gamma < 1$ *in* (iv) *can be replaced by*

$$\max\{1/p, 1/q'\} \leqslant \gamma < 2,$$

provided that in addition f is odd and integrable.

The introduction of the interpolation techniques first developed by M. Riesz, Thorin, and Marcinkiewicz, has led to considerable simplifications in the proofs of some of these theorems, and to further extensions (see Z II, pp. 109, 120). The most general result in this field, which includes (i), (ii), the case $1 < p \leqslant q < \infty$ of (iii), and (iv), has been obtained by E. M. Stein and G. Weiss, *Trans. Amer. Math. Soc.* 87 (1958), 159–72.

p. 190. A simpler proof of a result equivalent to Lemma 25 is given in 1927, 11.

p. 206. A simpler proof of Theorem 15 is given by T. M. Flett, *Pac. J. Math.* 25 (1968), 463–94.

NOTES ON THE THEORY OF SERIES (XIII): SOME NEW PROPERTIES OF FOURIER CONSTANTS

G. H. HARDY *and* J. E. LITTLEWOOD†.

[*Extracted from the Journal of the London Mathematical Society, Vol.* 6, *Part* 1.]

1. This note contains a preliminary account of some theorems which extend, and in certain respects complete, those of a paper in Vol. 97 of the *Mathematische Annalen*‡. We hope to publish full details before long elsewhere.

We use r, p, q for indices satisfying

$$r > 1, \quad 1 < p \leqslant 2, \quad q \geqslant 2$$

respectively, and $A(r)$, $A(p)$, $A(q)$ for positive numbers, not usually the same at different occurrences, depending only on the parameters indicated. A, without argument, is a positive absolute constant.

† Received 7 May, 1930; read 15 May, 1930.

‡ Hardy and Littlewood, 1.

1931, 4 (with J. E. Littlewood) *Journal of the London Mathematical Society*, 6, 3–9.

Suppose that we are given a set of complex c_n defined for all integral n and tending to 0 when $|n| \to \infty$. We define c_n^* by saying that

$$c_0^*,\ c_1^*,\ c_{-1}^*,\ c_2^*,\ c_{-2}^*,\ c_3^*,\ \ldots$$

is the sequence of $|c_n|$ arranged in descending order of magnitude; if several $|c_n|$ are equal, then there are corresponding repetitions in the c_n^*.

We write

$$S_r(c) = \{\Sigma |c_n|^r (|n|+1)^{r-2}\}^{1/r}, \quad S_r^*(c) = \{\Sigma c_n^{*r} (|n|+1)^{r-2}\}^{1/r}, \tag{1.1}$$

and (for any measurable f)

$$J_r(f) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^r d\theta\right\}^{1/r}. \tag{1.2}$$

When the c_n are the (complex) Fourier constants of $f(\theta)$, we write $S_r(f)$, $S_r^*(f)$ for $S_r(c)$, $S_r^*(c)$. It is easy to see that

$$S_p^*(c) \geqslant S_p(c), \quad S_q^*(c) \leqslant S_q(c). \tag{1.3}$$

Our main theorems may now be stated as follows.

Theorem 1. *A necessary and sufficient condition that the c_n should be, for* every *variation of their arguments and arrangement, the Fourier constants of a function $f(\theta)$ of L^q, is that $S_q^*(c) < \infty$; and then*

$$J_q(f) \leqslant A(q)\, S_q^*(c) \tag{1.4}$$

for every such $f(\theta)$.

Theorem 2. *A necessary and sufficient condition that the c_n should be, for* some *variation of their arguments and arrangement, the Fourier constants of a function $f(\theta)$ of L^p, is that $S_p^*(c) < \infty$; and then*

$$S_p^*(c) \leqslant A(p)\, J_p(f) \tag{1.5}$$

for every such $f(\theta)$.

We may add that the $A(q)$ in (1.4) may be taken to be Aq, and the $A(p)$ in (1.5) to be Ap', where $p^{-1}+p'^{-1} = 1$ (so that p' is a q).

2. We insert here a few general remarks to explain the significance of these theorems. In our paper in the *Annalen* we showed that $S_q(c) < \infty$ is a *sufficient* condition that $f(\theta)$ should belong to L^q, and that then $J_q(f) \leqslant A(q)\, S_q(c)$, an inequality which (1.3) shows to be weaker than (1.4). Suppose now that we ask for a condition on the $|c_n|$ both necessary and sufficient for $f(\theta)$ to belong to L^q. The answer is negative;

there can be no such condition, since two functions, one of which belongs to L^q while the other does not, may have identical $|c_n|$. Thus of the two functions whose Fourier series are

$$(a)\quad \sum_1^\infty n^{-\frac{3}{4}} e^{ni\theta}, \qquad (b)\quad \sum_1^\infty n^{-\frac{3}{4}} e^{kin\log n}\, e^{ni\theta},$$

where $k > 0$, the first belongs to L^q if $q < 4$ only, while the second is continuous†. The second series shows that the sufficient condition $S_q(c) < \infty$, which requires $q < 4$, is certainly not also a necessary condition. The same example shows that the condition

$$\mathbf{S}_{q'}(c) = \{\Sigma\, |c_n|^{q'}\}^{1/q'} < \infty$$

of Young and Hausdorff, which is also sufficient, and is in this case also satisfied for $q < 4$ only, is not a necessary condition‡.

These considerations suggest a change in the problem, and the problem which we substitute, and solve by Theorem 1, is that of finding a necessary and sufficient condition that *all sets* c_n *which have the same* c_n^* (and so differ in arguments and arrangement only) should be sets of Fourier constants of functions of L^q. It is plain that such a condition must depend only on the c_n^*, so that it is $S_q^*(c)$ and not $S_q(c)$ which is relevant.

Since the Young-Hausdorff condition is also a condition on the c_n^* only, it might be thought likely to give the solution of our problem. That it does not do so follows from Theorem 1 itself; we have only to choose the c_n so that $S_q^*(c)$ is convergent and $\mathbf{S}_{q'}(c)$ divergent. If for example $q = 4$, we may take $c_n = (|n|+1)^{-\frac{3}{4}}\{\log(|n|+2)\}^{-\frac{1}{2}}$.

3. The proof of Theorem 1 is comparatively easy to describe when q is an even integer $2k$, and we state it shortly here. The details are a little simpler (though the ideas remain the same) when c_n is even in n and c_0 is the numerically largest c_n. In this case we say that c_n is *symmetrical*, and we state the arguments for this case, the results generally.

We have first

THEOREM 3. *If* a_n, b_n, ... *are* $2k$ *finite sets of positive numbers, tending to* 0 *as* $|n| \to \infty$, *and* a_n^*, b_n^*, ... *are defined in* § 1, *then*

$$\sum_{r+s+\ldots=0} a_r b_s \ldots \leqslant A(2k) \sum_{r+s+\ldots=0} a_r^* b_s^* \ldots.$$

† Hardy and Littlewood, **1**; Hille, **5**.

‡ See Hardy and Littlewood, **1**, or Hobson, **6**, for proofs and references to the original papers of Young and Hausdorff.

We proved this in the symmetrical case, when $A(2k) = 1$, in our paper **3.** The result obviously continues to hold for complex $a_n, b_n, \ldots$ when the left-hand side is replaced by its modulus.

THEOREM 4. *If $f(\theta) = \Sigma c_n e^{ni\theta}$ is a trigonometrical polynomial, and $f^*(\theta) = \Sigma c_n^* e^{ni\theta}$, then*

$$J_q(f) \leqslant A(q)\, J_q(f^*). \tag{3.1}$$

More generally, if the c_n^* are the Fourier constants of a function $f^*(\theta)$ of L^q, then the c_n are the Fourier constants of a function $f(\theta)$ for which (3.1) *is true.*

This follows immediately from Theorem 3 when $q = 2k$ and the functions are polynomials, and the passage from polynomials to general functions is straightforward.

The rest of the proof of Theorem 1 (for $q = 2k$) depends on special properties of functions of the type $f^*(\theta)$, *i.e.* functions whose Fourier constants are of the type c_n^*. In these properties there is no distinction between p and q.

4. In this section we are concerned *only* with sequences c_n^* and the corresponding functions, and we drop the asterisks and write a_n for c_n^*. Since a_n tends steadily to 0 when $n \to \infty$ by positive or by negative values, the series $\Sigma a_n e^{ni\theta}$ converges for all θ save multiples of 2π, and is therefore, by a known theorem, the Fourier series of its sum $f(\theta)$ whenever this sum is integrable.

THEOREM 5. *A necessary and sufficient condition that $f(\theta)$, of type $f^*(\theta)$, should belong to L^r is that $S_r(a) < \infty$, and then the ratio*

$$J_r(f) : S_r(a)$$

lies between two positive bounds $A(r)$.

We suppose the a_n symmetrical, so that

$$\tfrac{1}{2} f(\theta) = \tfrac{1}{2} a_0 + \Sigma a_n \cos n\theta.$$

It is easy to see that the problem is essentially unchanged if we drop the constant term, and that what we then have to prove is that the ratio

$$J_r(f) : U_r(a),$$

where
$$U_r(a) = \left(\sum_1^\infty n^{r-2} a_n^r\right)^{1/r},$$

has upper and lower bounds $A(r)$.

We use two inequalities which we have proved elsewhere†, viz.

$$\Sigma n^{-2} A_n^r \leqslant A(r) \Sigma n^{r-2} a_n^r, \tag{4.1}$$

where $A_n = a_1 + a_2 + \ldots + a_n$, and

$$\int_0^\pi \left\{\frac{\mathbf{F}(\theta)}{\theta}\right\}^r d\theta \leqslant A(r) \int_0^\pi |f(\theta)|^r d\theta, \tag{4.2}$$

where
$$\mathbf{F}(\theta) = \int_0^\theta |f(t)|\, dt.$$

(i) If $\pi/n > \theta > \pi/(n+1)$, we have

$$\tfrac{1}{2}|f(\theta)| \leqslant \sum_1^n a_m + \left|\sum_{n+1}^\infty a_m \cos m\theta\right| \leqslant A_n + \frac{\pi a_n}{\theta},$$

and so, by (4.1),

$$J_r^r(f) = \pi^r \sum_1^\infty \int_{\pi/(n+1)}^{\pi/n} |f|^r d\theta \leqslant A(r) \sum_1^\infty n^{-2}(A_n + n a_n)^r \leqslant A(r)\, U_r^r(a).$$

ii) On the other hand

$$\tfrac{1}{2}F(\theta) = \tfrac{1}{2}\int_0^\theta f(t)\, dt = \sum_1^\infty \frac{a_m}{m} \sin m\theta,$$

$$\tfrac{1}{2}F\left(\frac{\pi}{n}\right) = \sum_1^{n-1} \left(\frac{a_m}{m} - \frac{a_{m+n}}{m+n} + \frac{a_{m+2n}}{m+2n} - \ldots\right) \sin \frac{m\pi}{n}$$

$$\geqslant \sum_1^{n-1} \left(\frac{a_m}{m} - \frac{a_{m+n}}{m+n}\right) \sin \frac{m\pi}{n} \geqslant A \sum_{\frac{1}{4}n}^{\frac{3}{4}n} \frac{a_m}{m} \geqslant A a_n,$$

$$\sum_2^\infty n^{r-2} a_n^r \leqslant A(r) \sum_2^\infty n^{r-2} F^r\left(\frac{\pi}{n}\right) \leqslant A(r) \sum_2^\infty n^{r-2} \mathbf{F}^r\left(\frac{\pi}{n}\right)$$

$$\leqslant A(r) \sum_2^\infty \int_{\pi/n}^{\pi/(n-1)} \left\{\frac{\mathbf{F}(\theta)}{\theta}\right\}^r d\theta \leqslant A(r) \int_0^\pi \left\{\frac{\mathbf{F}(\theta)}{\theta}\right\}^r d\theta \leqslant A(r) \int_0^\pi |f(\theta)|^r d\theta,$$

by (4.2). Since $|a_1| \leqslant J_1(f) \leqslant J_r(f)$, it follows that $U_r(a) \leqslant A(r) J_r(f)$.

5. We can now complete the proof of Theorem 1 when $q = 2k$. In fact, combining Theorems 4 and 5, we obtain

$$J_q(f) \leqslant A(q) J_q(f^*) \leqslant A(q)\, S_q^*(c),$$

so that the condition of the theorem is *sufficient*. That it is necessary is evident from Theorem 5, since we may take $f(\theta) = f^*(\theta)$. The proof for general q, and the proof of Theorem 2 in any case, require further considerations of the same general character as those of §§ 5 and 7 of our

† See Hardy and Littlewood, **1**, **2**, and Hardy, **4**.

paper in the *Annalen.* An additional theorem which results from the argument is

Theorem 6. $J_p(f^*) \leqslant A(p) J_p(f).$

The proof of this theorem is curiously indirect.

6. There are two theorems which are in a sense the reciprocals of Theorems 1 and 2, and in which we consider "rearrangements" not of the Fourier constants of a function but of the values of the function. We suppose now that $f^*(\theta)$ is the "rearrangement of $|f(\theta)|$ as an even decreasing function"; the formal definition will be found in a recent note of F. Riesz†. The function $f^*(\theta)$ is "equimeasurable" with $|f(\theta)|$, in the sense that the sets of θ in which the two functions assume values lying in any given interval are of equal measure; and $f^*(\theta)$ has in general an infinite peak at the origin and decreases steadily as we pass away from it in either direction.

If now c_n is the Fourier constant of $f(\theta)$, and

$$T_r(f) = (\Sigma |c_n|^r)^{1/r}, \qquad I_r(f^*) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f^{*r}(\theta)\, |\theta|^{r-2} d\theta\right\}^{1/r},$$

then the theorems corresponding to Theorems 1 and 2 are

Theorem 7. *A necessary and sufficient condition that $T_q(f)$ should be finite for* all *$f(\theta)$ which have a given $f^*(\theta)$ is that $I_q(f^*)$ should be finite; and then*

$$T_q(f) \leqslant A(q)\, I_q(f^*).$$

Theorem 8. *A necessary and sufficient condition that $T_p(f)$ should be finite for* some *$f(\theta)$ with a given $f^*(\theta)$ is that $I_p(f^*)$ should be finite; and then*

$$I_p(f^*) \leqslant A(p)\, T_p(f).$$

[We have proved a number of other results, of which we may mention the following.

Theorem 9. *If $k > 1$, and $A(k)$ and c_n^* are defined as in § 1, then*

$$\int_{-\pi}^{\pi} e^{|f(\theta)|} d\theta \leqslant A(k) \sum_{1}^{\infty} n^{-2} \exp\left(k \sum_{-n}^{n} c_m^*\right).$$

Theorem 10. *Suppose that $\Sigma |c_n|/\log(1/|c_n|)$ is convergent. Then $\Sigma c_n e^{ni\theta}$ is the Fourier series of a function f such that* $\exp(k|f|)$ *is integrable for every value of k.*

† F. Riesz, 7.

THEOREM 11. *Suppose that* $f \sim \Sigma c_n e^{ni\theta}$ *and that* $|f| \log^+ |f|$ *belongs to L. Then* $\Sigma c_n^*/n$ *is convergent and* $\Sigma \exp(-k/|c_n|)$ *is convergent for every positive value of* k.

We mention finally a result concerned with "majorants" of $f(\theta)$.

THEOREM 12. *Suppose that* $p = 2k/(2k-1)$, *where* k *is a positive integer, and that* $f \sim \Sigma c_n e^{ni\theta}$ *belongs to* L^p. *Then there exists a set of* C_n, *satisfying* $C_n \geqslant |c_n|$ *for every* n, *which are the Fourier constants of a function* $F(\theta)$ *of* L^p.

We add that Mr. R. M. Gabriel, in a paper recently communicated to the Society, has shown that, when $q = 2k$, the factor $A(q)$ of Theorem 4 may be omitted.]

References.

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CORRECTION

p. 7, *line* 9. For π^r read π^{-1}.

COMMENTS

For further references concerning Theorems 1–8 see 1948, 1 and the comments on 1926, 7. Alternative proofs of Theorems 1, 2, 7, 8 are given in Z II, p. 127.

Proofs of Theorems 11 and 12 are given in 1935, 6 and 1935, 7 respectively.

Some new cases of Parseval's Theorem.

By

G. H. Hardy in Oxford and J. E. Littlewood in Cambridge.

1. Introduction.

1. In what follows we are concerned with a pair of analytic functions

$$(1.1)\qquad f(z)=\sum_1^\infty a_n z^n, \qquad g(z)=\sum_1^\infty b_n z^n,$$

regular for $r=|z|<1$. It is convenient to suppose that the series have no constant terms. We call

$$(1.2)\qquad h(z)=P(f,g)=\sum a_n b_n z^n$$

the "Faltung" or "Parseval function" of $f(z)$ and $g(z)$, and our problem is that of the convergence or summability of the series

$$(1.3)\qquad P=\sum a_n b_n e^{ni\theta},$$

the Faltung or Parseval series of the series

$$(1.4)\qquad \sum a_n e^{ni\theta}, \quad \sum b_n e^{ni\theta}.$$

It is classical that P is absolutely convergent whenever f and g are L^2. More generally, suppose that

$$p>1, \quad q=p'=\frac{p}{p-1}, \quad \frac{1}{p}+\frac{1}{q}=1,$$

and that f and g are L^p and L^q respectively. Then it was proved by M. Riesz that P is convergent[1]), uniformly in θ, but in general not absolutely convergent. Riesz's theorem holds for general Fourier series, whereas we have stated it for power series only; but it follows from his theorem on conjugate functions that there is no real loss of generality in this limitation. Here we shall be concerned with cases in which p or q is less than 1, and the limitation to power series will prove to be essential.

[1]) That P is summable $(C,1)$ had been proved earlier by Young (22). We have made extensions in a rather different direction; see Hardy and Littlewood (5).

If $p < 1$, q (as defined above) is negative, and Riesz's enunciation ceases to be significant. But there is a certain analogy between the Lebesgue class L^{γ} and the Lipschitz class $\operatorname{Lip}\left(-\frac{1}{\gamma}\right)$, a theorem true for functions of the first class when $\gamma > 0$ being often also true, with appropriate changes, for functions of the second class when $\gamma < 0$.[2]) Here this analogy suggests that, when f is L^p and $p < 1$, the appropriate class for g is $\operatorname{Lip}\left(\frac{1}{p} - 1\right)$. Guided by this analogy, we were led to the following theorems.

Theorem 1. *If*

(i) $0 < k \leqq 1$, $pk \geqq 1$ (*so that* $\mu = \frac{p}{p + pk - 1} \leqq 1$ *and* $p > 1$ *if* $k < 1$);

(ii) f *is* L^{μ};

(iii) g *is* $\operatorname{Lip}(k, p)$ [3]);

then P *is uniformly convergent, and indeed uniformly summable* $\left(C, -\frac{1}{p} + \delta\right)$ *for every positive* δ.

Theorem 2. *If*

(i) $0 < k \leqq 1$ $\left(\text{so that } \mu = \frac{1}{k+1} < 1\right)$;

(ii) f *is* L^{μ};

(iii) g *is* $\operatorname{Lip} k$;

then P *is uniformly summable* (C, δ) *for every positive* δ.

Theorem 3. *If, in addition to the conditions of Theorem* 1, $p \leqq 2$, *then* P *is absolutely convergent.*

It will be observed that Theorem 2 is the limiting case $p = \infty$ of Theorem 1. The question whether P is necessarily convergent in this case remains open. It is also to be observed that, in all the cases considered,

$$\mu \geqq \frac{1}{2}.$$

---

[2]) Witness for example Theorems 4 and 12 of our paper 7.

[3]) It may be convenient that we should repeat the definitions of the classes $\operatorname{Lip} k$ and $\operatorname{Lip}(k, p)$. The function $g(z)$ belongs to $\operatorname{Lip} k$ if

$$|g(z') - g(z)| \leqq C|z' - z|^k,$$

where C is independent of z and z', and to the class $\operatorname{Lip}(k, p)$ if

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(re^{i\theta+ih}) - g(re^{i\theta-ih})|^p \, d\theta \leqq C|h|^{pk},$$

where C is independent of r.

There are doubtless similar theorems in which $\mu < \frac{1}{2}$, but they will plainly demand definitions of Lipschitz classes of order greater than 1 (in which there is of course no difficulty of principle).

Our theorems are largely applications of theorems which we have proved in previous papers, and in particular in **20** (to which, and to the earlier papers of this series, we shall have to refer continually).

2. Convergence and summability: proof of Theorems 1 and 2.

2.1. In this section we shall make free use of theorems from **20**, and one preliminary remark is required. There, we defined the fractional integral $f_\alpha(z)$ by the formula

$$f_\alpha(z) = \sum \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)} a_n z^{n+\alpha}. \tag{2.11}$$

Here, we shall write instead

$$f_\alpha(z) = \sum_1^\infty n^{-\alpha} a_n z^n. \tag{2.12}$$

The reader will find no difficulty in satisfying himself that any property quoted for the one form of f_α is also possessed by the other.

Suppose, for example, that one of the functions belongs to L^p. The factor z^α in (2.11) is unimportant; and apart from this factor the two forms differ only by a factor of the type

$$1 + \frac{c_1}{n+1} + \frac{c_2}{(n+1)(n+2)} + \dots\text{[4]}$$

in the coefficient of z^n. If then we call the two functions φ and ψ, ψ is the sum of φ, a finite linear combination of ordinary integrals of φ, and a continuous function. By Theorem 33 of **20**, the integrals of φ belong to L^p (and indeed to higher classes). Hence, if φ belongs to L^p, ψ also belongs to L^p.

The advantage of the definition (2.12), for our present purpose, is that

$$h = P(f, g) = P(f_\alpha, g^\alpha) \tag{2.13}$$

for any real α.

2.2. Passing to the proof of Theorem 1, we dispose first of the case $p = 1$, $k = 1$, $\mu = 1$. In this case, if $f(e^{i\theta})$ and $g(e^{i\theta})$ are the boundary functions of $f(z)$ and $g(z)$, the first is integrable and the second (by Theorem 24 of 7) is of bounded variation; and

$$h(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi})\, g(e^{i\theta - i\varphi})\, d\varphi$$

[4]) A finite series followed by an error term of lower order than any preceding term. If $f(z)$ belongs to any L^p, then $a_n = O(n^K)$ for some K.

is also of bounded variation. It follows, by Theorem 16 of **4**, that $\sum |a_n b_n|$ is convergent. Finally, since $a_n b_n = O\left(\frac{1}{n}\right)$, the series (1.3) is summable $(C, -1+\delta)$ for every positive δ.

We may then suppose that $p > 1$. We prove that

(i) P is the Fourier (power) series of a continuous function,

(ii) P is the Fourier (power) series of a function of the class

$$\mathrm{Lip}\left(\frac{1}{p}, p\right)$$

or Λ_p. It will follow from (i) that P is uniformly summable $(C, 1)$, and from (ii), by Theorem 1 of **8**[5]), that it is uniformly summable $\left(C, -\frac{1}{p}+\delta\right)$ and in particular uniformly convergent.

2.3. Proof of (i). We shall prove rather more, viz. that

$$\int_0^1 |h'(r e^{i\theta})|\, dr$$

is uniformly convergent. *A fortiori*, $h(e^{i\theta})$ is continuous and P is its Fourier series.

If $r = \varrho^2 < 1$, we have

$$r h'(r e^{i\theta}) = \sum n a_n b_n \varrho^{2n} e^{ni\theta} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varrho e^{i\varphi})\, g'(\varrho e^{i\theta - i\varphi})\, d\varphi,$$

$$|r h'(r e^{i\theta})| \leqq M_{p'}(f)\, M_p(g').\ ^{6})$$

By Theorem 48 of **20** (Theorem 3 of **8**),

$$M_p(g') = O((1-\varrho)^{k-1}),$$

and our result will plainly follow if

$$\int_0^1 (1-\varrho)^{k-1} M_{p'}(f)\, d\varrho$$

is convergent. This follows from Theorem 31 of **20**, when we replace p by μ, q by p', and l by 1, so that $l\alpha$ is replaced by

$$\frac{1}{\mu} - \frac{1}{p'} = 1 + k - \frac{1}{p} - \frac{1}{p'} = k.$$

---

[5]) There is no reference to uniformity in **8**, where we are concerned with a particular point; but the whole argument is "uniform in θ", the difference between any two Cesàro means, of order greater than $-\frac{1}{p}$, being shown to tend uniformly to zero.

[6]) As usual $p' = \frac{p}{p-1}$.

2.4. Proof of (ii). By Theorem 33 of **20**, $f_{k-\frac{1}{p}}(z)$ belongs to L (since

$$\frac{\mu}{1-\left(k-\frac{1}{p}\right)\mu}=1);$$

and, by Theorem 49, $g^{k-\frac{1}{p}}(z)$ belongs to $\mathrm{Lip}\left(k-k+\frac{1}{p}, p\right)$ or Λ_p. Also h is the Faltung of $f_{k-\frac{1}{p}}$ and $g^{k-\frac{1}{p}}$. It is therefore sufficient to prove that the Faltung of a function F of L and a function G of Λ_p belongs to Λ_p.

But in these circumstances, writing $\Delta F(\theta)$ for $F(\theta+h)-F(\theta-h)$, and supposing $h>0$, we have

$$|\Delta H(\theta)| \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi}|F(\varphi)|\,|\Delta G(\theta-\varphi)|\,d\varphi,$$

$$\left(\int|\Delta H(\theta)|^p d\theta\right)^{\frac{1}{p}} \leqq \frac{1}{2\pi}\int|F(\varphi)|\,d\varphi\left(\int|\Delta G(\theta-\varphi)|^p d\theta\right)^{\frac{1}{p}} = O\left(h^{\frac{1}{p}}\right),$$

the result required. This completes the proof of Theorem 1.

2.5. Proof of Theorem 2. The argument of § 2.3 is valid as it stands in the limiting case $p=\infty$, if we replace $M_{p'}$ and M_p by M_1 and M_∞ or M. This is all that is required to prove the theorem.

The argument of § 2.4 fails, and we cannot assert convergence; but we have no 'Gegenbeispiel'.

3. Absolute convergence: proof of Theorem 3.

3. We shall give two different proofs of Theorem 3. The first, in which we make use of theorems from **20** and of what we have proved already, is much the shorter; but the second depends on subsidiary propositions which are interesting in themselves, and also proves more.

(i) We observe first that it is sufficient to prove the theorem in the case $p=2$. For suppose that g belongs to $\mathrm{Lip}(k, p)$ and that $p<q$. Then, by Theorem 5 of 8, g also belongs to $\mathrm{Lip}(l, q)$, where

$$l=k-\frac{1}{p}+\frac{1}{q};$$

and

$$\mu(l, q)=\frac{q}{q+ql-1}=\frac{p}{p+pk-1}=\mu(k, p).$$

It follows that, if Theorem 3 is true for q, l, μ, it is also true for p, k, μ. The same argument would show, of course, that Theorem 1 is true for a given p if it is true for some larger p.

(ii) We may therefore suppose that $p = 2$, when we have to prove that $\sum |a_n b_n|$ is convergent for an f and g belonging respectively to

$$L^{\frac{2}{1+2k}}, \quad \mathrm{Lip}(k, 2).$$

The hypothesis on g, viz.

$$\int |g(e^{i\theta+ih}) - g(e^{i\theta-ih})|^2 d\theta = O(h^{2k}),$$

or

$$\sum |b_n|^2 \sin^2 nh = O(h^{2k}),$$

is then equivalent to

$$\sum_1^n |\nu b_\nu|^2 = O(n^{2-2k}),$$

and is a hypothesis on the moduli $|b_n|$ only; the arguments $\arg b_n$ are at our disposal. It follows that $\sum a_n b_n$, which we know already to be convergent, is absolutely convergent.

Our second proof demands some preliminary theorems concerning series of positive terms. These have some interest in themselves, and we have developed them a little further than is necessary for our application.

4. Some theorems on series of positive terms.

4.1. Theorem 4. *If x_n, y_n, and λ_n are positive;*

$$p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad r > 1;$$

and

$$\sum x_n^p < \infty, \quad \sum y_n^q < \infty, \quad \sum_1^n \lambda_\nu^r = O(n);$$

then

$$\sum \sum \frac{x_m y_n}{m+n} \lambda_{m+n} < \infty.$$

The theorem becomes false if we replace r by 1.

The summations are over $m, n = 1, 2, 3, \ldots$. The case $\lambda_n = 1$, $p = q = 2$ is Hilbert's double series theorem.

We write

$$\Lambda_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n,$$

so that

$$\Lambda_n \leqq n^{\frac{1}{r'}} \left(\sum_1^n \lambda_\nu^r \right)^{\frac{1}{r}} = O(n).$$

We have[7])

$$S = \sum \sum \frac{x_m y_n}{m+n} \lambda_{m+n} = \sum \sum x_m \left(\frac{\lambda_{m+n}}{m+n} \right)^{\frac{1}{p}} \left(\frac{m}{n} \right)^s \cdot y_n \left(\frac{\lambda_{m+n}}{m+n} \right)^{\frac{1}{q}} \left(\frac{n}{m} \right)^s,$$

[7]) Compare Hardy, Littlewood, and Pólya (21, p. 280).

where s is a positive number to be chosen later. By Hölder's inequality

$$S \leqq P^{\frac{1}{p}} Q^{\frac{1}{q}},$$

where

$$P = \sum\sum x_m^p \left(\frac{m}{n}\right)^{sp} \frac{\lambda_{m+n}}{m+n}, \quad Q = \sum\sum y_n^q \left(\frac{n}{m}\right)^{sq} \frac{\lambda_{m+n}}{m+n};$$

and it is enough to prove the convergence of P and Q.

Now

$$(4.11) \qquad P = \sum_m x_m^p m^{sp} \sum_n n^{-sp} \frac{\lambda_{m+n}}{m+n} = \sum x_m^p m^{sp} T_m,$$

say. Summing partially, and observing that $\Lambda_n = O(n)$, we obtain

$$T_m = \sum_n \frac{n^{-sp}}{m+n} \{(\Lambda_{m+n} - \Lambda_m) - (\Lambda_{m+n-1} - \Lambda_m)\}$$

$$= \sum_n (\Lambda_{m+n} - \Lambda_m) \Delta \frac{n^{-sp}}{m+n},$$

where Δu_n denotes $u_n - u_{n+1}$. Now

$$\Delta \frac{n^{-sp}}{m+n} < K \frac{n^{-sp-1}}{m+n},$$

where K is independent of m and n. Hence

$$T_m \leqq K \sum_n (\Lambda_{m+n} - \Lambda_n) \frac{n^{-sp-1}}{m+n}.$$

But

$$\Lambda_{m+n} - \Lambda_m = \sum_{m+1}^{m+n} \lambda_\nu \leqq n^{\frac{1}{r'}} \Big(\sum_{m+1}^{m+n} \lambda_\nu^r \Big)^{\frac{1}{r}} < K n^{\frac{1}{r'}} (m+n)^{\frac{1}{r}};$$

and so

$$T_m \leqq K \sum_n n^{\frac{1}{r'} - sp - 1} (m+n)^{\frac{1}{r} - 1} = O(m^{-sp}),$$

provided only that $0 < sp < \frac{1}{r'}$. If s satisfies this condition, it follows from (4.11) that P is convergent.

Similarly Q is convergent if $0 < sq < \frac{1}{r'}$. Since we can choose s to satisfy both these conditions, the theorem is proved.

To prove that the result is false when $r = 1$, take $\lambda_n = n$ when $n = 2^\nu$ and $\lambda_n = 0$ otherwise. The theorem would then assert that

$$\sum_\nu \sum_{m+n=2^\nu} x_m y_n$$

is convergent whenever $\sum x_m^p$ and $\sum y_n^q$ are convergent, and this is false.

4.2. Theorem 5. *If*

$$p > 1, \quad X_n = x_1 + x_2 + \ldots + x_n,$$

and

$$\sum x_n^p < \infty, \quad \Lambda_n = O(n),$$

then

$$\sum \left(\frac{X_n}{n}\right)^p \lambda_n < \infty.$$

We do not require this theorem, but we insert it because it corresponds to a theorem of Hardy[8]) as Theorem 4 corresponds to Hilbert's theorem. It will be observed that a difference between the two theorems appears when (in the notation of Theorem 4) $r = 1$.

Suppose that $\Lambda_n \leqq Cn$. Then

$$\sum_1^N \left(\frac{X_n}{n}\right)^p \lambda_n = \sum_1^{N-1} \Lambda_n \Delta\left(\frac{X_n}{n}\right)^p + \Lambda_N \left(\frac{X_N}{N}\right)^p,$$

and the last term tends to zero because $\Lambda_N = O(N)$ and $X_N = o\left(N^{\frac{1}{p'}}\right)$.[8a]) Hence

$$S = \sum \left(\frac{X_n}{n}\right)^p \lambda_n = \sum \Lambda_n \Delta\left(\frac{X_n}{n}\right)^p \leqq CK \sum \left(\frac{X_n}{n}\right)^p + CK \sum \left(\frac{X_n}{n}\right)^{p-1} x_n$$

$$\leqq CK \sum \left(\frac{X_n}{n}\right)^p + CK \left(\sum \left(\frac{X_n}{n}\right)^p\right)^{\frac{1}{p'}} \left(\sum x_n^p\right)^{\frac{1}{p}} \leqq CK \sum x_n^p,$$

the K being positive numbers which depend only on p.

4.3. Theorem 6. *If* $p > 1$,

$$\chi_n = \sum_m e^{-\frac{m}{n}} x_m,$$

and

$$\sum x_n^p < \infty, \quad \Lambda_n = O(n),$$

then

$$\sum \left(\frac{\chi_n}{n}\right)^p \lambda_n < \infty.$$

This theorem is related to one of the theorems[9]) of our paper **19** (to which it reduces when $\lambda_n = 1$), as Theorems 4 and 5 are to the other known theorems to which we have referred.

We suppose again that $\Lambda_n \leqq Cn$. We can choose M so that

$$\sum_{M+1}^{\infty} x_m^p < \varepsilon^p,$$

[8]) $\sum \left(\frac{X_n}{n}\right)^p < \infty$.

[8a]) Compare the proof in § 4.3 of the corresponding property of χ_n.

[9]) Theorem 2 (2.54).

and then

$$\chi_n \leqq \sum_1^M e^{-\frac{m}{n}} x_m + \varepsilon \left(\sum_{M+1}^{\infty} e^{-\frac{p' m}{n}} \right)^{\frac{1}{p'}};$$

from which it follows that $\chi_n = o\left(n^{\frac{1}{p'}}\right)$. But

$$\sum_1^N \left(\frac{\chi_n}{n}\right)^p \lambda_n = \sum_1^{N-1} \Lambda_n \Delta \left(\frac{\chi_n}{n}\right)^p + \Lambda_N \left(\frac{\chi_N}{N}\right)^p,$$

and the last term tends to zero. Hence

$$S = \sum \left(\frac{\chi_n}{n}\right)^p \lambda_n = \sum \Lambda_n \Delta \left(\frac{\chi_n}{n}\right)^p \leqq CKS_1 + CKS_2,$$

where

$$S_1 = \sum \left(\frac{\chi_n}{n}\right)^p, \qquad S_2 = \sum \left(\frac{\chi_n}{n}\right)^{p-1} |\Delta \chi_n|^{10)},$$

and the K depend only on p. Since $S_1 < \infty$ (by the theorem of **19** just referred to), and

$$S_2 \leqq \left(\sum \left(\frac{\chi_n}{n}\right)^p \right)^{\frac{1}{p'}} \left(\sum |\Delta \chi_n|^p \right)^{\frac{1}{p}},$$

it is sufficient to prove that

$$T = \sum |\Delta \chi_n|^p < \infty.$$

But

$$|\Delta \chi_n| = \sum_m \left(e^{-\frac{m}{n+1}} - e^{-\frac{m}{n}}\right) x_m \leqq \frac{4}{(2n)^2} \sum m e^{-\frac{m}{2n}} x_m \leqq \frac{4}{(2n)^2} \sum K n e^{-\frac{m}{4n}} x_m = K \frac{\chi_{4n}}{4n},$$

and so

$$T = \sum |\Delta \chi_n|^p \leqq K \sum \left(\frac{\chi_{4n}}{4n}\right)^p \leqq K \sum \left(\frac{\chi_n}{n}\right)^p \leqq K \sum x_m^p.$$

This completes the proof of the theorem.

5. Direct proof of Theorem 3.

5.1. We return to Theorem 3, proving, as we said in § 3, rather more.

Theorem 7. *If* $\mu < 1$ *and*

(i) $f(z)$ *is* L^μ,

(ii) $|c_1| + |c_2| + \ldots + |c_n| = O(n)$,

then

$$\sum n^{\mu-2} |a_n|^\mu |c_n|$$

is convergent.

[10] $\Delta \chi_n$ is negative.

When $c_n = 1$, this reduces to Theorem 16 of 4 [11]).

We may plainly suppose that c_n is positive. It is also sufficient (on grounds to which we have appealed many times in this series of papers, and need not repeat again) to prove the theorem for an $f(z)$ which has no zeros in the unit circle.

We write

$$k = \frac{2}{\mu} > 2, \qquad r = 2 - \mu = \frac{2}{k'} > 1$$

(so that $(k-1)\mu = r$), and

$$f = \varphi^k, \qquad \varphi = \sum u_n z^n$$

(so that φ belongs to L^2). Let

$$\varrho = e^{-\frac{1}{k' n}}.$$

Then

$$|a_n| \leqq \frac{\varrho^{-n}}{2\pi} \int_{-\pi}^{\pi} |f(\varrho e^{i\theta})| \, d\theta \leqq K \int_{-\pi}^{\pi} |\varphi|^k \, d\theta \leqq K \left(\sum (|u_m| \varrho^m)^{k'} \right)^{k-1},$$

by Hausdorff's theorem, K depending only on k. If now we write $x_m = |u_m|^{k'}$, $\lambda_m = |c_m|$, and define χ_n as in Theorem 6, we have

$$|a_n| \leqq K \chi_n^{k-1}$$

and

$$\sum n^{\mu-2} |a_n|^\mu |c_n| \leqq K \sum \chi_n^{(k-1)\mu} n^{\mu-2} \lambda_n = K \sum \left(\frac{\chi_n}{n}\right)^r \lambda_n.$$

But

$$\sum x_n^r = \sum |u_n|^2$$

is convergent, since φ is L^2; and our conclusion therefore follows from Theorem 6.

Theorem 8. *If the conditions of Theorem 7 are satisfied, then*

$$\sum n^{-\frac{1}{\mu}} |a_n| \, |c_n|$$

is convergent.

This is included in Theorem 7, since

$$a_n = o\left(n^{\frac{1}{\mu} - 1}\right),$$

by Theorem 28 of **20**.

5. 2. When $\mu < 1$ (*i. e.* when $pk > 1$) Theorem 3 may be deduced from Theorem 8. For, by Hausdorff's theorem and Theorem 48 of **20**,

$$\left(\sum (|n b_n| r^n)^{p'}\right)^{\frac{1}{p'}} \leqq M_p(g') = O((1-r)^{k-1}).$$

---

[11]) Stated as a convergence theorem.

Hence, taking $r = e^{-\frac{1}{N}}$, we see that

$$\sum_1^N |n b_n|^{p'} = O(N^{p'(1-k)}),$$

$$\sum_1^N |n^{\frac{1}{\mu}} b_n|^{p'} = O\left(N^{p'\left(\frac{1}{\mu}-k\right)}\right) = O(N),$$

and *a fortiori*

$$\sum_1^N |n^{\frac{1}{\mu}} b_n| = O(N).$$

We may therefore take $c_n = n^{\frac{1}{\mu}} b_n$ in Theorem 8, and this proves Theorem 3.

5.3. Theorems 7 and 8 are false when $\mu = 1$. In fact, in the notation of § 5.1, $k = 2$, $f = \varphi^2$, and

$$\sum_n n^{-1} |a_n| \, |c_n| = \sum_n n^{-1} |c_n| \, \Big| \sum_{\mu+\nu=n} u_\mu u_\nu \Big|;$$

and it is plain (since the condition on f or φ is a condition on the moduli $|u_n|$ only) that each of the theorems is equivalent to the forbidden case $r = 1$ in Theorem 4.

Theorem 3 is however still true, since the c_n of § 5.2 satisfies

$$\sum_1^N |c_n|^r = O(N)$$

for $r = p' > 1$, so that the theorem follows from Theorem 4.

5.4. We shall not attempt to show systematically that the theorems which we have proved are the 'best possible' of their kind, but we shall discuss shortly some examples whose general tendency is in this direction.

(i) We stated in § 1 that the limitation of our theorems to power series is essential. To illustrate this, consider the analogue for general Fourier series of the case $pk = 1$, $\mu = 1$ of Theorem 1. This would imply that the Faltung of any two real functions $F(\theta)$ and $G(\theta)$, of which F is L and G is Λ_p, is continuous; and this would imply that any G of Λ_p is bounded. That this is untrue is shown by the example

$$G(\theta) = \log \frac{1}{|\theta|},$$

which is Λ_p for every $p > 1$.

The same point may be illustrated rather differently. Take

$$F(r, \theta) = 1 + 2\sum r^n \cos n\theta, \qquad G(r, \theta) = \sum \frac{r^n}{n} \cos n\theta.$$

Then F is L, since $M_1(F) = 1$, and G is Λ_p, but the Parseval series diverges for $\theta = 0$.

If we desire similar examples with $\mu < 1$, it is natural to use the function

$$F(r,\theta) = \left(\frac{d}{d\theta}\right)^{\beta}\left(1 + 2\sum r^n \cos n\theta\right) = 2\sum n^{\beta} r^n \cos\left(n\theta + \frac{1}{2}n\beta\right),$$

where β is a positive integer. It is easily verified that F is $L^{\frac{1}{1+\beta}}$. In this case, however, $\mu \leqq \frac{1}{2}$, so that either $k > 1$ (when the example is not relevant here) or $k = 1$, $p = \infty$ (in which case it is used in (2) below).

(2) In Theorem 2 we left the question of *convergence* unsettled. The function just mentioned, with $\beta = 1$, enables us to answer the corresponding question for *harmonic* functions negatively. Take

$$F(r,\theta) = \sum n r^n \sin n\theta,$$

and for $G(\theta)$ any odd function of Lip 1 (*i. e.* any odd function which is the integral of a bounded function). Then $k = 1$, $p = \infty$, $\mu = \frac{1}{2}$, and the conditions corresponding to those of Theorem 2 are satisfied. It is not true that $\sum n b_n$ is necessarily convergent (since the Fourier series of a bounded function is not necessarily convergent).

(3) The example

$$f(z) = \sum \frac{n^{\beta} z^n}{(\log n)^{\gamma}}, \quad g(z) = \sum \frac{e^{ain\log n}}{n^{k+\frac{1}{2}}} z^n,$$

where

$$a > 0, \quad 0 < k < 1, \quad p > 2, \quad \beta = k - \frac{1}{p}, \quad \gamma > \frac{1}{\mu},$$

shows that Theorem 3 is false when $p > 2$. Here f is L^{μ} and g is Lip k, *a fortiori* Lip (k, p); and $\sum |a_n b_n|$ is divergent.

(4) In Theorems 1—3 we have always $pk \geqq 1$ and so $\mu \leqq 1$. If $pk < 1$ then $\mu > 1$; and we give an example to illustrate the failure of the theorems in this case. Suppose that $0 < p < 2$ and $k = \frac{1}{p} - \frac{1}{2}$, so that $\mu = 2$, and take $g(z) = (1-z)^{-\frac{1}{2}}$; it is easily verified that g is Lip (k, p). If the conclusion of Theorem 1 were true, it would follow that the convergence of $\sum |a_n|^2$ implies that of $\sum n^{-\frac{1}{2}} a_n$; and this is false.

6. On a theorem of W. H. Young.

6.1. The theorems of this and the next section are of a different type, but also depend on theorems from **20**.

It was proved by Young[12]) that if

$$p \geqq 1, \quad q \geqq 1, \quad \frac{1}{p} + \frac{1}{q} > 1, \tag{6.11}$$

$F(\theta)$ and $G(\theta)$ are any functions of L^p and L^q, and $H(\theta)$ is their Parseval function or Fourier Faltung, then $H(\theta)$ belongs to L^{μ}, where

$$\mu = \frac{pq}{p+q-pq}. \tag{6.12}$$

[12]) Young (**23**).

In particular, when $F(\theta)$ and $G(\theta)$ are the boundary functions $f(e^{i\theta})$ and $g(e^{i\theta})$, we have

Theorem 9. *If p and q satisfy* (6.11), *$f(z)$ is L^p, and $g(z)$ is L^q, then $h(z)$ is L^μ and*

(6.13)
$$M_\mu(h) \leqq M_p(f)\, M_q(g).$$

The question arises whether this theorem for power series remains true when p or q is less than 1, and, if not, what replaces it. The answer to the first question is easily seen to be negative; if

$$f(z) = \sum_2^\infty \frac{n^\alpha}{(\log n)^\gamma} z^n,$$

where α and γ are positive and $\frac{\gamma}{1+\alpha} > 1$, and

$$g(z) = \sum 2^{-\beta n} z^{2^n},$$

where $0 < \beta < \alpha$, then f belongs to $L^{\frac{1}{1+\alpha}}$ and g is continuous, but

$$h(z) = \frac{1}{(\log 2)^\gamma} \sum \frac{2^{n(\alpha-\beta)}}{n^\gamma} z^{2^n}$$

does not belong to any Lebesgue class.

It is known that if h belongs to L^μ, where $\mu > 0$, then

(6.14)
$$\int_0^1 |h|^\mu\, dr$$

is (uniformly) convergent[13]), and that, if $0 < \mu \leqq 2$, and

$$\mathfrak{H}(r) = \sum |a_n|\,|b_n|\, r^n,$$

then

(6.15)
$$\int_0^1 \mathfrak{H}^\mu(r)\, dr$$

is convergent[14]), the converse propositions being false. This suggests that the appropriate conclusion, when we allow p or q to be less than 1 in Young's theorem, may be the convergence of (6.14) or (6.15).

Theorem 10. *If*

$$p > 0, \quad q > 0, \quad \frac{1}{p} + \frac{1}{q} > 1,$$

f is L^p and g is L^q, then (6.14) *is (uniformly) convergent.*

Theorem 11. *If, in addition to the conditions of Theorem* 10, *$p \leqq 2$, $q \leqq 2$, then* (6.15) *is convergent.*

To prove Theorem 10 we observe that, since $\frac{1}{p} + \frac{1}{q} > 1$, we can choose k so that

$$k > p, \quad k' > q, \quad \frac{1}{k} + \frac{1}{k'} = 1.$$

[13]) Fejér and F. Riesz (**18**).

[14]) Hardy and Littlewood (**4**, Theorem 15).

Then, if $r = \varrho^2 < 1$, we have

$$(6.16)\quad |h(re^{i\theta})| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varrho e^{i\varphi})\, g(\varrho e^{i\theta - i\varphi})\, d\varphi \right| \leqq M_k(\varrho, f)\, M_{k'}(\varrho, g),$$

$$\int_0^1 |h(re^{i\theta})|^{\mu}\, d\varrho \leqq \int_0^1 M_k^{\mu}(f)\, M_{k'}^{\mu}(g)\, d\varrho$$

$$\leqq \left(\int_0^1 M_k^{\frac{pk}{k-p}}(f)\, d\varrho \right)^{\frac{(k-p)\mu}{pk}} \left(\int_0^1 M_{k'}^{\frac{qk'}{k'-q}}(g)\, d\varrho \right)^{\frac{(k'-q)\mu}{qk'}} < \infty,$$

by Theorem 31 of **20**, provided only that $k > p$, $k' > q$, and

$$\frac{(k-p)\mu}{pk} + \frac{(k'-q)\mu}{qk'} = 1,$$

all of which conditions are satisfied.

If $p < 2$, $q < 2$ we can suppose that $k = 2$, and then we can replace (6.16) by

$$\mathfrak{H}(r) = \sum |a_n|\,|b_n|\, \varrho^{2n} \leqq M_2(\varrho, f)\, M_2(\varrho, g).$$

We then obtain Theorem 11 by the same argument as before. The case in which p or q is 2 requires a trivial modification of the argument which may be left to the reader.

If $p \geqq 1$, $q \geqq 1$, these theorems are naturally corollaries of Theorem 9, in virtue of the theorems of Fejér-Riesz and Hardy-Littlewood just referred to.

References.

(For numbers less than **18** see the list of papers at the end of **20**.)

18. L. Fejér und F. Riesz, Über einige funktiontheoretische Ungleichungen, Math. Zeitschr. **11** (1921), S. 305—314.

19. G. H. Hardy and J. E. Littlewood, Elementary theorems concerning power series with positive coefficients and moment constants of positive functions, Journ. für Math. **157** (1927), S. 141—158.

20. — — Some properties of fractional integrals (II), Math. Zeitschrift **34** (1931), S. 403—439.

21. — — G. Pólya, The maximum of a certain bilinear form, Proc. London Math. Soc. (2) **25** (1926), p. 265—282.

22. W. H. Young, On a class of parametric integrals and their application in the theory of Fourier series, Proc. Roy. Soc. (A), **85** (1911), p. 401—414.

23. — On the multiplication of successions of Fourier constants, ibidem **87** (1912), p. 331—339.

(Eingegangen am 27. April 1931.)

COMMENTS

p. 621. Parts of Theorems 1 and 3 correspond to the case $s = \infty$ of Theorem 3 of 1937, 4 (see T. M. Flett, *Pac. J. Math.* 25 (1968), 463–94).

The question whether P is convergent under the hypotheses of Theorem 2 is settled (negatively) in 1932, 6.

It has been shown by C.-T. Loo, *Duke Math. J.* 12 (1945), 373–80, that if, in Theorem 1, $p > 2$, and the hypothesis (iii) is replaced by

$$M_p(g') = O((1-\rho)^{k-1}(\log 1/(1-\rho))^{-\delta}),$$

where $\delta > 1$ (cf. p. 623, *line* 6 *from below*), then P is summable $|C, \alpha|$ for $\alpha > \frac{1}{2} - p$.

p. 622, *two lines following* (2.12). See 1932, 4, § 3.2.

An additional note on Parseval's Theorem.

By

G. H. Hardy and J. E. Littlewood in Cambridge.

1. In our paper "Some new cases of Parseval's Theorem"[1]) we left open a question to which we have since found the answer.

Suppose that $0 < \alpha \leqq 1$, that

$$f(z) = \sum_1^\infty a_n z^n, \qquad g(z) = \sum_1^\infty b_n z^n,$$

and that $f(z)$ and $g(z)$ belong to the classes

$$L^{\frac{1}{1+\alpha}}, \qquad \mathrm{Lip}\,\alpha$$

respectively. Is it true that $\sum a_n b_n$ is necessarily convergent? The answer is (as was to be expected) negative.

2. We shall prove (what is, except when $\alpha = 1$, rather more) that $\sum a_n b_n$ is not necessarily convergent when f belongs to $L^{\frac{1}{1+\alpha}}$ and g is the α-th integral of a bounded function. Since the Parseval series of f and g is identical with that of f_α and g^α, it is the same thing to prove that the proposition

(A) "$\sum a_n b_n$ is convergent whenever f^α belongs to $L^{\frac{1}{1+\alpha}}$ and g is bounded"

is false.

The "inequality" form of (A) would be

$$\left|\sum_1^N a_n b_n\right| \leqq K(\alpha) \operatorname{Max} |g| \left(\int_{-\pi}^{\pi} |f^\alpha|^{\frac{1}{1+\alpha}} d\theta\right)^{1+\alpha}, \tag{1}$$

and we begin by disproving this. The disproof of (A) itself is then a matter of familiar routine.

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[1]) Math. Zeitschr. **34** (1931), S. 620–633. We refer to this paper as P.

1932, 6 (with J. E. Littlewood) *Mathematische Zeitschrift*, 34, 634–6.

Suppose first that $\alpha = 1$. It is known[2]) that there are functions g for which $|g| \leqq 1$ and

(2) $$|B_N| = |b_1 + b_2 + \dots + b_N| > K \log N$$

for a constant positive K and an infinity of N. We choose such a g and such an N, and define $f(z) = f_N(z)$ by

(3) $$f^1(z) = a_1 z + 2 a_2 z^2 + 3 a_3 z^3 + \dots = \frac{z}{N+1}\left(\frac{1 - z^{N+1}}{1 - z}\right)^3.\,^{3)}$$

Then, writing ν for $N + 1$, we have

(4) $$\int |f^1|^{\frac{1}{2}} d\theta \leqq \frac{1}{\sqrt{\nu}} \int_{-\pi}^{\pi} \left|\frac{\sin \frac{1}{2}\nu\theta}{\sin \frac{1}{2}\theta}\right|^{\frac{3}{2}} d\theta \leqq \frac{K}{\sqrt{\nu}} \int_0^{\frac{1}{\nu}} \nu^{\frac{3}{2}} d\theta + \frac{K}{\sqrt{\nu}} \int_{\frac{1}{\nu}}^{\infty} \theta^{-\frac{3}{2}} d\theta < K.$$

On the other hand, the first N terms of $f(z)$ are

$$\frac{1}{2(N+1)} \sum_1^N (n+1) z^n,$$

and so

(5) $$\sum_1^N a_n b_n = \frac{2 b_1 + 3 b_2 + \dots + (N+1) b_N}{2(N+1)}$$

$$= \frac{1}{2} B_N - \frac{B_1 + B_2 + \dots + B_{N-1}}{2(N+1)} > K \log N,$$

since $B_N > K \log N$, by (2), and the second term, which is substantially a Cesàro mean formed from the bounded function g, is bounded. It follows from (4) and (5) that (1) is false when $\alpha = 1$ and f and g are chosen suitably.

The same example shows the falsity of (1) for general α, since

$$\left(\int |f^\alpha|^{\frac{1}{1+\alpha}} d\theta\right)^{1+\alpha} \leqq K(\alpha) \left(\int |f^1|^{\frac{1}{2}} d\theta\right)^{2}.\,^{4)}$$

3. The falsity of (A) itself may now be proved as follows. We suppose for simplicity of writing that $\alpha = 1$.

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[2]) H. Bohr, „Über die Koeffizientensumme einer beschränkten Potenzreihe", Göttinger Nachrichten (1916), S. 276—291 and (1917), S. 119—128. See also a paper by L. Neder, with the same title, Math. Zeitschr. **11** (1921), S. 115—123.

[3]) $f^1(z)$ is not quite the ordinary derivative $f'(z)$, just as $f^\alpha(z)$ is not quite the Liouville or Hadamard derivative of fractional order; but the differences are, for our purposes, trivial, and we ignore them. See § 2. 1 of P.

[4]) This follows (if we pay attention to the glosses in § 2. 1 of P) from Theorem 33 of our paper 21 (in the bibliography of P), with $1 - \alpha$ for α, $\frac{1}{2}$ for p, and so $\frac{1}{1+\alpha}$ for $q = \frac{p}{1 - p\alpha}$.

We define $g(z)$ and $f_N(z)$ as in § 2, but now take

$$f(z) = \sum_1^{\infty} \frac{1}{r^3} f_{N_r}(z), \tag{6}$$

where (N_r) is an increasing sequence of numbers which satisfy (2) and tend to infinity so rapidly that

$$\frac{\log N_r}{r^3(N_1 + N_2 + \ldots + N_{r-1})} \to \infty \tag{7}$$

and *a fortiori*

$$\frac{N_r}{N_{r-1}^2} \to \infty. \tag{8}$$

Then, first,

$$\int |f^1|^{\frac{1}{2}} d\theta \leqq \sum_1^{\infty} r^{-\frac{3}{2}} \int |f^1_{N_r}|^{\frac{1}{2}} d\theta < K. \tag{9}$$

If we write

$$f_{N_r}(z) = \sum a_{n,r} z^n$$

(so that $a_{n,r} \geqq 0$ and $a_{n,r} = 0$ if $n > 3N_r + 1$), we have

$$\sum a_{n,r} = \frac{1}{N_r + 1} \int_0^1 \left(\frac{1 - z^{N_r+1}}{1 - z} \right)^3 dz < K N_r, \tag{10}$$

and

$$\sum_1^{N_r} a_{n,s} = \frac{1}{2(N_s + 1)} \sum_1^{N_r} (n + 1) < K \frac{N_r^2}{N_s} \tag{11}$$

if $s > r$. Also $|b_n| \leqq 1$. Hence, if we write

$$\sum_1^{N_r} a_n b_n = \sum_1^{N_r} b_n \sum_{s=1}^{\infty} \frac{a_{n,s}}{s^3} = T_1 + T_2 + T_3, \tag{12}$$

where T_1, T_2, and T_3 are the contributions of the terms for which $s < r$, $s = r$ and $s > r$ respectively, we have

$$|T_1| \leqq \sum_{s=1}^{r} \frac{1}{s^3} \sum a_{n,s} < K(N_1 + N_2 + \ldots + N_{r-1}), \tag{13}$$

by (10), and

$$|T_3| \leqq \sum_{s=r+1}^{\infty} \frac{1}{s^3} \sum_{n=1}^{N_r} a_{n,s} < K \frac{N_r^2}{N_{r+1}}, \tag{14}$$

by (11). It follows from (14) and (8) that $T_3 \to 0$. Also, by (5),

$$|T_2| > K \frac{\log N_r}{r^3}; \tag{15}$$

and (13), (15), and (7) show that $\sum a_n b_n$ is divergent.

(Eingegangen am 5. September 1931.)

NOTES ON THE THEORY OF SERIES (XVIII): ON THE CONVERGENCE OF FOURIER SERIES

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1. In this note we give first our proof of a theorem (Theorem 1) which we stated in Note XIII. We then prove a new theorem (Theorem 2) which leads to another proof of the main theorem of Note XVII.

The first of these theorems requires some preliminary explanations. We are concerned with an integrable function $f(\theta)$ with the period 2π. We write

$$f(\theta) \sim \sum_{-\infty}^{\infty} c_n e^{ni\theta},$$

$\Sigma c_n e^{ni\theta}$ being the complex Fourier series of $f(\theta)$.

If $f(\theta)$ is measurable and $|f(\theta)| \log^+ |f(\theta)|$, where $\log^+ x = \max(0, \log x)$, is integrable, we say that $f(\theta)$ belongs to Z. A function of Z is necessarily integrable (belongs to L).

The numbers c_n^+ are the numbers $|c_n|$ rearranged so that

$$c_0^+ \geqq c_{-1}^+ \geqq c_1^+ \geqq c_{-2}^+ \geqq c_2^+ \geqq c_{-3}^+ \geqq \dots *.$$

Since $c_n \to 0$ when $|n| \to \infty$, such a rearrangement is possible.

We can now state our first theorem in the form

THEOREM 1†. *If $f(\theta)$ belongs to Z, then*

$$\Sigma \frac{c_n^+}{|n|+1} < \infty, \tag{1·1}$$

and

$$\Sigma \exp\left(-\frac{a}{|c_n|}\right) < \infty \tag{1·2}$$

for every positive a.

2. We show first that (1·2) is a corollary of (1·1). It follows from (1·1) that

$$\sum_{N-1}^{n} \frac{c_\nu^+}{\nu+1} < \epsilon,$$

* The notation is that of Gabriel (**1**): see also Hardy, Littlewood, and Pólya (**5**, Ch. 10).

† Our statement of the theorem differs slightly in form from that in **2**.

The theorem has been proved in another way, and extended to more general orthogonal series, by Zygmund: see Zygmund (**7**, pp. 234–5, Theorems 4, 5, 6).

where $N=[n^{\frac{1}{2}}]$, for $n>n_0(\epsilon)$; and *a fortiori* that

$$c_n^+ \sum_{N-1}^{n} \frac{1}{\nu+1} < \epsilon.$$

But
$$\sum_{N-1}^{n} \frac{1}{\nu+1} > \int_N^n \frac{dx}{x} = \log\frac{n}{N} \geqq \tfrac{1}{2}\log n,$$

and so
$$c_n^+ = o\left(\frac{1}{\log|n|}\right) \tag{2·1}$$

for large positive n; and similar reasoning shows that (2·1) is true also for negative n. Hence
$$e^{-a/c_n^+} \to 0,$$
when $n\to\infty$, more rapidly than any power of $|n|$, and
$$\Sigma e^{-a/|c_n|} = \Sigma e^{-a/c_n^+} < \infty$$
for every positive a.

3. For the proof of (1·1), we need two lemmas.

LEMMA A. *If*
$$f_R(\theta) = \sum_{-R}^{R} c_n e^{ni\theta}, \quad f_R^+(\theta) = \sum_{-R}^{R} c_n^+ e^{ni\theta}, \tag{3·1}$$

then
$$\int_{-\pi}^{\pi} |f_R(\theta)|^{2k} d\theta \leqq \int_{-\pi}^{\pi} |f_R^+(\theta)|^{2k} d\theta \tag{3·2}$$

for every positive integral k; and
$$\int_{-\pi}^{\pi} e^{b|f_R(\theta)|} d\theta \leqq 2\int_{-\pi}^{\pi} e^{b|f_R^+(\theta)|} d\theta \tag{3·3}$$

for every positive b.

We proved (3·2) for *cosine* polynomials (i.e. with $c_{-n}=c_n$) in **2**. The general inequality is Gabriel's*. To deduce (3·3) we have only to observe that

$$\int e^{b|f|} d\theta \leqq 2\int \cosh b|f|\, d\theta = 2\Sigma \frac{1}{2k!}\int (b|f|)^{2k} d\theta$$
$$\leqq 2\Sigma \frac{1}{2k!}\int (b|f^+|)^{2k} d\theta = 2\int \cosh b|f^+|\, d\theta \leqq 2\int e^{b|f^+|} d\theta.$$

LEMMA B. *If* $-\pi \leqq \theta \leqq \pi$ *and*
$$h(\theta) = \sum_{1}^{N} \frac{\cos n\theta}{n+1},$$

then
$$|h(\theta)| < \log\frac{1}{|\theta|} + A,$$

where A is a constant, for all θ and N.

Suppose that θ is positive and $m=[1/\theta]$. If $N \leqq m$, then
$$|h(\theta)| \leqq \sum_{1}^{N} \frac{1}{n+1} \leqq \log N \leqq \log\frac{1}{\theta}.$$

* Gabriel (**1**, p. **41**, Theorem **4**).

If $N > m$, then

$$|h(\theta)| \leqq \sum_{1}^{m} \frac{1}{n+1} + \left| \sum_{m+1}^{N} \frac{\cos n\theta}{n+1} \right|$$

$$< \log m + \frac{1}{m+2} \max_{m<\nu \leqq N} \left| \sum_{m+1}^{\nu} \cos n\theta \right|$$

$$< \log \frac{1}{\theta} + \frac{1}{m+2} |\operatorname{cosec} \tfrac{1}{2}\theta| < \log \frac{1}{\theta} + A.$$

4. Passing to the proof of Theorem 1, we define ν_n by

$$|c_n| = c^+_{\nu_n};$$

ν_n is a "permutation function" which assumes every integral value just once for an integral n. Then

$$S_N = \sum_{-N}^{N} \frac{c_n^+}{|n|+1} = \Sigma' \frac{|c_n|}{|\nu_n|+1} = \sum_{-M}^{M} \lambda_n |c_n|,$$

say, where

$$M = M(N), \quad M \geqq N,$$

Σ' is a sum over those n for which $|\nu_n| \leqq N$, and λ_n takes, in some order, the values

$$1 \text{ (once)}; \ \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N} \text{ (twice)}; \ 0 \, (2M - 2N \text{ times}).$$

Hence

$$S_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) g(\theta) \, d\theta, \tag{4·1}$$

where

$$g(\theta) = \sum_{-M}^{M} \lambda_n \operatorname{sgn} \bar{c}_n e^{-ni\theta} = \sum_{-M}^{M} \lambda_{-n} \operatorname{sgn} \bar{c}_{-n} e^{ni\theta}.$$

Now*

$$uv = l\left(u \frac{v}{l}\right) \leqq lu \log u + le^{(v-l)/l} \leqq lu \log^+ u + le^{(v-l)/l} \tag{4·2}$$

for $u > 0$, $l > 0$, and all real v. We take

$$u = |f(\theta)|, \quad v = |g(\theta)|, \quad l = 4.$$

It then follows from (4·1) and (4·2) that

$$S_N \leqq \frac{2}{\pi} \int_{-\pi}^{\pi} |f(\theta)| \log^+ |f(\theta)| \, d\theta + A \int_{-\pi}^{\pi} e^{\frac{1}{4}|g(\theta)|} \, d\theta, \tag{4·3}$$

where A is again a constant. The first term is independent of N, and the second is less than

$$A \int_{-\pi}^{\pi} e^{\frac{1}{4}|g^+(\theta)|} \, d\theta,$$

by Lemma A. But $g^+(\theta) = 1 + 2h(\theta)$, where $h(\theta)$ is the function of Lemma B, and so

$$\int_{-\pi}^{\pi} e^{\frac{1}{4}|g^+(\theta)|} \, d\theta < A \int_{-\pi}^{\pi} |\theta|^{-\frac{1}{2}} \, d\theta < A.$$

Hence

$$S_N \leqq \frac{2}{\pi} \int_{-\pi}^{\pi} |f(\theta)| \log^+ |f(\theta)| \, d\theta + A,$$

which proves the theorem.

* 'Young's inequality'. See for example Hardy, Littlewood, and Pólya (**5**, p. 61, Theorem 63, and p. 107).

5. In what follows we are concerned with the behaviour of the Fourier series of $f(\theta)$ at a particular point, and make the usual formal simplifications. We suppose that $f(\theta)$ is real and even, that

$$f(\theta) \sim \tfrac{1}{2}a_0 + \sum_{1}^{\infty} a_n \cos n\theta,$$

that the particular value of θ to be considered is 0, and that the sum of the series for $\theta = 0$ is (if it exists) also to be 0.

We write $$s_n = \tfrac{1}{2}a_0 + a_1 + \dots + a_n$$
and denote by $$s_1^*, s_2^*, \dots, s_n^*$$
the values of $|s_1|, |s_2|, \dots, |s_n|$ rearranged in decreasing order. It is to be observed that

$$s_\nu^* = s_\nu^*(n) \quad (1 \leqq \nu \leqq n)$$

is a function of both n and ν.

THEOREM 2. *If*
$$f(\theta) = o\left(\log \frac{1}{\theta}\right)^{-1} \tag{5·1}$$
for small positive θ, then
$$\sum_{0}^{n} \frac{s_\nu^*}{\nu+1} = o(\log n). \tag{5·2}$$

6. We suppose that $|s_m| = s_{\nu_m}^* \quad (0 \leqq m \leqq n)$,
so that ν_m is now a permutation function of the indices $0, 1, 2, \dots, n$.

We have
$$s_m = \frac{2}{\pi}\int_0^\pi \frac{f(\theta)}{2\sin\frac{1}{2}\theta} \sin(m+\tfrac{1}{2})\theta \, d\theta \tag{6·1}$$
and so
$$S_n = \sum_{0}^{n} \frac{s_m^*}{m+1} = \sum_{0}^{n} \frac{|s_m|}{\nu_m+1} = \frac{2}{\pi}\int_0^\pi F(\theta)\, g(\theta)\, d\theta, \tag{6·2}$$
where
$$F(\theta) = \frac{f(\theta)}{2\sin\frac{1}{2}\theta}$$
and
$$g(\theta) = \sum_{0}^{n} \frac{\operatorname{sgn} s_m}{\nu_m+1} \sin(m+\tfrac{1}{2})\theta. \tag{6·3}$$
We choose δ so that $|f(\theta)| \leqq 1$ for $0 < \theta < \delta$, write
$$S_n = \frac{2}{\pi}\left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_\delta^\pi\right) F(\theta)\, g(\theta)\, d\theta = J_1 + J_2 + J_3, \tag{6·4}$$
and consider the three integrals separately.

7. (1) If $n \geqq n_0 = n_0(\epsilon)$, then, in J_1,
$$|f(\theta)| < \frac{\epsilon}{\log(1/\theta)} \leqq \frac{\epsilon}{\log n},$$
$$\left|\frac{g_1(\theta)}{2\sin\frac{1}{2}\theta}\right| \leqq \tfrac{1}{2}\pi \left|\frac{g(\theta)}{\theta}\right| \leqq \tfrac{1}{2}\pi(n+\tfrac{1}{2}) \sum_{0}^{n} \frac{1}{\nu_m+1} < An\log n,$$
and so
$$|J_1| \leqq A\epsilon n \int_0^{1/n} d\theta = A\epsilon$$
for $n \geqq n_0$. Hence
$$J_1 = o(1). \tag{7·1}$$

8. (2) By the Riemann-Lebesgue theorem, we can choose $m_0(\epsilon)$ so that

$$|j_m| = \left|\frac{2}{\pi}\int_\delta^\pi F(\theta)\sin(m+\tfrac{1}{2})\theta\,d\theta\right| < \epsilon$$

for $m \geqq m_0$. Then

$$|J_3| = \left|\sum_0^n \frac{\operatorname{sgn} s_m}{\nu_m+1} j_m\right| \leqq \frac{2}{\pi}\int_\delta^\pi |F(\theta)|\,d\theta \sum_0^{m_0-1}\frac{1}{\nu_m+1} + \epsilon\sum_{m_0}^n \frac{1}{\nu_m+1}$$

$$\leqq \frac{2m_0}{\pi}\int_\delta^\pi |F(\theta)|\,d\theta + \epsilon\log n < (\epsilon + o(1))\log n;$$

and so
$$J_3 = o(\log n). \tag{8·1}$$

9. (3) It is sufficient, after (6·4), (7·1), and (8·1), to prove that

$$J_2 = o(\log n); \tag{9·1}$$

and we prove this by an argument very much like that of § 4.

We apply (4·2) to J_2, taking

$$u = |F(\theta)| = \left|\frac{f(\theta)}{2\sin\frac{1}{2}\theta}\right|, \quad v = |g(\theta)|, \quad l = 2.$$

Then
$$|J_2| \leqq \frac{4}{\pi}\int_{1/n}^\delta |F(\theta)|\log^+|F(\theta)|\,d\theta + A\int_{1/n}^\delta e^{\frac{1}{2}|g(\theta)|}\,d\theta = K_2 + L_2, \tag{9·2}$$

say. Since $|f(\theta)| < 1$ if $\theta < \delta$, we have

$$\log^+|F(\theta)| \leqq \log\frac{1}{2\sin\frac{1}{2}\theta} < A\log\frac{1}{\theta}$$

and so
$$K_2 = \frac{4}{\pi}\int_{1/n}^\delta o\left(\frac{1}{\theta\log(1/\theta)}\right) O\left(\log\frac{1}{\theta}\right) d\theta = o\left(\int_{1/n}^\delta \frac{d\theta}{\theta}\right) = o(\log n). \tag{9·3}$$

On the other hand, if

$$\frac{\operatorname{sgn} s_m}{\nu_m+1} = \rho_m,$$

we have
$$g(\theta) = \mathfrak{I}(e^{\frac{1}{2}i\theta}\gamma(\theta)), \quad |g(\theta)| \leqq |\gamma(\theta)|,$$

where
$$\gamma(\theta) = \sum_0^n \rho_m e^{mi\theta},$$

a polynomial of degree n whose coefficients have (in some order) the absolute values

$$1, \frac{1}{2}, \ldots, \frac{1}{n+1}.$$

Hence, in the notation of § 1,

$$\gamma^+(\theta) = 1 + \tfrac{1}{2}e^{-i\theta} + \tfrac{1}{3}e^{i\theta} + \tfrac{1}{4}e^{-2i\theta} + \ldots,$$

the last term having modulus $1/(n+1)$ and an argument depending upon the parity of n. The imaginary part of $\gamma^+(\theta)$ is bounded in n and θ, and the modulus of the real part is less than

$$\log\left(\frac{1}{|\theta|}\right) + A,$$

by Lemma B; and so
$$|\gamma^+(\theta)| < \log\left(\frac{1}{|\theta|}\right) + A. \tag{9·4}$$

Using (9·4) and Lemma A, we obtain

$$L_2 = A\int_{1/n}^{\delta} e^{\frac{1}{4}|g(\theta)|}\,d\theta \leqq A\int_{-\pi}^{\pi} e^{\frac{1}{4}|g(\theta)|}\,d\theta \leqq A\int_{-\pi}^{\pi} e^{\frac{1}{2}|\gamma(\theta)|}\,d\theta$$
$$\leqq A\int_{-\pi}^{\pi} e^{\frac{1}{2}|\gamma^{+}(\theta)|}\,d\theta < A\int_{-\pi}^{\pi} |\theta|^{-\frac{1}{2}}\,d\theta < A. \tag{9·5}$$

The theorem follows from (9·2), (9·3), and (9·4).

10. THEOREM 3. *Suppose that $f(\theta)$ satisfies* (5·1), *and that*

$$\nu = \nu(\delta, n)$$

is the number of values of m for which

$$m \leqq n, \quad |s_m| \geqq \delta.$$

Then
$$\nu = O(n^{\epsilon}) \tag{10·1}$$
for every positive δ and ϵ.

This is a corollary of Theorem 2; for

$$\sum_1^n \frac{s_m^+}{m} \geqq \delta \sum_1^{\nu} \frac{1}{m} > \delta \log \nu,$$

and so
$$\log \nu = o(\log n),$$
for every positive δ.

THEOREM 4. *Under the same conditions*

$$\sum_1^n |s_m|^q = o(n) \tag{10·2}$$

for every positive q.

This is a theorem of "strong summability", and says the more the larger q. It is a corollary of Theorem 3. For

$$s_n = O(\log n),$$

since $f(\theta)$ is bounded near the origin*. We choose δ so that $\delta^q < \eta$, and take $\epsilon = \frac{1}{2}$. Then

$$\sum_1^n |s_m|^q \leqq \sum_1^{\nu} O(\log n)^q + \delta^q n \leqq O(n^{\frac{1}{2}}(\log n)^q) + \eta n < 2\eta n$$

for large n.

11. THEOREM 5. *Suppose that $f(\theta)$ satisfies* (5·1), *and that*

$$a_n > -An^{-\zeta} \tag{11·1}$$

for a positive A and ζ. Then $s_n \to 0$.

If not, then one of
$$s_n > 2\delta, \quad s_n < -2\delta \tag{11·2}$$
is true for a positive δ and an infinity of n. Suppose, for example, that

$$s_n > 2\delta \quad (n = n_i,\ i = 1, 2, \ldots).$$

* Indeed (5·1) involves $s_n = o(\log\log n)$.

Then
$$s_n > 2\delta + a_{n_i+1} + \ldots + a_n > 2\delta - A \sum_{n_i+1}^{n} m^{-\zeta}$$
$$> 2\delta - A\,(n - n_i)\, n_i^{-\zeta} > \delta$$
for
$$n_i \leqq n < n_i + \frac{\delta}{A} n_i^{\zeta},$$
i.e. for a block of terms, of order n_i^{ζ}, near $n = n_i$. This contradicts Theorem 3.

The second of the possibilities (11·2) may be disposed of similarly (using a block of terms to the left of $n = n_i$).

12. The main theorem of Note XVII asserted that $s_n \to 0$ when (i) $f(\theta)$ satisfies (5·1) and (ii)
$$a_n = O\,(n^{-\zeta}) \tag{12·1}$$
for a positive ζ. We have now given three proofs of this theorem, viz. (*a*) a "direct" proof, which is substantially that given in Zygmund's book*, (*b*) a "Tauberian" proof, written out in **4**, and (*c*) the proof given here.

We have also proved two generalizations of the theorem†. We may replace (5·1) by
$$\int_0^{\theta} |f(t)|\, dt = o\left(\frac{t}{\log\,(1/t)}\right), \tag{12·2}$$
and we may replace (12·1) by the one-sided condition (11·1). Proof (*a*) effects the first generalization, but not the second, while proof (*b*) effects both, but requires, for the second, a very difficult Tauberian theorem. Our present method (*c*) covers the second, and more difficult, generalization (which appears in Theorem 5), and it is a little curious that it does not cover the first. We do not know whether Theorem 2 remains true when (5·1) is replaced by (12·2).

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(**5**) G. H. Hardy, J. E. Littlewood and G. Pólya: *Inequalities* (Cambridge, 1934).

(**6**) G. W. Morgan: "On the convergence criteria for Fourier series of Hardy and Littlewood", *Annali d. R. Scuola Normale Sup. di Pisa* (2).

(**7**) A. Zygmund: *Trigonometrical series* (Warsawa-Lwów, 1935).

* Zygmund (**7**, 35).

† Hardy and Littlewood (**3**, Theorem 3, and **4**, Theorems 3 and 9). Morgan (**6**) has extended the theorem in other directions.

CORRECTION

p. 322, *line* 5. For (9.4) read (9.5).

COMMENT

p. 323. In an attempt to answer the question raised in the last two lines, O. Szász, *Bull. Amer. Math. Soc.* 48 (1942), 705–11, proved that if f satisfies (12.2) and there exists $c > 0$ such that $f(\theta) = O(|\theta|^{-c})$ as $\theta \to 0$, then (5.2) holds.

NOTES ON THE THEORY OF SERIES (XIX): A PROBLEM CONCERNING MAJORANTS OF FOURIER SERIES

By G. H. HARDY (*Cambridge*) and J. E. LITTLEWOOD (*Cambridge*)

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1. Statement of the problem

1.1. SUPPOSE that $f(\theta)$ is a complex and integrable function of the real variable θ, with period 2π, and that

$$c_m = c_m(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-mi\theta}\,d\theta$$

is its typical complex Fourier constant, so that

$$\mathbf{S}(f) = \sum_{-\infty}^{\infty} c_m e^{mi\theta} \tag{1.1.1}$$

is the Fourier series of $f(\theta)$. We call any formal series

$$\sum_{-\infty}^{\infty} C_m e^{mi\theta} \tag{1.1.2}$$

in which

$$C_m \geqslant |c_m|$$

a majorant of (1.1.1). If (1.1.2) is itself a Fourier series, say the Fourier series $\mathbf{S}(F)$ of $F(\theta)$, then we say that F is a majorant of f; and we write

$$F \succ f, \qquad f \prec F, \qquad \mathbf{S}(F) \succ \mathbf{S}(f), \qquad \mathbf{S}(f) \prec \mathbf{S}(F).$$

If

$$C_m = |c_m|$$

for every m, we say that F is the exact majorant of f.

1.2. We consider relations of inequality between

$$J_r(f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^r\,d\theta\right)^{1/r},$$

where $r > 1$, and $J_r(F)$.† There are special cases in which it is easy to establish such relations.

(*a*) If $r = 2$ then

$$J_r(f) = (\sum |c_m|^2)^{\frac{1}{2}} \leqslant (\sum C_m^2)^{\frac{1}{2}} = J_r(F).$$

In this case there is equality when F is the exact majorant.

† If f or F is not L^r, then $J_r(f)$ or $J_r(F)$ is to be interpreted as ∞. This gloss does not affect our results and need not be referred to again.

1935, 7 (with J. E. Littlewood) *Quarterly Journal of Mathematics*, 6, 304–15.

(b) If $r = 2k$, where k is a positive integer, and $J_{2k}(f) < \infty$, then the Fourier series of f^k is $\sum c_m^{(k)} e^{mi\theta}$, where

$$c_m^{(k)} = \sum_{m_1+m_2+\ldots+m_k=m} c_{m_1} c_{m_2} \ldots c_{m_k};$$

and
$$|c_m^{(k)}| \leqslant C_m^{(k)}, \qquad f^k \prec F^k.$$

Hence
$$J_{2k}(f) = J_2^{2/k}(f^k) \leqslant J_2^{2/k}(F^k) = J_{2k}(F) \tag{1.2.1}$$

for any majorant F.

(c) Let $r = 1$, $\phi = |f|^{\frac{1}{2}}$, and let Φ be the exact majorant of ϕ. Then $F = \Phi^2$ is a majorant of f, and for this special majorant

$$J_1(F) = J_2^2(\Phi) = J_2^2(\phi) = J_1(f).$$

1.3. In what follows we suppose

$$r > 1,\dagger \qquad 1 < p \leqslant 2, \qquad q > 2.$$

As usual, we write
$$r' = \frac{r}{r-1}, \qquad \frac{1}{r}+\frac{1}{r'} = 1.$$

Here r may be a p or a q; p' is a q and q' a p.

The first suggestion of (1.2.1) is that

$$J_q(f) \leqslant J_q(F) \tag{1.3.1}$$

for every q and every majorant F. This is untrue and, since it is the falsity of (1.3.1) which first reveals the difficulties of our problem, we prove it at once by an example.

If
$$f(z) = 1+z-az^3,$$

where a is real and positive, then

$$\phi = f^{\frac{3}{2}} = 1+\tfrac{3}{2}z+\tfrac{3}{8}z^2-cz^3+\ldots,$$

where
$$c = \tfrac{1}{16}+\tfrac{3}{2}a,$$

for small z. Hence, if $z = re^{i\theta}$ and r is small,‡

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z)|^3\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\phi(z)|^2\, d\theta = 1+\tfrac{9}{4}r^2+\tfrac{9}{64}r^4+c^2r^6+\ldots,$$

which is greater than it would be if a were replaced by $-a$. In other words, if

$$f(\theta) = 1-re^{i\theta}-ar^3e^{3i\theta}$$

and F is the exact majorant of f, then

$$J_3(f) > J_3(F)$$

for sufficiently small r; and this contradicts (1.2.1).

† (c) of § 1.2 contains all we have to say about the case $r = 1$.

‡ This use of r is temporary and will not cause confusion.

1.4. It remains possible that

$$J_q(f) \leqslant A_q J_q(F), \tag{1.4.1}$$

where A_q is a function of q only (and $A_q = 1$ for $q = 2, 4, 6, \ldots$). Whether this is true or not we cannot say, but it is a quite plausible conjecture; and the conjecture leads naturally to another. In problems of this character there is very usually a 'skew symmetry', resulting in a reversal of the sign of inequality, about the index 2;† and it is natural to suppose that, if (1.4.1) is true, then also

$$J_p(F) \leqslant A_p J_p(f), \tag{1.4.2}$$

not indeed for every majorant F (which is plainly impossible), but at any rate for *some* majorant.

1.5. Our main theorem here (Theorem 1) contains a partial solution of this problem. We cannot prove (1.4.1) or refute it, but we can show that (1.4.1) implies (1.4.2). More precisely, the truth of (1.4.1), for a particular q (and all majorants F), implies that of (1.4.2) for $p = q'$ (and some majorant F). Since (1.4.1) is true at any rate when $q = 2k$, we obtain a proof of (1.4.2), which is not trivial in any case (apart from $p = 2$), for $p = \frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \ldots$.

It is instructive to contrast our argument with those used, by other writers and by ourselves, in similar problems such as the Young-Hausdorff problem or M. Riesz's problem of conjugate functions.‡ In all these problems the case $p = q = 2$ is trivial, and the full proof may be divided into three stages. We have to establish two propositions

$$P(p), \qquad Q(q);$$

Q may be (as in Riesz's problem) of the same form as P, but more usually (as here) it is not. We proceed as follows.

(*a*) We prove that $\qquad Q(q) \rightarrow P(q')$.§

This proposition ('reciprocation') reduces the problem to the proof of $Q(q)$.

(*b*) We prove that $\qquad Q(q_1) . Q(q_2) \rightarrow Q(q)$

for $q_1 \leqslant q \leqslant q_2$. This proposition ('interpolation') reduces the proof

† For example in the Hausdorff inequalities

$$J_{p'}(f) \leqslant \left(\sum |c_m|^p\right)^{1/p}, \quad \left(\sum |c_m|^q\right)^{1/q} \leqslant J_{q'}(f);$$

or in our own inequalities involving

$$J_p(f), \ \left(\sum (|m|+1)^{p-2} |c_m|^p, \ldots, \ldots \right.$$

‡ The original model is Hausdorff's (Hausdorff, **4**).

§ q' is a p. The symbol $\rightarrow$, when standing between *propositions*, is Hilbert's symbol of implication.

of $Q(q)$ either to a proof of $Q(q)$ in the two extreme cases, $q = 2$ and $q = \infty$,† or to its proof for some particular sequence of values of q, usually the even integers.‡ When the operations or functionals involved in the problem are *linear*, 'interpolation' may be inferred from M. Riesz's theorem concerning the convexity of linear functionals.§

(*c*) We prove, say, $Q(2k)$, and this generally requires some special trick. It is here that the individuality of the problem is likely to declare itself most clearly, the other stages of the proof being usually effected by appeals to general theorems or arguments now of standardized types.

This is the standard position; the position here is abnormal. Here stage (*c*) of the argument is trivial, as we saw in § 1.2. Stage (*b*), on the other hand, collapses altogether. The operations involved here are not linear; F, even the exact majorant, is not a linear functional of f. There is therefore no general theorem to which we can appeal for the proof of 'interpolation', and we are unable to prove it by any special device. Indeed 'interpolation', in the most obvious sense (and the sense in which it would be true if the operations *were* linear), is *false*, since (1.2.2) is true for $q = 2k$ and not for general q.

It is perhaps rather surprising in these circumstances that, in spite of the non-linearity of our operations, we can put through stage (*a*) of the argument and so prove Theorem 1. The arguments generally follow familiar lines, but the non-linearity gives them some curious twists.

2. Proof of the theorem

2.1. Our main theorem‖ is

THEOREM 1. *If p' is a q for which* (1.4.1) *is true, for every f and* EVERY *majorant F, and in particular if*

$$p = \frac{2k}{2k-1}, \qquad p' = 2k \qquad (k = 1, 2, 3,...),$$

then (1.4.2) *is true for every f and* SOME *majorant F.*

We write $$f_n = \sum_{-n}^{n} c_m e^{mi\theta}.$$

† As in F. Riesz's proof of Hausdorff's theorems; see F. Riesz (**6**).

‡ As in Hausdorff's own proof of his theorems (where the case $q = 2k$ is derived from Young), in one of M. Riesz's proofs of the theorem about conjugate functions, or in our own proof of the theorems referred to on p. 306, note †. See Hausdorff (**4**), M. Riesz (**7**), Hardy and Littlewood (**1**).

§ M. Riesz (**7**); see also Hardy, Littlewood, and Pólya (**3**), 214–19, Theorem **295**).

‖ Which we stated without proof in (**2**).

Thus f_n is the 'Fourier polynomial' of f of degree n. We use a similar notation for other functions.

We denote by $F^{(n)}$ any integrable function such that $f_n \prec (F^{(n)})_n$, i.e. such that
$$|c_m| \leqslant C_m \qquad (|m| \leqslant n).$$
Thus $F^{(n)}$ depends to some extent upon f_n, but has an arbitrary 'tail', in which the coefficients may be complex. When $n \to \infty$, $F^{(n)}$ becomes a majorant of f. We define $\lambda_q(n)$ by
$$\lambda_q(n) = \max \frac{J_q(f_n)}{J_q(F^{(n)})}. \tag{2.1.1}$$
Here 'max' means 'upper bound', and the upper bound is taken over all f_n and all $F^{(n)}$ associated with each f_n. We shall see in a moment that this upper bound exists.

Next, let g be any function of L^p, G_n any polynomial of degree n which majorizes g_n, and
$$\mu_p(g,n) = \min \frac{J_p(G_n)}{J_p(g)}$$
(the lower bound for all such G_n). This lower bound is attained for one or many G_n, which we call G_n^*; and we can select one G_n^* so as to be uniquely defined by g.† We now define $\mu_p(n)$ by
$$\mu_p(n) = \max \mu_p(g,n) = \max \frac{J_p(G_n^*)}{J_p(g)}, \tag{2.1.2}$$
the upper bound being taken over all g and the uniquely corresponding G_n^*. In this case also we shall see that the upper bound exists.

2.2. LEMMA 1. *The bounds $\lambda_q(n)$, $\mu_p(n)$ exist.*

(1) Given f_n, $F^{(n)}$, we have‡
$$|c_m| = |c_m(f_n)| \leqslant c_m(F^{(n)}) \leqslant J_1(F^{(n)}) \leqslant J_q(F^{(n)}) \qquad (|m| \leqslant n),$$
$$|f_n| \leqslant (2n+1)\max_{|m|\leqslant n}|c_m| \leqslant (2n+1)J_q(F^{(n)}),$$
$$\frac{J_q(f_n)}{J_q(F^{(n)})} \leqslant \frac{\max|f_n|}{J_q(F^{(n)})} \leqslant 2n+1.$$
Hence $\lambda_q(n) \leqslant 2n+1$.

† The existence of one G_n^* at any rate is shown by the 'Bolzano-Weierstrass' argument, applied to the space of $2(2n+1)$ real coordinates defined by the real and imaginary parts of the $2n+1$ coefficients.

If there are many G_n^*, let $u+iv$ be the central coefficient of a G_n^*, and select those for which u has its maximum value. If this is not enough to define a unique polynomial, apply the same process to v, and then, if necessary, to other coefficients.

‡ We use $c_m(h)$ generally for the mth Fourier coefficient of h. Without an explicit argument, c_m means $c_m(f)$.

(2) Given g, let G_n be the *exact* majorant of g_n, i.e.

$$G_n = \sum_{-n}^{n} |c_m(g)| e^{mi\theta}.$$

By the definition of G_n^*,

$$J_p(G_n^*) \leqslant J_p(G_n).$$

Also
$$c_m(G_n) = |c_m(g)| \leqslant J_1(g) \leqslant J_p(g),$$

$$J_p(G_n) \leqslant \max|G_n| \leqslant (2n+1)\max c_m(G_n) \leqslant (2n+1)J_p(g),$$

and so
$$\frac{J_p(G_n^*)}{J_p(g)} \leqslant \frac{J_p(G_n)}{J_p(g)} \leqslant 2n+1.$$

Thus both $\lambda_q(n)$ and $\mu_p(n)$ exist (and do not exceed $2n+1$).

2.3. In what follows we write generally

$$\chi_r(\zeta) = |\zeta|^{r-1}\operatorname{sgn}\bar{\zeta},$$

where the bar denotes the conjugate. Thus $\chi_r(\zeta) = \rho^{r-1}e^{-i\alpha}$ when $\zeta = \rho e^{i\alpha}$.

LEMMA 2. *If h is a trigonometrical polynomial, and t and γ are real, then*

$$\left(\frac{d}{dt}\frac{1}{2\pi}\int_{-\pi}^{\pi} |h+te^{i\gamma+mi\theta}|^r\, d\theta\right)_{t=0} = r\mathbf{R}\{e^{i\gamma}c_{-m}[\chi_r(h)]\}. \qquad (2.3.1)$$

We may differentiate under the integral sign. Also, if $h = \rho e^{i\alpha}$, we have

$$|h_t| = |h+te^{i\gamma+mi\theta}| = \sqrt{\{\rho^2+t^2+2\rho t\cos(m\theta+\gamma-\alpha)\}},$$

$$\frac{d}{dt}|h_t|^r = r|h_t|^{r-2}\{t+\rho\cos(m\theta+\gamma-\alpha)\},$$

$$\frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |h_t|^r\, d\theta\right)_{t=0} = \frac{r}{2\pi}\int_{-\pi}^{\pi} |h|^{r-1}\cos(m\theta+\gamma-\alpha)\, d\theta$$

$$= \frac{r}{2\pi}\mathbf{R}\left(\frac{e^{i\gamma}}{2\pi}\int_{-\pi}^{\pi} |h|^{r-1}\operatorname{sgn}\bar{h}\, e^{mi\theta}\, d\theta\right),$$

the formula required.

2.4. Our next lemma is decidedly more difficult, and its proof contains the kernel of the proof of Theorem 1.

LEMMA 3. *For every q and n*

$$\lambda_q(n) = \mu_{q'}(n). \qquad (2.4.1)$$

(1) We show first that $\lambda_q \leqslant \mu_{q'}$; (2.4.2)

this is the easier half of the proof.

We start from an f, an f_n, not null, and an associated $F^{(n)}$, take

$$g = \chi_q(f_n) = |f_n|^{q-1}\operatorname{sgn}\bar{f}_n,$$

and denote by G_n^* the function associated with g in the manner of § 2.1. We write

$$k_m = c_m(g), \qquad K_m^* = c_m(G_n^*).$$

Then
$$J_q^q(f_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_n g\, d\theta = \sum_{-n}^{n} c_m k_{-m}$$

$$\leqslant \sum_{-n}^{n} |c_m||k_{-m}| \leqslant \sum_{-n}^{n} C_m K_{-m}^*, \tag{2.4.3}$$

since $|c_m| \leqslant C_m$ and $|k_{-m}| \leqslant K_{-m}^*$, by the definitions of $F^{(n)}$ and G_n^* respectively. But

$$\sum_{-n}^{n} C_m K_{-m}^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^{(n)} G_n^*\, d\theta \leqslant J_q(F^{(n)}) J_{q'}(G_n^*). \tag{2.4.4}$$

Hence, by the definition of $\mu_{q'}$,

$$J_q^q(f_n) \leqslant J_q(F^{(n)}) J_{q'}(G_n^*) \leqslant J_q(F^{(n)}).\mu_{q'} J_{q'}(g) = \mu_{q'} J_q(F^{(n)}) J_q^{q-1}(f_n),$$

$$\frac{J_q(f_n)}{J_q(F^{(n)})} \leqslant \mu_{q'}, \tag{2.4.5}$$

which proves (2.4.2).

(2) Secondly, we prove that

$$\mu_{q'} \leqslant \lambda_q. \tag{2.4.6}$$

Suppose that g is L^p, G_n^* (not null) is the function associated with g, and

$$k_m = c_m(g), \qquad K_m^* = c_m(G_n^*). \tag{2.4.7}$$

We write†
$$H^{(n)} = \chi_p(G_n^*) = |G_n^*|^{p-1}\operatorname{sgn}\bar{G}_n^*, \tag{2.4.8}$$

$$L_m = c_m(H^{(n)}),$$

and we begin by proving some properties of the L_m.

(i) It follows from (2.4.8) and (2.4.7) that‡

$$H^{(n)} = |G_n^*|^{p-2}\bar{G}_n^* = |G_n^*|^{p-2}\sum_{-n}^{n} K_m^* e^{-mi\theta} \tag{2.4.9}$$

(since K_m^* is real). Since

$$|G_n^*(-\theta)| = |\overline{G_n^*(\theta)}| = |G_n^*(\theta)|,$$

† It will be shown later that $H^{(n)}$ is related to an h_n as $F^{(n)}$ is to f_n in § 2.1: the notation anticipates this.

‡ Unless $G_n^* = 0$, which can happen at most for a finite number of values of θ.

the first factor on the right of (2.4.9) is real and even, and has, therefore real Fourier coefficients. The second factor also has real coefficients, and so therefore has $H^{(n)}$. Hence

(*a*) *L_m is real for all m.*

(ii) If
$$K_m^* > |k_m|, \tag{2.4.10}$$
then, by the minimal property of G_n^*,
$$J_p^p(G_n^*+te^{mi\theta}) \geqslant J_p^p(G_n^*) \tag{2.4.11}$$
for small real t of either sign. Hence
$$\left\{\frac{d}{dt}J_p^p(G_n^*+te^{mi\theta})\right\}_{t=0} = 0;$$
and
$$L_{-m} = \mathbf{R}(L_{-m}) = \mathbf{R}\{c_{-m}(\chi_p(G_n^*))\} = 0,$$
by Lemma 2 (with $\gamma = 0$). Thus

(*b*) *$L_{-m} = 0$ if $K_m^* > |k_m|$.*

(iii) This argument fails when $K_m^* = |k_m|$, since $K_m = c_m(G)$ has to satisfy $K_m \geqslant |k_m|$, and t cannot be negative. But (2.4.11) is still true for *positive* t (and $|m| \leqslant n$), and so $L_{-m} \geqslant 0$. Hence

(*c*) *$L_{-m} \geqslant 0$ for $|m| \leqslant n$.*

Hence
$$(H^{(n)})_n = \sum_{-n}^{n} L_m e^{mi\theta}$$
has non-negative coefficients. And if we write
$$l_m = L_m \operatorname{sgn} \bar{k}_{-m} \quad (|m| \leqslant n), \qquad l_m = 0 \quad (|m| > n), \tag{2.4.12}$$
then $(H^{(n)})_n$ is a majorant of
$$h_n = \sum_{-n}^{n} l_m e^{mi\theta},$$
and

(*d*) *$H^{(n)}$ is of the type $H^{(n)}$ of § 2.1, i.e. is related to h_n as $F^{(n)}$ is to f_n in § 2.1.*

Now
$$J_p^p(G_n^*) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G_n^* H^{(n)}\, d\theta = \sum_{-n}^{n} K_m^* L_{-m} = \sum_{-n}^{n} |k_m| L_{-m},$$
on account of (*b*). This, by (2.4.12), is
$$\sum_{-n}^{n} k_m l_{-m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g h_n\, d\theta \leqslant J_p(g) J_{p'}(h_n).$$

Combining these results, and using the definition of $\lambda_{p'}$, we obtain

$$J_p^p(G_n^*) \leqslant J_p(g).\lambda_{p'} J_{p'}(H^{(n)}) = \lambda_{p'} J_p(g) J_p^{p-1}(G_n^*),$$

$$\frac{J_p(G_n^*)}{J_p(g)} \leqslant \lambda_{p'},$$

and therefore (2.4.6). This completes the proof of Lemma 3.

2.5. Lemma 4. *For every r, f, and n,*

$$J_r(f_n) \leqslant A_r J_r(f). \tag{2.5.1}$$

This is one of M. Riesz's theorems.†

Lemma 5. *If* (1.4.1) *is true for a particular q, all f, and all majorants F of f, then*

$$J_q(f_n) \leqslant A_q J_q(F^{(n)}) \tag{2.5.2}$$

for every f_n and every $F^{(n)}$ associated with f_n as in § 2.1. *In particular* (2.5.2) *is true when $q = 2k$.*

Since $F_n^{(n)}$ is a majorant of f_n,

$$J_q(f_n) \leqslant A_q J_q(F_n^{(n)}) \leqslant A_q J_q(F^{(n)}),$$

by Lemma 4.

2.6. We can now prove Theorem 1. Since (2.5.2) is true for all f_n and $F^{(n)}$,

$$\lambda_q(n) \leqslant A_q,$$

and therefore, by Lemma 3,

$$\mu_{q'}(n) \leqslant A_q.$$

If then $p = q'$, $q = p'$, we have

$$J_p(G_n^*) \leqslant A_{p'} J_p(g)$$

for every n and every associated pair g and G_n^*. There is therefore‡ a sequence (n_ν) such that $G_{n_\nu}^*$ converges weakly to a G^* for which

$$J_p(G^*) \leqslant A_{p'} J_p(g).$$

Finally

$$c_m(G_{m_\nu}^*) \geqslant |c_m(g)|$$

if $n_\nu \geqslant m$, so that

$$c_m(G^*) = \lim_{\nu\to\infty} c_m(G_{n_\nu}^*) \geqslant |c_m(g)|$$

for all m, and G^* is a majorant of g.

† M. Riesz (**8**), Theorem 5.

‡ F. Riesz (**5**), § 7.

3. An appendix

3.1. We have now completed the substance of the paper, but we add a few minor observations. These do not take us far, but, since they may have some bearing on the unsolved problem, are probably worth recording. It is inevitable in the circumstances that they should seem rather disjointed.

We require

LEMMA 6. *If f is L^r,*

$$g = \chi_r(f) = |f|^{r-1}\operatorname{sgn}\bar{f},$$

and $$k_{-m} = c_{-m}(g) = 0$$

for a particular m, then $J_r(f)$ is increased strictly by any alteration of c_m.

Let $$f_1 = f + ce^{mi\theta}$$

where $c \neq 0$. If $J_r(f) = 0$, f is null, f_1 not null, and $J_r(f) < J_r(f_1)$. Otherwise we have

$$J_r^r(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} fg\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1 g\,d\theta\dagger$$

$$\leqslant J_r(f_1)J_{r'}(g) = J_r(f_1)J_r^{r-1}(f),$$

and so $J_r(f) \leqslant J_r(f_1)$. Equality demands

$$|f_1|^r \equiv |g|^{r'} \equiv |f|^r$$

and also $$\operatorname{sgn} f_1 \equiv \operatorname{sgn}\bar{g} \equiv \operatorname{sgn} f;$$

so $f_1 \equiv f$, which is false.

3.2. THEOREM 2. *If*

$$|c_m| \leqslant C_m \quad (|m| \leqslant n)$$

and, among all polynomials f_n which satisfy these conditions, f_n^* gives a maximum for $J_r(f_n)$, then

$$|c_m^*| = C_m \quad (|m| \leqslant n).$$

If $$g = \chi_r(f_n), \qquad k_m = c_m(g),$$

and $$k_{-m} = 0, \tag{3.2.1}$$

then, by Lemma 6, any change in c_m increases $J_r(f_n)$ strictly. Hence f_n^* cannot give the maximum unless $C_m = 0$, when there is nothing to prove.

If $|c_m^*| < C_m$ then

$$J_r(f_n^* + te^{i\gamma}e^{mi\theta}) \leqslant J_r(f_n^*)$$

† Since $k_{-m} = 0$.

for all real γ and small positive t. Hence, by Lemma 2,

$$\mathbf{R}(e^{i\gamma}k_{-m}) = 0$$

for all real γ. This implies $k_{-m} = 0$, which is impossible because $C_m \neq 0$.

3.3. From Theorem 2 we can deduce another theorem bearing on our main problem.

It is easy to see that there are 'maximal pairs' f_n, $F^{(n)}$ for which

$$J_q(f_n) = \lambda_q(n) J_q(F^{(n)}).\dagger$$

For such a pair f_n, $F^{(n)}$, all of (2.4.3)–(2.4.5) become equalities. From this it follows that g, G_n^* are a maximal pair in the conjugate problem, and that

$$G_n^* \equiv \lambda_q \chi_q(F^{(n)}) = \lambda_q |F^{(n)}|^{q-1} \operatorname{sgn} \bar{F}^{(n)}.$$

Also $$|c_m||k_{-m}| = C_m K^*_{-m},$$

and so $$|c_m| = C_m \qquad (|m| \leqslant n)$$

unless $$k_{-m} = 0.$$

Theorem 2, however, enables us to say more.

THEOREM 3. *If f_n and $F^{(n)}$ are a maximal pair, then $F_n^{(n)}$ is the exact majorant of f_n.*

For, if this were not so, we could leave $F^{(n)}$ fixed, and increase $J_q(f_n)/J_q(F^{(n)})$ by changing f_n.

Finally we prove

THEOREM 4. *If* $$0 \leqslant c_m \leqslant C_m,$$

and, among all polynomials f_n which satisfy these conditions, f_n^* gives a maximum for $J_r(f_n)$, then every c_m^* is 0 or C_m.

In the first place, as in § 2.4 (2) (i),

$$g = \chi_r(f) = |f|^{r-1} \operatorname{sgn} \bar{f}$$

has real Fourier coefficients.

Next, as in the proof of Theorem 2,

$$k_{-m} = c_{-m}(g) = 0,$$

† We select a maximizing sequence $f_n^{(N)}$, $F^{(n,N)}$. Then $f_n^{(N)}$ has a limit function f_n, when $N \to \infty$ appropriately. Also $F^{(n,N)}$ has a weak limit $F^{(n)}$ for a subsequence of N, and $$J_q(F^{(n)}) \leqslant \underline{\lim}\, J_q(F^{n,N}).$$

Hence $$J_q(f_n) \geqslant \lambda_q(n) J_q(F^{(n)});$$

and the opposite inequality is a consequence of the definition of $\lambda_q(n)$.

if $0 < c_m < C_m$; and then, by Lemma 6, any change in c_m will increase $J_r(f_n)$. Hence, for the maximal polynomial, c_m is 0 or C_m.

There are other proofs of Theorems 2–4, but we have chosen those connected most closely with the analysis of § 2.

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CORRECTIONS

p. 305, (1.2.1). The indices $2/k$ should be $1/k$.

p. 305, (*c*). The argument here seems to be incorrect, since a majorant of $|f|$ need not be a majorant of f (take, for example, $f(\theta) = \operatorname{sgn}\theta$). The argument can be corrected by defining ϕ by $\phi(\theta) = |f(\theta)|^{\frac{1}{2}}e^{\frac{1}{2}i\arg f(\theta)}$, where $\arg f(\theta)$ has its principal value (i.e. $-\pi < \arg f(\theta) \leqslant \pi$, and $\arg f(\theta) = 0$ when $f(\theta) = 0$). However, the measurability of this ϕ is not quite obvious, and to complete the proof we need the following lemma.

LEMMA. *Let f be a complex-valued function measurable on $[-\pi,\pi]$, and let g be the function with domain $[-\pi,\pi]$ whose value at each $\theta \in [-\pi,\pi]$ is the principal value of the argument of $f(\theta)$. Then g is measurable.*

Let $f = u+iv$, where u and v are real-valued, and let E_j ($j = 1, 2,\ldots, 6$) be the subsets of $[-\pi,\pi]$ in which respectively $u(\theta) > 0$, $u(\theta) < 0$ and $v(\theta) \geqslant 0$, $u(\theta) < 0$ and $v(\theta) < 0$, $u(\theta) = 0$ and $v(\theta) > 0$, $u(\theta) = 0$ and $v(\theta) < 0$, and $u(\theta) = v(\theta) = 0$. These six sets E_j are clearly measurable. For points θ belonging to E_1, E_2, E_3, E_4, E_5, E_6, $g(\theta)$ is equal to

$$\tan^{-1}\{v(\theta)/u(\theta)\},\ \pi+\tan^{-1}\{v(\theta)/u(\theta)\},\ -\pi+\tan^{-1}\{v(\theta)/u(\theta)\},\ \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi,\ 0,$$

respectively. The restriction of g to E_1 is the composite of the continuous

function $\tan^{-1}$ and a measurable function, and is therefore measurable on E_1. Similarly the restrictions of g to E_2 and E_3 are measurable on E_2 and E_3 respectively. Since g is constant on E_4, E_5, E_6, it follows that g is measurable, as required.

p. 305, *second line of* § 1.3. For $q > 2$ read $q \geqslant 2$.

p. 305, *line* 4 *from below*. Read $f(\theta) = 1+re^{i\theta}-ar^3e^{3i\theta}$.

p. 305, *last line*. Read (1.3.1).

p. 306, *note* †, *line* 2. The first sum should be $\sum |c_m|^p$, and similarly for the second. In line 4 transpose $(\sum$.

COMMENTS

§ 1.2. The results of (a) and (b) can be strengthened to the following forms.

(α) *If $f \in L^2$, then f has an exact majorant $F \in L^2$, and $J_2(F) = J_2(f)$.*

(β) *Let k be a positive integer. If $f \in L$, and there exists a majorant F of f such that $F \in L^{2k}$, then $f \in L^{2k}$, and $J_{2k}(f) \leqslant J_{2k}(F)$.*

Here (α) and the case $k = 1$ of (β) are trivial consequences of the Riesz–Fischer theorem, so that it is enough to prove (β) for $k > 1$. The argument of § 1.2 (b) shows that if F is a majorant of f, and both f and F belong to L^{2k}, then $J_{2k}(f) \leqslant J_{2k}(F)$. Now let $f \in L$, let F be a majorant of f such that $F \in L^{2k}$, and let s_n be the nth partial sum of the Fourier series of f. Then F is a majorant of s_n, whence $J_{2k}(s_n) \leqslant J_{2k}(F)$, by the case already proved. It follows that a subsequence (s_{n_r}) of the sequence (s_n) converges weakly in L^{2k} to a function $g \in L^{2k}$ such that $J_{2k}(g) \leqslant J_{2k}(F)$. Moreover, $c_m(g) = \lim c_m(s_{n_r}) = c_m(f)$ for all m, so that $f(x) = g(x)$ p.p., and this proves the result.

There is also a result corresponding to $k = +\infty$, namely:

(γ) *If F is a bounded function with non-negative Fourier coefficients C_n, then*

$$\sum_{-\infty}^{\infty} C_n \leqslant \operatorname{ess\,sup} |F|.$$

Hence if F is a majorant of f, then f is bounded and

$$\operatorname{ess\,sup} |f| \leqslant \operatorname{ess\,sup} |F|.$$

Here the Fourier series of F at 0 is bounded (C, 1), and, since it has non-negative coefficients, is therefore convergent.

§ 1.3. R. P. Boas, *J. d'Analyse Math.* 10 (1962–3), 253–71, has shown that (1.3.1) is false for every q greater than 2 that is not an even integer.

§ 2. The argument of this section actually proves the equivalence of results of the form of (1.4.1) and (1.4.2). More precisely, the argument gives the following:

Let $1 < p < +\infty$, and let $q = p' = p/(p-1)$. Then if either of the statements (I_p), (II_q) *below is true, so is the other.*

(I_p) *For each f of L^p there exists a majorant F of f (belonging to L^p) such that $J_p(F) \leqslant C_1(p)J_p(f)$, where $C_1(p)$ depends only on p.*

(II_q) *Each integrable f which possesses a majorant belonging to L^q itself belongs to L^q, and there exists a constant $C_2(q)$, depending only on q, such that*

$$J_q(f) \leqslant C_2(q)J_q(F)$$

for all majorants F of f.

Since (II_q) is true when $q = 2k$ $(k = 1, 2, \ldots)$, this implies that (I_p) is true when $p = 2k/(2k-1)$.

The result that $(\mathrm{I})_p$ implies $(\mathrm{II})_q$ has been proved independently by R. P. Boas, *loc. cit.* Boas includes also the case $p = 1$.

We can also show that (I_p) is false for $2 < p < +\infty$, so that (II_q) is false for $1 < q < 2$. To prove the falsity of (I_p) for $2 < p < +\infty$, we use some results for power series. Given a function ϕ regular in the unit circle, with Taylor series $\sum c_n z^n$, we say that a function Φ regular in the unit circle is a *majorant* of ϕ if Φ has Taylor series $\sum C_n z^n$ and $C_n \geqslant |c_n|$ for all n. We call the function Φ^* with Taylor series $\sum |c_n| z^n$ the *exact majorant* of ϕ.

The following result is an immediate consequence of M. Riesz's theorem on conjugate functions (see Z I, p. 253).

If (I_p) holds, where p is any given number such that $1 < p < +\infty$, then for each $\phi \in H^p$ there exists a majorant Φ of ϕ belonging to H^p such that

$$\int_{-\pi}^{\pi} |\Phi(e^{i\theta})|^p \, d\theta \leqslant A(p) \int_{-\pi}^{\pi} |\phi(e^{i\theta})|^p \, d\theta,$$

where $A(p)$ depends only on p.

We note now that if Φ belongs to H^p, then, by an inequality of L. Fejér and F. Riesz (*Math. Zeitschrift*, 11 (1921), 305–14),

$$\int_{0}^{1} |\Phi(\rho)|^p \, d\rho \leqslant B(p) \int_{-\pi}^{\pi} |\Phi(e^{i\theta})|^p \, d\theta.$$

Hence if (I_p) holds for any p satisfying $1 < p < +\infty$, then for each $\phi \in H^p$ there exists a majorant Φ of ϕ such that

$$\int_{0}^{1} |\Phi(\rho)|^p \, d\rho \leqslant C(p) \int_{-\pi}^{\pi} |\phi(e^{i\theta})|^p \, d\theta. \tag{1}$$

Since also $\Phi(\rho) \geqslant \Phi^*(\rho) \geqslant 0$, where Φ^* is the exact majorant of ϕ, (1) implies that

$$\int_{0}^{1} \{\Phi^*(\rho)\}^p \, d\rho \leqslant C(p) \int_{-\pi}^{\pi} |\phi(e^{i\theta})|^p \, d\theta. \tag{2}$$

However, this inequality (2) is known to be false when $p > 2$, a counterexample being given by

$$\phi(z) = \sum n^{-\frac{1}{2}-\delta} e^{i\alpha n \log n} z^n \quad (\alpha > 0,\ \delta > 0)$$

(see 1926, 7, p. 206; the inequality is true for $p \leqslant 2$). Hence (I_p) is false for $2 < p < +\infty$, and therefore (II_q) is false for $1 < q < 2$.

It should be mentioned also that Kahane, *Proc. Koninklijke Ned. Akad.*, A. 60, No. 3 (1957), 268–71, has given an example of an integrable f with Fourier series $\sum_{-\infty}^{\infty} c_n e^{ni\theta}$ such that the series $\sum_{-\infty}^{\infty} |c_n| e^{ni\theta}$ is not a Fourier series.

NOTE ON THE THEORY OF SERIES (XXIII): ON THE PARTIAL SUMS OF FOURIER SERIES

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1. In what follows $f(\theta)$ is a periodic function of L^2,

$$T(\theta) = \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta) = \tfrac{1}{2}A_0(\theta) + \sum_1^\infty A_n(\theta)$$

is the Fourier series of $f(\theta)$, and

$$s_n(\theta) = \tfrac{1}{2}A_0(\theta) + \sum_1^n A_m(\theta) = \sum_0^{n}{}' A_m(\theta)$$

is the nth partial sum of $T(\theta)$. We denote by $n(\theta)$ any function of θ which is measurable, is finite p.p.†, and assumes non-negative integral values only, and by $n(\theta, H)$ an $n(\theta)$ none of whose values exceeds H. We shall sometimes write N for $n(\theta)$, and $N(H)$ for $n(\theta, H)$, to simplify the set-up of formulae.

We define $S_n(\theta)$ and $S(\theta)$ by

$$S_n(\theta) = \operatorname*{Max}_{m \leq n} |s_m(\theta)|, \quad S(\theta) = \lim_{n\to\infty} S_n(\theta):$$

$S(\theta)$ may be infinite. Finally, we write

$$\lambda_n = (ln)^{-\frac{1}{2}} = \{\log(n+2)\}^{-\frac{1}{2}}, \quad A_n^*(\theta) = \lambda_n A_n(\theta),$$

and derive $s_n^*(\theta)$, $S_n^*(\theta)$, $S^*(\theta)$ from $A_n^*(\theta)$ as we derived $s_n(\theta)$, $S_n(\theta)$, $S(\theta)$ from $A_n(\theta)$.

Our object here is to give a fairly simple proof that

$$\text{(A)} \qquad \int_{-\pi}^{\pi} |s_{n(\theta)}^*(\theta)|^2 d\theta \leqq C \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta,$$

where C is an absolute constant, for every $n(\theta)$. The inequality is equivalent to

$$\text{(A}'\text{)} \qquad \int_{-\pi}^{\pi} |S^*(\theta)|^2 d\theta \leqq C \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta.\ddagger$$

2. Some preliminary remarks are desirable to clear the ground and show the relations of (A) or (A′) to inequalities proved already by other writers.

(1) We may suppose without loss of generality that $f(\theta)$, a_n, b_n are real, $a_0 = 0$, and

$$\int_{-\pi}^{\pi} f^2 d\theta = 1.$$

† Almost everywhere (*presque partout*).

‡ If we choose $N = n(\theta) \leqq H$ so as to make $|s_N^*(\theta)|$ as large as possible for each θ, we obtain an inequality like (A′) for $S_H^*(\theta)$; and (A′) itself follows when $H \to \infty$.

1944, 1 (with J. E. Littlewood) *Proceedings of the Cambridge Philosophical Society*, 40, 103–7.

(2) The inequality (A) is the special case $p = 2$ of

(B) $$\int_{-\pi}^{\pi} |s^*_{n(\theta)}(\theta)|^p\,d\theta \leqq C(p)\int_{-\pi}^{\pi} |f(\theta)|^p\,d\theta,$$

where $1<p\leqq 2$, $s^*_n(\theta)$ involves a convergence factor $(ln)^{-1/p}$, and $C(p)$ depends only on p. This is proved by Littlewood and Paley(**4**), who mention that we had proved (A) before. The proof of (B) is so intricate that it seems worth while to print our much simpler, and still unpublished, proof of (A).

(3) It is the last, and strongest, of a system of nine inequalities, most of which appear somewhere in the literature, namely

$$(1)\ \left|\int s_{N(H)}d\theta\right| \leqq C\sqrt{(lH)},\quad (2)\ \int |s_{N(H)}|\,d\theta \leqq C\sqrt{(lH)},\quad (3)\ \int s^2_{N(H)}d\theta \leqq ClH,$$

$$(4)\ \left|\int \frac{s_N}{\sqrt{(lN)}}\,d\theta\right| \leqq C,\qquad (5)\ \int \frac{|s_N|}{\sqrt{(lN)}}\,d\theta \leqq C,\qquad (6)\ \int \frac{s^2_N}{lN}\,d\theta \leqq C,$$

$$(7)\ \left|\int s^*_N d\theta\right| \leqq C,\qquad (8)\ \int |s^*_N|\,d\theta \leqq C,\qquad (9)\ \int s^{*2}_N d\theta \leqq C.$$

Some of the relations between these inequalities are obvious. Thus the implications

$$3\to 2\to 1,\quad 6\to 5\to 4,\quad 9\to 8\to 7,\quad 6\to 3,\quad 5\to 2$$

are all trivial; and $$1\to 2,\quad 4\to 5,\quad 7\to 8$$

are all but trivial †. Also

$$|s_n(\theta)| = \left|\sum_0^n{}' \lambda_m \frac{A_m(\theta)}{\lambda_m}\right| \leqq \frac{2}{\lambda_n}\operatorname*{Max}_{m\leqq n}|s^*_m(\theta)| = \frac{2}{\lambda_n}S^*_n(\theta),$$

by partial summation, so that $9\to 6,\quad 8\to 5.$

Thus (9) implies all the other inequalities; actually we shall find that it lies a little deeper than the rest.

(4) The inequalities (1), (2), (4), (5), (7) and (8) are due to Kolmogoroff and Seliverstoff(**2, 3**) and Plessner(**5**). The first concern of these writers was to prove the theorem

(C) '$T^*(\theta) = \Sigma' A^*_n(\theta)$ *converges p.p.*',

and the inequalities contain the kernel of the proof. Actually Kolmogoroff and Seliverstoff used (1) and (2) in (**2**), in which they prove only the convergence of $\Sigma'\lambda_n^{1+\epsilon}A_n(\theta)$, and (4) and (5) in (**3**); Plessner proved (7) and (8) in (**5**); and Zygmund (**6**, pp. 252–5) follows Plessner. It should be observed that, although Kolmogoroff and Seliverstoff

† Suppose, for example, that we have proved (7), so that, in particular,

$$\left|\int s^*_{N(H)}(\theta)\,d\theta\right| \leqq C,$$

with a C independent of H. If $\phi_H(\theta)$ and $\psi_H(\theta)$ are the lower and upper bounds of $s^*_n(\theta)$ for $n\leqq H$, then $\phi_H(\theta)\leqq s^*_N(\theta)\leqq \psi_H(\theta)$; and $\phi_H(\theta)\leqq 0$, $\psi_H(\theta)\geqq 0$, since $a_0=0$ and so $s^*_0(\theta)=0$. Hence

$$|s^*_{N(H)}(\theta)| \leqq \psi_H(\theta)-\phi_H(\theta),\quad \int |s^*_{N(H)}(\theta)|\,d\theta \leqq \left|\int \psi_H(\theta)\,d\theta\right| + \left|\int \phi_H(\theta)\,d\theta\right| \leqq 2C.$$

Making $H\to\infty$, we obtain (8), with $2C$ for C.

proved less in (2), this paper contains the fundamental identity (3·8), which is the first key to all these theorems.

Considered merely as a weapon for the proof of (C), (9) has no advantage over (8)†, though each has some advantage over (5) or (6). But it proves a good deal more, in particular that $\Sigma A_n^*(\theta)$ converges 'dominatedly (L^2)'.

(5) We prove (9) in two stages, proving first (6) and then (6)→(9).

3. *Proof of* (6). We have supposed f real and

$$\int f^2 d\theta = 1.\ddagger \tag{3·1}$$

We have to prove that

$$\int U^2 d\theta \leqq C, \tag{3·2}$$

where

$$U = \lambda_{n(\theta)} s_{n(\theta)}(\theta) = \lambda_N s_N(\theta); \tag{3·3}$$

and it is sufficient to prove

$$\int U_H^2 d\theta \leqq C, \tag{3·4}$$

where

$$U_H = \lambda_{N(H)} s_{N(H)}(\theta) \tag{3·5}$$

and C is independent of H. For, if we have proved (3·4), and define W_H and W by

$$W_H = \operatorname*{Max}_{m \leqq H} \lambda_m \,|\, s_m(\theta)\,|,\S \quad W = \lim_{H\to\infty} W_H,$$

we have

$$\int W_H^2 d\theta \leqq C, \quad \int W^2 d\theta \leqq C.$$

and the last inequality is equivalent to (3·2).

Thus what we have to prove is (3·4), and from this point we may suppress the H, it being understood that all suffixes which occur are bounded by H. It is necessary and sufficient for the truth of (3·4) that

$$|\,J\,| = \left|\int U\phi\, d\theta\right| \leqq C \tag{3·6}$$

for all $\phi = \phi(\theta)$ with

$$\int \phi^2 d\theta = 1. \tag{3·7}$$

We write

$$C_n(\theta) = \tfrac{1}{2} + \cos\theta + \ldots + \cos n\theta = \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}.$$

Then

$$J = \frac{1}{\pi}\int \lambda_N \phi(\theta)\, d\theta \int f(t)\, C_N(t-\theta)\, dt = \frac{1}{\pi}\int f(t)\, dt \int \lambda_N \phi(\theta)\, C_N(t-\theta)\, d\theta;$$

and hence, by Schwarz's inequality and (3·1),

$$J^2 \leqq \frac{1}{\pi^2}\int \left\{\int \lambda_N \phi(\theta)\, C_N(t-\theta)\, d\theta\right\}^2 dt$$

$$= \frac{1}{\pi^2}\int dt \int \Lambda_1 \phi(\theta_1)\, C_{N_1}(t-\theta_1)\, d\theta_1 \int \Lambda_2 \phi(\theta_2)\, C_{N_2}(t-\theta_2)\, d\theta_2,$$

where

$$N_1 = n(\theta_1), \quad N_2 = n(\theta_2), \quad \Lambda_1 = \lambda_{N_1}, \quad \Lambda_2 = \lambda_{N_2}.$$

† Since $S_n^*(\theta)$ increases with n, it follows at once from either (8) or (9) that $s_n^*(\theta)$ is *bounded* p.p. If $\Sigma(a_n^2+b_n^2)<\infty$ then there is a μ_n, increasing to ∞ with n, such that $\Sigma\mu_n^2(a_n^2+b_n^2)<\infty$. Hence the partial sums of $\Sigma\lambda_n\mu_n A_n(\theta)$ are bounded p.p., and therefore $\Sigma\lambda_n A_n(\theta)$ converges p.p.

‡ All integrals are over $(-\pi, \pi)$ unless the contrary is indicated.

§ Not $\lambda_H S_H(\theta)$, because $\lambda_m S_m(\theta)$ is usually not monotone.

Changing the order of integration, and observing that

$$\frac{1}{\pi}\int C_{N_1}(t-\theta_1)\,C_{N_2}(t-\theta_2)\,dt = C_{N_{1,2}}(\theta_1-\theta_2), \tag{3·8}$$

where
$$N_{1,2} = \mathrm{Min}\,(N_1, N_2),$$

we obtain
$$J^2 \leqq \frac{1}{\pi}\iint \Lambda_1\Lambda_2\,|\phi(\theta_1)|\,|\phi(\theta_2)|\,|C_{N_{1,2}}(\theta_1-\theta_2)|\,d\theta_1 d\theta_2.$$

Now
$$\Lambda_1\Lambda_2\,|\phi(\theta_1)|\,|\phi(\theta_2)| \leqq \tfrac{1}{2}\{\Lambda_1^2\phi^2(\theta_1)+\Lambda_2^2\phi^2(\theta_2)\}$$

and
$$|C_{N_{1,2}}(\theta_1-\theta_2)| = \left|\frac{\sin\{(N_{1,2}+\frac{1}{2})(\theta_1-\theta_2)\}}{2\sin\frac{1}{2}(\theta_1-\theta_2)}\right| \leqq \frac{C}{|\theta_1-\theta_2|+(N_{1,2}+2)^{-1}}.$$

Hence
$$J^2 \leqq C(J_1+J_2),$$

where
$$J_1 = \int \Lambda_1^2\phi^2(\theta_1)\,d\theta_1 \int \frac{d\theta_2}{|\theta_1-\theta_2|+(N_{1,2}+2)^{-1}}.$$

But
$$N_{1,2} \leqq N_1, \quad \frac{1}{|\theta_1-\theta_2|+(N_{1,2}+2)^{-1}} \leqq \frac{1}{|\theta_1-\theta_2|+(N_1+2)^{-1}},$$

and hence (since N_1 is independent of θ_2)

$$\int_{-\pi}^{\pi}\frac{d\theta_2}{|\theta_1-\theta_2|+(N_{1,2}+2)^{-1}} \leqq \int_{-2\pi}^{2\pi}\frac{d\theta}{|\theta|+(N_1+2)^{-1}} = 2\log\{2\pi(N_1+2)+1\} \leqq ClN_1 = C\Lambda_1^{-2}.$$

Thus
$$J_1 \leqq C\int \phi^2(\theta_1)\,d\theta_1 \leqq C.$$

Similarly $J_2 \leqq C$, and (3·4) follows.

4. *Deduction of* (9) *from* (6). To deduce (9), we must use one of the theorems of our paper (**1**), viz.

$$\int_{-\pi}^{\pi} \mathrm{Max}\,\sigma_n^2(\theta)\,d\theta \leqq C\int_{-\pi}^{\pi} f^2(\theta)\,d\theta, \tag{4·1}$$

where $\sigma_n(\theta)$ is the first Cesaro mean of $s_n(\theta)$, and $\mathrm{Max}\,\sigma_n^2(\theta)$ the upper bound of $\sigma_n^2(\theta)$ for all n†. It will be observed that (9) is the only one of the nine inequalities which depends upon a theorem of this kind.

We may suppose $N > 1$. Then two partial summations give

$$\begin{aligned} s_N^*(\theta) &= \sum_0^N{}' \lambda_m A_m(\theta) = \sum_0^{N-1} s_m(\theta)\,\Delta\lambda_m + \lambda_N s_N(\theta) \\ &= \sum_0^{N-2}(m+1)\sigma_m(\theta)\,\Delta^2\lambda_m + N\sigma_{N-1}(\theta)\,\Delta\lambda_{N-1} + \lambda_N s_N(\theta) = P_N + Q_N + R_N, \end{aligned}$$

say; and so
$$s_N^{*2} \leqq 3(P_N^2+Q_N^2+R_N^2). \tag{4·2}$$

First
$$\int R_N^2\,d\theta = \int \frac{s_N^2}{lN}\,d\theta \leqq C, \tag{4·3}$$

by (6). Secondly
$$0 \leqq \Delta\lambda_{N-1} \leqq \frac{C}{N(lN)^{\frac{3}{2}}} \leqq \frac{C}{N},$$

and so
$$\int Q_N^2\,d\theta \leqq C\int \sigma_{N-1}^2(\theta)\,d\theta \leqq C, \tag{4·4}$$

† (**1**), pp. 110–11 (Theorem 21), where more is proved.

by (4·1). Finally
$$0 \leqq \Delta^2 \lambda_m \leqq \frac{C}{(m+1)^2 (lm)^{\frac{3}{2}}},$$

and so
$$|P_N| \leqq C \operatorname*{Max}_{m \leqq N} |\sigma_m(\theta)| \sum_0^\infty \frac{1}{(m+1)(lm)^{\frac{3}{2}}} \leqq C \operatorname{Max} |\sigma_n(0)|,$$

$$\int P_N^2 \, d\theta \leqq C \int \operatorname{Max} \sigma_n^2(\theta) \, d0 \leqq C, \tag{4·5}$$

again by (4·1). Finally, (9) follows from (4·2)–(4·5).

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(4) Littlewood, J. E. and Paley, R. E. A. C. *Proc. London Math. Soc.* (2), 43 (1937), 105–26.
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COMMENTS

p. 104. The inequalities (1)–(9) were the fruits of attempts to decide whether the Fourier series of a function of L^2 converges p.p. This question has now been answered affirmatively by L. Carleson, *Acta Math.* 116 (1966), 135–57, and to this extent the inequalities (1)–(9) have been superseded.

(*d*) Special Trigonometric Series

INTRODUCTION TO PAPERS ON SPECIAL TRIGONOMETRIC SERIES

Trigonometric series $\sum a_n \cos n\theta$ and $\sum a_n \sin n\theta$ with coefficients a_n which decrease to the limit 0 have a number of special properties, and in 1928, 9 and 1931, 1 Hardy dealt with the case in which $a_n \sim An^{-\alpha}$, where $A \neq 0$ and $0 < \alpha < 1$, obtaining asymptotic relations for the sums of the series as $\theta \to 0$. He also dealt with the case $\alpha = 1$ for the sine series, and in 1943, 2, in collaboration with Rogosinski, he returned to this case and gave a simplified treatment of his own earlier theorem and of other results. The culminating paper of this group is 1945, 1, again written in collaboration with Rogosinski, in which the results for series with decreasing coefficients of order $n^{-\alpha}$ are extended to the case where the coefficients are in some sense slowly varying.

The literature of this subject is fragmentary, since different writers have tended to consider slightly different problems, and no attempt is made below to give references to related work. Accounts of the histories of the theorems considered by Hardy, and by Hardy and Rogosinski, can be found in 1931, 1, § 6 and 1945, 1, § 6. Other references are given in Z I, Chapter 5, and R. P. Boas, *Integrability Theorems for Trigonometric Transforms*, Berlin, 1967.

The remaining paper of this group, 1941, 3, is on a different topic, and connects with Hardy's work on transforms and integral equations.

A THEOREM CONCERNING TRIGONOMETRICAL SERIES

G. H. HARDY*.

[*Extracted from the Journal of the London Mathematical Society, Vol.* 3, *Part* 1.]

1. The theorem which follows was suggested by Mr. Haslam-Jones's note in a recent number of the *Journal*†, and is in fact a "transform"‡ of the theorem of Bromwich§ which he quotes. It is curious that, in spite of its simplicity, it has apparently been overlooked.

THEOREM 1. *If a_n is positive and decreasing, and $a_n \sim An^{-\alpha}$, where $A > 0$, $0 < \alpha < 1$, when $n \to \infty$, then*

$$f(\theta) = \sum_1^\infty a_n \cos n\theta \sim A \sin \tfrac{1}{2}\alpha\pi\, \Gamma(1-\alpha)\, \theta^{\alpha-1}$$

when $\theta \to 0$ through positive values.

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* Received 15 October, 1927; read 10 November, 1927.

† U. S. Haslam-Jones, "A note on the Fourier coefficients of unbounded functions", *Journal London Math. Soc.*, 2 (1927), 151-154.

‡ See G. H. Hardy and J. E. Littlewood, "On the strong summability of Fourier series", *Proc. London Math. Soc.* (2), 26 (1927), 273–286 (278–279), for an explanation of the meaning of the word.

§ T. J. I'A. Bromwich, *Infinite series* (ed. 2, 1926), 518.

1928, 9 *Journal of the London Mathematical Society*, 3, 12–13.

We choose a (large) positive number c and write

$$(1)\qquad f(\theta) = A\sum_1^\infty n^{-a}\cos n\theta - A\sum_{\mu+1}^\infty n^{-a}\cos n\theta + \sum_1^\mu (a_n - An^{-a})\cos n\theta + \sum_{\mu+1}^\infty a_n\cos n\theta$$

$$= S_1+S_2+S_3+S_4,$$

say, where $\mu = [c/\theta]$. Since a_n is monotonic, we have

$$(2)\qquad |S_4| \leqslant a_{\mu+1}\operatorname*{Max}_{\nu>\mu}\left|\sum_{\mu+1}^{\nu}\cos n\theta\right| < K\left(\frac{c}{\theta}\right)^{-a}\frac{1}{\theta} = Kc^{-a}\theta^{a-1},$$

where K is independent of θ and c. A similar argument gives the same upper bound for $|S_2|$. We can therefore choose c so that

$$(3)\qquad |S_2+S_4| < \epsilon\theta^{a-1}$$

for all θ in question. But, when c is fixed and $\theta \to 0$, we have

$$(4)\qquad S_3 = \sum_1^\mu o(n^{-a}) = o(c^{1-a}\theta^{a-1}) = o(\theta^{a-1})$$

and

$$(5)\qquad S_1 \sim A\sin\tfrac{1}{2}a\pi\,\Gamma(1-a)\,\theta^{a-1};$$

and the theorem follows from (1), (3), (4), and (5).

2. There is naturally a similar theorem for the corresponding sine series, and more general results in which $a_n \sim An^{-a}(\log n)^{-\beta}$, but it is unnecessary to state them formally, since the theorems and their proofs will easily be supplied by anyone who has read Haslam-Jones's note. A more interesting observation is this: Theorem 1 (like Bromwich's theorem) stands in a certain relation to the most familiar test for the convergence of a Fourier series, and it is to be expected that there will be a (still simpler) theorem bearing the same relation to Fejér's theorem, in which no condition of "monotony" will be required. This is

Theorem 2. *If $a_n \sim An^{-a}$, where $0 < a < 1$, then*

$$\sum_1^\infty a_n\frac{\sin n\theta}{n} \sim \frac{A}{a}\sin\tfrac{1}{2}a\pi\,\Gamma(1-a)\,\theta^a.$$

The theorem is all but trivial, and its proof may be left to the reader.

COMMENTS

The result of this paper is included in Theorem 2 of 1945, 1. A converse is proved in 1931, 1.

SOME THEOREMS CONCERNING TRIGONOMETRICAL SERIES OF A SPECIAL TYPE

By G. H. Hardy.

[Received 10 May, 1930.—Read 15 May, 1930.]

[*Extracted from the Proceedings of the London Mathematical Society, Ser.* 2, *Vol.* 32, *Part* 6.]

1. In this note I develop more systematically a group of theorems some of which have been proved already in the *Journal* or *Proceedings*. They concern primarily the series

$$(1.1)\quad \Sigma a_n \cos n\theta = f(\theta), \qquad (1.2)\quad \Sigma a_n \sin n\theta = g(\theta),$$

in which $n = 1, 2, 3, \ldots$, and the a_n are positive and decrease steadily to the limit 0. It is, of course, familiar that the sine series is convergent for all θ, and the cosine series for all θ save multiples of 2π, and that they are the Fourier series of their sums when these are integrable. They also possess a number of interesting and less obvious properties, among which I may mention the following.

(1) It was proved by Chaundy and Jolliffe* that *a necessary and sufficient condition for the uniform convergence of* (1 . 2) *is that* $na_n \to 0$; *more precisely, the condition is sufficient for the uniform convergence of the series and necessary for the continuity of its sum.*

(2) In a recent note in the *Journal*† I proved that *if*

$$(1.3)\qquad a_n \sim An^{-\alpha},$$

where $A > 0$, $0 < \alpha < 1$, *and the formula is to be interpreted as meaning* $a_n = o(n^{-\alpha})$ *when* $A = 0$, *then*

$$(1.4)\qquad f(\theta) \sim A \sin \tfrac{1}{2}\alpha\pi\Gamma(1-\alpha)\,\theta^{\alpha-1},$$

$$(1.5)\qquad g(\theta) \sim A \cos \tfrac{1}{2}\alpha\pi\Gamma(1-\alpha)\,\theta^{\alpha-1}$$

(*with a similar gloss*), *when* $\theta \to 0$ *by positive values.*

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* Chaundy and Jolliffe, **2**; Jolliffe, **8**. Only the first form of the theorem is given in the first paper. In the second Jolliffe considers also series of more general types.

† Hardy, **3**.

1931, 1 *Proceedings of the London Mathematical Society* (2), 32, 441–8.

(3) Littlewood and I have shown still more recently that *a necessary and sufficient condition that* $f(\theta)$ *or* $g(\theta)$ *should belong to the Lebesgue class* L^r, *where* $r > 1$, *is that* $\Sigma n^{r-2} a_n^r$ *should be convergent.*

The last theorem is of a different character and I shall not refer to it again. My object here is to go further in the direction indicated by (1) and (2).

2. THEOREM 1. *Suppose that* a_n *is positive and decreases steadily to the limit* 0, *that* $f(\theta)$ *and* $g(\theta)$ *are defined by* (1 . 1) *and* (1 . 2), *and that* $A > 0$, $0 < \alpha < 1$. *Then* (1 . 3) *is a necessary and sufficient condition for the truth of either* (1 . 4) *or* (1 . 5).

I proved in my former note that the condition is sufficient, and it remains only to prove it necessary. The proofs for (1 . 1) and (1 . 2) are not quite the same, but we require in either case a lemma of a "Tauberian" character.

LEMMA. *If* a_n *is positive and decreasing, or, more generally, if* $n^{-k} a_n$ *is decreasing for some* k, *and if* (1 . 3) *holds in some Cesàro sense, then* (1 . 3) *holds in the ordinary sense.*

By saying that (1 . 3) holds in some Cesàro sense, say (C, r), where r is a positive integer, we mean that, if

$$a_n^0 = a_n, \quad a_n^1 = a_1^0 + \ldots + a_n^0, \quad a_n^2 = a_1^1 + \ldots + a_n^1, \quad \ldots,$$

then

$$a_n^r \sim A \frac{\Gamma(1-\alpha)}{\Gamma(r+1-\alpha)} n^{r-\alpha}. \tag{2 . 1}$$

It is obviously sufficient to prove that (1 . 3) holds $(C, r-1)$ if it holds (C, r).

Suppose then that (2 . 1) holds for a particular r, and write A_n for a_n^{r-1} and C for the constant on the right-hand side. If $\delta > 0$ and $\nu = [n + \delta n]$, then

$$A_{n+1} + A_{n+2} + \ldots + A_\nu \sim C\{(1+\delta)^{r-\alpha} - 1\} n^{r-\alpha}. \tag{2 . 2}$$

If $r > 1$, A_n increases with n, so that

$$\delta n A_n \leqslant C\{(1+\delta)^{r-\alpha} - 1\} n^{r-\alpha} + o(n^{r-\alpha}),$$

$$\frac{A_n}{n^{r-1-\alpha}} \leqslant C \frac{(1+\delta)^{r-\alpha} - 1}{\delta} + o(1),$$

$$\bar{l} = \overline{\lim} \frac{A_n}{n^{r-1-\alpha}} \leqslant C \frac{(1+\delta)^{r-\alpha} - 1}{\delta},$$

and so, since δ is as small as we please, $\overline{l} \leqslant (r-a)C$. Similarly, considering a block of A_m to the left of A_n, we find that $\underline{l} \geqslant (r-a)C$. Hence $\overline{l} = \underline{l} = (r-a)C$, and (1 . 3) holds $(C, r-1)$.

If $r = 1$, (2 . 2) is

$$a_{n+1}+a_{n+2}+\ldots+a_\nu \sim \frac{A}{1-a}\{(1+\delta)^{1-a}-1\}\, n^{1-a}. \tag{2.3}$$

We have now (supposing that $k > 0$, obviously the most unfavourable case)

$$a_\mu \leqslant \left(\frac{\mu}{n}\right)^k a_n \leqslant (1+\delta)^k a_n \quad (n < \mu \leqslant \nu),$$

and we deduce that

$$\underline{l} = \underline{\lim}\, n^a a_n \geqslant \frac{A}{1-a}\,\frac{(1+\delta)^{1-a}-1}{(1+\delta)^k\,\delta},$$

and so $\underline{l} \geqslant A$. Similarly $\overline{l} \leqslant A$, and (1 . 3) holds in the ordinary sense.

Actually we require only the cases of the lemma in which r is 1 or 2 and k is 0 or 1.

3. (i) Considering now the cosine series, we prove first that

$$a_n = \frac{2}{\pi}\int_0^\pi f(\theta)\cos n\theta\, d\theta, \tag{3.1}$$

i.e. that the series is the Fourier series of its sum*. In fact, since the series is uniformly convergent in (ϵ, π) for any positive ϵ, we have

$$F(\theta) = \int_\theta^\pi f(u)\, du = -\Sigma a_n \frac{\sin n\theta}{n},$$

and the series on the right is uniformly convergent for all θ†. Hence

$$\frac{a_n}{n} = -\frac{2}{\pi}\int_0^\pi F(\theta)\sin n\theta\, d\theta = -\frac{2}{\pi}\lim_{\epsilon\to 0}\int_\epsilon^\pi F(\theta)\sin n\theta\, d\theta$$

$$= \frac{2}{n\pi}\lim_{\epsilon\to 0}\left\{-F(\epsilon)\cos n\epsilon + \int_\epsilon^\pi f(\theta)\cos n\theta\right\} = \frac{2}{n\pi}\int_0^\pi f(\theta)\cos n\theta\, d\theta.$$

The integral here is in the first instance a "Cauchy" integral, but it plainly exists also as a Lebesgue integral.

* This is a corollary of general theorems of the "Riemannian" theory. It is more appropriate here to give a direct proof.

† Since $\Sigma \frac{\sin n\theta}{n}$ is boundedly convergent.

We now deduce from (3 . 1) in the usual manner

$$a_n^1 = \frac{1}{\pi}\int_0^\pi f(\theta)\,\frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\,d\theta, \qquad a_n^2 = \frac{1}{\pi}\int_0^\pi f(\theta)\,\frac{\sin^2\frac{1}{2}(n+1)\theta}{\sin^2\frac{1}{2}\theta}\,d\theta.$$

The kernel in the last integral being positive, the asymptotic value of a_n^2 may be calculated in the obvious way. We find that

$$a_n^2 \sim \frac{4}{\pi}\,A\,\Gamma(1-\alpha)\sin\tfrac{1}{2}\alpha\pi\int_0^\infty \theta^{\alpha-3}\sin^2\tfrac{1}{2}n\theta\,d\theta = \frac{A}{(1-\alpha)(2-\alpha)}\,n^{2-\alpha};$$

the justification of the process in detail is a straightforward piece of routine which may be left to the reader. Applying the lemma, with $r=2$, $k=0$, we obtain (1 . 3).

(ii) Passing to the sine series, we observe first that the series $\Sigma\, a_n \sin n\theta \sin m\theta$ is uniformly convergent, since

$$\left|\sum_p^q a_n \sin n\theta \sin m\theta\right| \leqslant |m\theta|\left|\sum_p^q a_n \sin n\theta\right| \leqslant 2m\,\frac{\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\,a_p.$$

It follows that*

$$a_m = \frac{2}{\pi}\int_0^\pi g(\theta)\sin m\theta\,d\theta.$$

Hence

$$a_n^1 = \frac{1}{\pi}\int_0^\pi g(\theta)\,\frac{1-\cos(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\,d\theta - \frac{1}{\pi}\int_0^\pi g(\theta)\,\frac{1-\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\,d\theta.$$

The second term is independent of n, and the first, having again a positive kernel, may be evaluated by substituting the asymptotic value of $g(\theta)$ and replacing the upper limit π by ∞. We find that

$$a_n^1 \sim \frac{An^{1-\alpha}}{1-\alpha},$$

and (1 . 3) again follows from the lemma, with $r=1$, $k=0$.

4. The argument fails when $\alpha = 1$, but the result still holds for the *sine* series with the obvious modifications.

THEOREM 2. *If a_n decreases steadily to the limit 0, then $na_n \to A$ is a necessary and sufficient condition that $g(\theta) \to \frac{1}{2}\pi A$, where $\theta \to 0$ by positive values.*

* This again follows from general theorems; the simple proof given here was suggested to me by Dr. Zygmund. See Zygmund, **10**, 105.

The case $A = 0$ is the theorem of Chaundy and Jolliffe. The general case is not immediately reducible to the special case, since $a_n - A/n$ is not necessarily monotonic.

(i) If $na_n \to A$, we have

$$g(\theta)-g_0(\theta) = \Sigma a_n \sin n\theta - A\, \Sigma \frac{\sin n\theta}{n}$$

$$= \sum_{n \leqslant c/\theta} a_n \sin n\theta - A \sum_{n \leqslant c/\theta} \frac{\sin n\theta}{n} + \sum_{n \geqslant C/\theta} a_n \sin n\theta$$

$$- A \sum_{n \geqslant C/\theta} \frac{\sin n\theta}{n} + \sum_{c/\theta < n < C/\theta} \left(a_n - \frac{A}{n}\right) \sin n\theta$$

$$= g_1(\theta)+g_2(\theta)+g_3(\theta)+g_4(\theta)+g_5(\theta),$$

say, c and C being positive (small and large respectively) and independent of θ. Then

$$g_1(\theta) = \sum_{n \leqslant c/\theta} O\left(\frac{1}{n}\right) O(n\theta) = O\left(\theta \cdot \frac{c}{\theta}\right) = O(c),$$

$$g_3(\theta) = O\left(a_{[C/\theta]} \operatorname*{Max}_{q > p \geqslant C/\theta} \left| \sum_p^q \sin n\theta \right| \right) = O\left(\frac{\theta}{C} \cdot \frac{1}{\theta}\right) = O\left(\frac{1}{C}\right),$$

where the constants of the O's are independent of c and C. There are plainly similar upper bounds for $g_2(\theta)$ and $g_4(\theta)$; and we may therefore choose c and C so that

$$|\, g(\theta) - g_0(\theta) - g_5(\theta)\, | < \epsilon$$

for all θ. Since $g_0(\theta) \to \frac{1}{2}\pi A$ when $\theta \to 0$, and

$$g_5(\theta) = \sum_{c/\theta < n < C/\theta} o\left(\frac{1}{n}\right) = o\left(\log \frac{C}{c}\right) = o(1)$$

when c and C are fixed, it follows that $g(\theta) \to \frac{1}{2}\pi A$.

(ii) To prove the condition necessary, it seems most convenient to use Poisson's integral. We have

$$\sum_1^\infty a_n r^n \sin n\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1-r^2)\, g(\phi)\, d\phi}{1-2r\cos(\phi-\theta)+r^2}$$

for $r < 1$, and so, differentiating with respect to θ and putting $\theta = 0$,

$$\sum_1^\infty n a_n r^n = \frac{2r}{\pi}\int_0^\pi \frac{(1-r^2)g(\phi)\sin\phi\, d\phi}{(1-2r\cos\phi+r^2)^2}$$

$$\sim 2A\int_0^\infty \frac{(1-r)\,\phi\, d\phi}{\{(1-r)^2+\phi^2\}^2} = \frac{A}{1-r}$$

when $r \to 1$*. Since $a_n \geqslant 0$, it follows† that

$$a_1+2a_2+\ldots+na_n \sim An, \qquad b_n = na_n \to A \quad (C,\, 1).$$

Since b_n/n decreases, it follows from the lemma (with $r = 1$, $k = 1$) that $b_n \to A$.

There is no precise analogue of Theorem 2 for the *cosine* series. It is easy to prove that $na_n \to A$ implies that

$$f(\theta) \sim A \log\frac{1}{\theta},$$

but the converse is false. Thus, if we take

$$a_n = 2^{-\nu} \quad (2^\nu \leqslant n < 2^{\nu+1},\ \nu = 0, 1, \ldots),$$

we find without difficulty that $f(\theta)$ differs from

$$\psi(\theta) = \frac{1}{\theta}\sum_1^\infty 2^{-\nu}\sin 2^\nu\theta$$

by a bounded function, and that $\psi(\theta) \sim \log(1/\theta)$ when $\theta \to +0$. But plainly na_n does not tend to a limit.

5. There is a theorem "reciprocal" to Theorem 1 in which a_n is not restricted to be monotonic but the series are supposed to be Fourier series of monotonic functions.

THEOREM 3. *If the series* (1 . 1) *is the Fourier series of* $f(\theta)$, *or* (1 . 2) *of* $g(\theta)$, *and* $f(\theta)$, *or* $g(\theta)$, *decreases steadily as* θ *increases from* 0, *then* (1 . 3) *is a necessary and sufficient condition for the truth of* (1 . 4), *or* (1 . 5).

The proof is very much like that of Theorem 1 (the rôles of θ and n being roughly interchanged), and I need hardly write it out at length. One half of the theorem, that (1 . 4), or (1 . 5), implies (1 . 3), is nearly, but not exactly, the same as a theorem of Bromwich generalised by Haslam-

* Here again I omit some details which the reader will easily supply.

† Hardy and Littlewood, **4**; Hobson, **7**, 185; Landau, **9**, 50.

Jones* in a recent note in the *Journal*. Bromwich supposes, for example, that $f(\theta) = \theta^{-\alpha}\phi(\theta)$, where $\phi(\theta)$ is of bounded variation. If $\phi(\theta)$ decreases steadily, the result is included in Theorem 3. We could replace the hypothesis that $f(\theta)$ decreases by the hypothesis that its variation in (θ, π) is $O(\theta^{-\alpha})$, and then our result would include the more general form of Bromwich's theorem.

We may also modify the theorem by assuming that (1 . 1) converges to $f(\theta)$, or to $\bar{f}(\theta) = \frac{1}{2}\{f(\theta+0)+f(\theta-0)\}$. It is then easily proved that the series is the Fourier series of $f(\theta)$.

6. Theorem 2 reduces to the Chaundy-Jolliffe theorem when $A = 0$. The most sophisticated part of the theorem is that which asserts that $na_n \to 0$ is a *necessary* condition for the continuity of $g(\theta)$. It may be worth while, in conclusion, to reproduce Jolliffe's ingenious proof, which does not depend in any way on the theory of Fourier series, in its simplest form, since the point is rather obscured in his paper by the greater generality of the series considered.

Let $\theta = \pi/(2m+1)$, where m is integral, and write

$$g(\theta) = \sum_{1}^{m} + \sum_{m+1}^{2m} + \sum_{2m+1}^{\infty} = g_1(\theta)+g_2(\theta)+g_3(\theta).$$

Then

$$g_1(\theta) \geqslant a_m \sum_{1}^{m} \sin n\theta = \frac{a_m}{2\sin\frac{1}{2}\theta}\cos\tfrac{1}{2}\theta,$$

$$g_2(\theta) \geqslant a_{2m} \sum_{m+1}^{2m} \sin n\theta = -\frac{a_{2m}}{2\sin\frac{1}{2}\theta}\cos(2m+\tfrac{1}{2})\,\theta,$$

$$|\,g_3(\theta)\,| \leqslant a_{2m+1}\,\mathrm{Max}\left|\sum_{2m+1}^{p} \sin n\theta\right| \leqslant \frac{a_{2m}}{2\sin\frac{1}{2}\theta}\{1-\cos(2m+\tfrac{1}{2})\,\theta\}.$$

Hence
$$2\sin\tfrac{1}{2}\theta\, g(\theta) \geqslant a_m\cos\tfrac{1}{2}\theta - a_{2m}.$$

The left-hand side is $o(1/m)$, and we can omit the $\cos\frac{1}{2}\theta$ with error $O(1/m^2) = o(1/m)$. Hence

$$a_m - a_{2m} = o\left(\frac{1}{m}\right)$$

and

$$|\,a_m\,| \leqslant |\,a_m - a_{2m}\,| + |\,a_{2m}-a_{4m}\,| + \ldots < \frac{\epsilon}{m}(1+\tfrac{1}{2}+\tfrac{1}{4}+\ldots) = \frac{2\epsilon}{m}$$

if m is sufficiently large.

* Bromwich, **1**, 494 ; Haslam-Jones, **6**.

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COMMENT

An easier proof of Theorem 2 is given in 1943, 2.

NOTES ON SPECIAL SYSTEMS OF ORTHOGONAL FUNCTIONS (IV): THE ORTHOGONAL FUNCTIONS OF WHITTAKER'S CARDINAL SERIES

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1. Suppose that n runs through all integral values, that (ϕ_n) is a system of normal orthogonal functions for the interval $(-\infty, \infty)$, and that ψ_n is the Fourier transform of ϕ_n. Then, by Parseval's theorem for Fourier transforms†,

$$\int \psi_m \overline{\psi}_n \, dx = \int \phi_m \overline{\phi}_n \, dx,$$

and (ψ_n) is also a normal orthogonal system.

The system (ϕ_n) is *complete* if there is no non-null function of L^2 orthogonal to every ϕ_n. It is *closed* if the class of 'polynomials' in the ϕ_n (the class of finite linear combinations $\Sigma a_\nu \phi_\nu$) is dense in L^2; i.e. if we can make

$$\int | f - \Sigma a_\nu \phi_\nu |^2 \, dx < \epsilon^2,$$

for every f of L^2 and every ϵ. The properties of completeness and closure are equivalent.

If f and g are a pair of Fourier transforms of L^2, then

$$\int | f - \Sigma a_\nu \phi_\nu |^2 \, dx = \int | g - \Sigma a_\nu \psi_\nu |^2 \, dx.$$

It follows that (ψ_n) is complete (closed) if and only if (ϕ_n) is complete (closed).

2. Next, suppose that (χ_n) is a complete normal orthogonal system for a finite interval $(-a, a)$, and define ϕ_n by

$$\phi_n = \chi_n \quad (|x| \leqslant a), \qquad \phi_n = 0 \quad (|x| > a).$$

Then (ϕ_n) is normal and orthogonal in $(-\infty, \infty)$, but it is plain that it is not usually complete. For $\int f \overline{\phi}_n \, dx = 0$ is

$$\int_{-a}^{a} f \overline{\chi}_n \, dx = 0, \tag{2·1}$$

and the truth of this for all n tells us nothing about the values of f outside $(-a, a)$.

† The bar denotes the conjugate. When no range of integration is shown, it is $(-\infty, \infty)$. For the theory of Fourier transforms of L^2, see Titchmarsh (8), ch. 3.

1941, 3 *Proceedings of the Cambridge Philosophical Society*, 37, 331–48.

If (as in § 1) ψ_n is the Fourier transform of ϕ_n, then (ψ_n) is normal and orthogonal. If g is L^2, and $\int g\bar{\psi}_n dx = 0$ for all n, then $\int f\bar{\phi}_n dx = 0$ for all n, i.e. (2·1) is true. It follows that $f \equiv 0$ in $(-a, a)$. If $f \equiv 0$ also outside $(-a, a)$, then f and g are null. Thus (ψ_n) is complete for functions g whose Fourier transforms vanish outside $(-a, a)$. Such a g is of the form

$$g(x) \doteq \frac{1}{\sqrt{(2\pi)}} \int_{-a}^{a} f(t)\, e^{-xit}\, dt,$$

where f is L^2. It is an integral function of order 1 and type a, and is L^2 on the real axis. It is natural to call such a function a 'Paley-Wiener' function†.

3. If we take
$$a = \pi, \quad \chi_n = \frac{e^{nix}}{\sqrt{(2\pi)}},$$
then
$$\psi_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{nit - xit}\, dt = \frac{(-1)^n}{\pi} \frac{\sin \pi x}{x - n}, \tag{3·1}$$

and (ψ_n) is a normal orthogonal system, complete in the class of Paley-Wiener functions. It is odd that, although these functions occur repeatedly in analysis, especially in the theory of interpolation, it does not seem to have been remarked explicitly that they form an orthogonal system‡.

† Functions of this type play a very important part in Paley and Wiener (5). See in particular Theorem X, p. 13.

‡ Levinson (4) has shown (as an improvement of an earlier theorem of Paley and Wiener) that if

$$G(z) = (z - \lambda_0) \prod_{1}^{\infty} \left\{ \left(1 - \frac{z}{\lambda_n}\right)\left(1 - \frac{z}{\lambda_{-n}}\right) \right\},$$

where
$$|\lambda_n - n| \leqslant D < \tfrac{1}{4}$$
for all n; if p_n is defined by

$$p_n = \frac{e^{\lambda_n i x}}{\sqrt{(2\pi)}} \quad (|x| \leqslant \pi), \qquad p_n = 0 \quad (|x| > \pi):$$

and q_n as the Fourier transform of

$$\frac{G(x)}{G'(\lambda_n)(x - \lambda_n)},$$

which is 0 when $|x| > \pi$; then (p_n) and (q_n) are biorthogonal in $(-\pi, \pi)$ or (what is here the same thing) in $(-\infty, \infty)$.

The Fourier transform of p_n is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\lambda_n i t - x i t} dt = \frac{\sin \pi(x - \lambda_n)}{\pi(x - \lambda_n)}.$$

Hence the two systems
$$\frac{G(x)}{G'(\lambda_n)(x - \lambda_n)}, \quad \frac{\sin \pi(x - \lambda_n)}{\pi(x - \lambda_n)}$$

are biorthogonal. When $\lambda_n = n$, each system reduces to the orthogonal system of the text.

It is easy to verify the orthogonality of the system (3·1) by Cauchy's theorem. If $m \neq n$ then

$$\int_{-\infty}^{\infty} \psi_m \psi_n \, dx = \frac{(-1)^{m+n}}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin^2 \pi x}{(x-m)(x-n)} \, dx$$
$$= \frac{(-1)^{m+n}}{2\pi^2} \Re \int_{-\infty}^{\infty} \frac{1-e^{2\pi i x}}{(x-m)(x-n)} \, dx = 0.$$

If $m = n$ then
$$\int_{-\infty}^{\infty} \psi_n^2 \, dx = \frac{1}{2\pi^2} \Re \int_{-\infty}^{\infty} \frac{1-e^{2\pi i x}}{(x-n)^2} \, dx = 1,$$

the last integral being a principal value at $x = n$.

4. Suppose now that (ψ_n) is the system of § 3, that f is L^2, and that

$$a_n = \int f \psi_n \, dx$$

so that $\Sigma a_n \psi_n$ is the Fourier series of f with respect to the system (ψ_n). If

$$s_n = \sum_{-n}^{n} a_m \psi_m$$

is the 'Fourier polynomial' of f, then there is a function F of L^2, whose Fourier constants are a_n, such that
$$\int (s_n - F)^2 \, dx \to 0.$$

This F usually differs from f, but is f when f is a Paley-Wiener function. The general theory shows that s_n converges to F in mean square; actually, as we shall see, the convergence is uniform in any finite interval.

There is little more to be said about f of L^2, and in what follows I shall consider more general classes of functions. It will be worth while to consider first two examples of expansions of special functions.

5. I begin by giving the values of four definite integrals, two of which will be wanted in this section and the third later. I suppose λ and μ positive, a and b real. Then the four integrals are†

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \lambda(x-a)}{x-a} \frac{\sin \mu(x-b)}{x-b} \, dx = \frac{\sin \nu(a-b)}{a-b}, \tag{5·1}$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1-\cos \lambda(x-a)}{x-a} \frac{\sin \mu(x-b)}{x-b} \, dx = \frac{1-\cos \nu(a-b)}{a-b}, \tag{5·2}$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1-e^{-\lambda i x}}{x} \frac{\sin \mu(x-b)}{x-b} \, dx = \frac{1-e^{-\nu i b}}{b}, \tag{5·3}$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} e^{cix} \frac{\sin \mu(x-b)}{x-b} \, dx = \begin{matrix} e^{cib} & (0 < c < \mu), \\ 0 & (c > \mu), \end{matrix} \tag{5·4}$$

† See Hardy(2).

where in the first three formulae

$$\nu = \operatorname{Min}(\lambda, \mu).$$

The integrals may be evaluated in many ways. For example, we may prove (5·4) by putting $x-b=y$ and using familiar formulae. If we then write it in the form

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{ci(x-a)}\frac{\sin\mu(x-b)}{x-b}\,dx = \begin{matrix} e^{ci(b-a)} & (0<c<\mu), \\ 0 & (c>\mu), \end{matrix}$$

and integrate with respect to c between the limits 0 and λ, we obtain (5·1) and (5·2), and (5·3) follows immediately.

(i) If
$$f(x) = \frac{\sin m(x-a)}{x-a} \quad (m>0),$$
then, by (5·1),

$$\int_{-\infty}^{\infty} f\psi_n\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin m(x-a)}{x-a}\frac{\sin\pi(x-n)}{x-n}\,dx = \frac{\sin\nu(n-a)}{n-a},$$

where $\nu = \operatorname{Min}(m,\pi)$. Hence the Fourier series of $f(x)$ is

$$\frac{\sin\pi x}{\pi}\sum_{-\infty}^{\infty}(-1)^n\frac{\sin\nu(n-a)}{(n-a)(x-n)}.$$

In this case f is L^2. The series converges to $f(x)$ if $m<\pi$.

(ii) If
$$f(x) = e^{cix} \quad (0<c<\pi),$$
then f is not L^2, and the integral

$$\int_{-\infty}^{\infty} f\phi_n\,dx = \frac{(-1)^n}{\pi}\int_{-\infty}^{\infty} e^{cix}\frac{\sin\pi x}{x-n}\,dx = e^{cin}$$

is not absolutely convergent. We may, however, regard e^{cin} as a generalized Fourier coefficient of f, and

$$\frac{\sin\pi x}{\pi}\sum\frac{(-1)^n}{x-n}e^{cin}$$

as its generalized Fourier series. The series converges to $f(x)$.

6. We shall say that f belongs to M if

$$\int_{-\infty}^{\infty}\frac{|f(x)|}{2+|x|}\,dx<\infty, \tag{6·1}$$

and to M^*
$$\int_{-\infty}^{\infty}\frac{|f(x)|\log(2+|x|)}{2+|x|}\,dx<\infty. \tag{6·2}$$

Plainly M includes M^*.

If f belongs to M then
$$a_n = \int f\psi_n\,dx$$

exists (as a Lebesgue integral), and we can speak of the Fourier coefficient, series, and polynomial of f. The Fourier polynomial is given by the formula

$$s_n(x) = \sum_{-n}^{n} a_m \psi_m(x) = \int_{-\infty}^{\infty} f(y)\, S_n(x,y)\, dy, \tag{6·3}$$

where
$$S_n(x,y) = \sum_{-n}^{n} \psi_m(x)\,\psi_m(y) = \frac{\sin \pi x \sin \pi y}{\pi^2} \sum_{-n}^{n} \frac{1}{(x-m)(y-m)}. \tag{6·4}$$

We shall require the properties of $S_n(x,y)$ set out in the following theorem.

THEOREM 1. *If $S_n(x,y)$ is defined by* (6·4), *then*

(i)
$$|S_n| < A;$$

(ii)
$$S_n \to \frac{\sin \pi(x-y)}{\pi(x-y)}$$

when $n \to \infty$, for any fixed x and y;

(iii) *this convergence is uniform for $|x| \leqslant X$ and all y (or $|y| \leqslant Y$ and all x);*

(iv)
$$|S_n| \leqslant B(X) \frac{\log(2+|y|)}{2+|y|}$$

for $|x| \leqslant X$ and all y;

(v) *if*
$$S_n^1 = \frac{1}{n} \sum_{0}^{n} S_m = \sum_{-n}^{n} \left(1 - \frac{|m|}{n}\right) \psi_m(x)\,\psi_m(y)$$

is the Cesàro mean of S_n, then

$$|S_n^1| \leqslant \frac{B(X)}{2+|y|}$$

for $|x| \leqslant X$ and all y. Here A is a constant and $B(X)$ a number depending on X only, and any term which is undefined for a particular x or y is then to be interpreted as a limit†.

7. We begin by expressing S_n as a definite integral. Since

$$\psi_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{mit - xit}\, dt, \quad \psi_m(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-miu + yiu}\, du,$$

we have

$$4\pi^2 S_n(x,y) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-xit + yiu} \sum_{-n}^{n} e^{mi(t-u)}\, dt\, du$$
$$= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-xit + yiu} \frac{\sin(n+\frac{1}{2})(t-u)}{\sin\frac{1}{2}(t-u)}\, dt\, du.$$

If here we make the substitution $u = t + w$, and then change the order of integration, we obtain

$$\int_0^{2\pi} \frac{\sin(n+\frac{1}{2})w}{\sin\frac{1}{2}w} e^{yiw}\, dw \int_{-\pi}^{\pi - w} e^{-(x-y)it}\, dt$$
$$+ \int_{-2\pi}^{0} \frac{\sin(n+\frac{1}{2})w}{\sin\frac{1}{2}w} e^{yiw}\, dw \int_{-\pi-w}^{\pi} e^{-(x-y)it}\, dt.$$

† Thus $\psi_m(x) = 0$ if x is an integer other than m, and $\psi_m(m) = 1$.

Performing the inner integrations, and writing

$$\xi = \tfrac{1}{2}(x+y), \quad \eta = \tfrac{1}{2}(x-y),$$

we obtain, after a little calculation,

$$S_n(x,y) = \frac{1}{2\pi^2\eta}\int_0^{2\pi} \frac{\sin(n+\frac{1}{2})w}{\sin\frac{1}{2}w} \cos\xi w \sin\eta\,(2\pi-w)\,dw, \tag{7·1}$$

or $$S_n(x,y) = \frac{1}{2\pi^2\eta}\int_0^{\pi} \frac{\sin(n+\frac{1}{2})w}{\sin\frac{1}{2}w} \{\cos\xi w \sin\eta\,(2\pi-w) + \cos\xi(2\pi-w)\sin\eta w\}\,dw. \tag{7·2}$$

8. We now write N for $n+\frac{1}{2}$ and express S_n, from (7·2), as the sum of the three terms

$$U_n = \frac{\sin 2\pi\eta}{2\pi^2\eta}\int_0^{\pi} \frac{\sin Nw}{\sin\frac{1}{2}w}\cos\xi w \cos\eta w\,dw, \tag{8·1}$$

$$V_n = \frac{\sin 2\pi\xi}{2\pi^2}\int_0^{\pi} \frac{\sin Nw}{\sin\frac{1}{2}w}\sin\xi w \frac{\sin\eta w}{\eta}\,dw, \tag{8·2}$$

$$W_n = \frac{\cos 2\pi\xi - \cos 2\pi\eta}{2\pi^2}\int_0^{\pi} \frac{\sin Nw}{\sin\frac{1}{2}w}\cos\xi w \frac{\sin\eta w}{\eta}\,dw. \tag{8·3}$$

We prove first that in each of these integrals we may replace the $\sin\frac{1}{2}w$ in the denominator by $\frac{1}{2}w$. The errors so introduced are of the types

$$\frac{\sin 2\pi\eta}{2\pi^2\eta}\int_0^{\pi} \sin Nw \cos\xi w \cos\eta w\,\chi(w)\,dw,\ \ldots,\ \ldots, \tag{8·4}$$

where $\chi(w)$ is regular. We show that these errors (i) are uniformly bounded, (ii) tend to zero when $N\to\infty$, (iii) tend to zero uniformly for $|x| \leqslant X$, (iv) are $O(|y|^{-1})$, for large $|y|$, uniformly for $|x| \leqslant X$.

Of these propositions (i), (ii) and (iv) are obvious, the last because

$$|\eta| \geqslant \tfrac{1}{2}(|y| - X).$$

As regards (iii), we write the first of (8·4) in the form

$$-\frac{\sin 2\pi\eta}{2\pi^2\eta}\frac{1}{N}\int_0^{\pi} \frac{d}{dw}(\cos Nw)\cos\xi w \cos\eta w\,\chi(w)\,dw$$

and integrate by parts. We thus obtain four terms† of the types

$$O\left|\frac{1}{N}\frac{\sin 2\pi\eta}{\eta}\right|, \quad O\left|\frac{1}{N}\frac{\sin 2\pi\eta}{\eta}\xi\right|, \quad O\left|\frac{1}{N}\frac{\sin 2\pi\eta}{\eta}\eta\right|, \quad O\left|\frac{1}{N}\frac{\sin 2\pi\eta}{\eta}\right|,$$

where the O's are uniform in all the variables. All of these except the second are $O(N^{-1})$ uniformly for all x and y. The second is

$$O\left|\frac{1}{N}\frac{\sin 2\pi\eta}{\eta}(|y|+X)\right| = O\left|\frac{1}{N}\frac{\sin 2\pi\eta}{\eta}(|\eta|+X)\right|$$

and is $O(N^{-1})$ uniformly for $|x| \leqslant X$.

† One from the terms integrated out, the rest from differentiating $\cos\xi w \cos\eta w\,\chi(w)$.

The other errors (8·4) may be dealt with similarly, if we remember that $|\sin\eta w| \leqslant |\eta w|$. Thus, when we consider the second error, the most troublesome term (corresponding to that just discussed) is

$$O\left\{\frac{1}{N}\int_0^\pi\left|\xi\frac{\sin\eta w}{\eta}\right|dw\right\} = O\left\{\frac{1}{N}\int_0^\pi(|\eta|+X)\left|\frac{\sin\eta w}{\eta}\right|dw\right\},$$

and is again $O(N^{-1})$ uniformly for $|x| \leqslant X$.

We have now to discuss the integrals U_n^*, V_n^*, W_n^* obtained by replacing $\sin\frac{1}{2}w$ by $\frac{1}{2}w$ in U_n, V_n, W_n.

9. (i) Thus

$$U_n^* = \frac{\sin 2\pi\eta}{\pi^2\eta}\int_0^\pi\frac{\sin Nw}{w}\cos\xi w\cos\eta w\,dw = \frac{\sin 2\pi\eta}{2\pi^2\eta}\int_0^\pi\frac{\sin Nw}{w}(\cos xw+\cos yw)\,dw$$
$$= \frac{\sin 2\pi\eta}{4\pi^2\eta}\int_0^\pi\left\{\frac{\sin(N+x)w}{w}+\frac{\sin(N-x)w}{w}+\frac{\sin(N+y)w}{w}+\frac{\sin(N-y)w}{w}\right\}dw. \tag{9·1}$$

It is plain that U_n^* is uniformly bounded, tends to the limit

$$\frac{\sin 2\pi\eta}{2\pi\eta} = \frac{\sin\pi(x-y)}{\pi(x-y)},$$

and is $O(|y|^{-1})$, uniformly for $|x| \leqslant X$.

We have still to prove that U_n^* tends to its limiting value uniformly for $|x| \leqslant X$. It is plain that it does so uniformly in any domain in which $|x|$ and $|y|$ are *both* bounded. We may therefore suppose (taking y, for example, positive) that $y > 3X$, in which case ξ and $-\eta$ are greater than $\frac{1}{3}y$.

It follows from the last formula in (9·1) that U_n^* is the sum of four integrals each of which, except perhaps the last, tends to its limit uniformly for $|x| \leqslant X$. The last is

$$O\left(\frac{1}{y}\right)\int_0^{(N-y)\pi}\frac{\sin u}{u}du \quad (y \leqslant N), \qquad O\left(\frac{1}{y}\right)\int_0^{(y-N)\pi}\frac{\sin u}{u}du \quad (y > N);$$

and differs from its limit by

$$O\left(\frac{1}{y}\right)\int_{(N-y)\pi}^{\infty}\frac{\sin u}{u}du \quad (y \leqslant N), \qquad O\left(\frac{1}{y}\right)\int_{(y-N)\pi}^{\infty}\frac{\sin u}{u}du \quad (y > N) \tag{9·2}$$

(all these O's being, as before, uniform in all the variables). If, as we are supposing, $y > 3X$, then the second of (9·2) is $O(y^{-1})$ and so $O(N^{-1})$. The first is $O(N^{-1})$ if $y \leqslant \frac{1}{2}N$, and $O(y^{-1}) = O(N^{-1})$ if $y > \frac{1}{2}N$. In any case the last of the four integrals in U_n^* tends to its limit uniformly.

(ii) Next

$$V_n^* = \frac{\sin 2\pi\xi}{\pi^2\eta}\int_0^\pi\frac{\sin Nw}{w}\sin\xi w\sin\eta w\,dw$$
$$= \frac{\sin 2\pi\xi}{4\pi^2\eta}\int_0^\pi\left\{\frac{\sin(N+y)w}{w}+\frac{\sin(N-y)w}{w}-\frac{\sin(N+x)w}{w}-\frac{\sin(N-x)w}{w}\right\}dw. \tag{9·3}$$

It is therefore uniformly bounded, and $O(|y|^{-1})$ uniformly for $|x| \leqslant X$. Its limit is 0. It is also plain that it tends to its limit uniformly in any bounded domain of x and y, and the proof that it does so uniformly for $|x| \leqslant X$ may then be completed as before.

(iii) Finally

$$W_n^* = \frac{\cos 2\pi\xi - \cos 2\pi\eta}{\pi^2} \int_0^\pi \sin Nw \cos \xi w \frac{\sin \eta w}{\eta w}\, dw \tag{9·4}$$

is also uniformly bounded and has the limit 0. We have to prove that W_n^* satisfies clauses (iii) and (iv) of the theorem; and in doing so we may suppose, as before, that $y > 3X$, and also that $N > 3X$.

We can write W_n^* in the form

$$W_n^* = \frac{\cos 2\pi\eta - \cos 2\pi\xi}{2\pi^2\eta} \int_0^\pi \frac{\sin Nw}{w} (\sin yw - \sin xw)\, dw$$
$$= K(L_n - M_n) = P_n - Q_n, \tag{9·5}$$

where
$$K = \frac{\cos 2\pi\eta - \cos 2\pi\xi}{4\pi^2\eta} = O\left(\frac{1}{y}\right), \tag{9·6}$$

$$L_n = \int_0^\pi \frac{\cos (N-y)\, w - \cos (N+y)\, w}{w}\, dw = \int_{|N-y|\,\pi}^{(N+y)\pi} \frac{1 - \cos u}{u}\, du, \tag{9·7}$$

and similarly
$$M_n = \int_{(N-x)\,\pi}^{(N+x)\,\pi} \frac{1-\cos u}{u}\, du. \tag{9·8}$$

We dispose first of $Q_n = KM_n$. It follows from (9·8) that

$$M_n = O\left(\log \frac{N+|x|}{|N-|x||}\right) = O\left(\frac{1}{N}\right).$$

Hence both of
$$|Q_n| \leqslant \frac{B(X)}{y}, \quad |Q_n| \leqslant \frac{B(X)}{N}$$

are true for $|x| \leqslant X$, $y > 3X$, $N > 3X$; and Q_n satisfies our requirements.

As regards P_n, it follows from (9·7), on the one hand that

$$L_n = O\{\log (N+y)\}, \tag{9·9}$$

and on the other that
$$L_n = O\left(\log \frac{N+y}{|N-y|}\right). \tag{9·10}$$

We distinguish the three cases

$$(a)\ y \leqslant \tfrac{1}{2}N, \quad (b)\ \tfrac{1}{2}N < y \leqslant 2N, \quad (c)\ y > 2N.$$

In case (a), (9·10) shows that

$$L_n = O\left(\frac{y}{N}\right), \quad P_n = O\left(\frac{1}{N}\right) = O\left(\frac{1}{y}\right),$$

$$|P_n| \leqslant \frac{B(X)}{y}, \quad |P_n| \leqslant \frac{B(X)}{N}. \tag{9·11}$$

In case (b), (9·9) shows that

$$L_n = O(\log y), \quad P_n = O\left(\frac{\log y}{y}\right) = O\left(\frac{\log N}{N}\right),$$

$$|P_n| \leqslant B(X)\frac{\log y}{y}, \quad |P_n| \leqslant B(X)\frac{\log N}{N}. \tag{9·12}$$

In case (c), (9·10) shows that

$$L_n = O(1), \quad P_n = O\left(\frac{1}{y}\right) = O\left(\frac{1}{N}\right);$$

and P_n again satisfies the inequalities (9·11). Hence in any case P_n satisfies our requirements; and this completes the proof of clauses (i)–(iv) of the theorem.

There remains clause (v). To prove this we observe (a) that what is true for S_n is true a fortiori for S_n^1, and (b) that the logarithm in (iv) arises only from

$$L_n = K\int_0^\pi \frac{\sin Nw}{w}\sin yw\,dw.$$

The effect of averaging is to replace $\sin Nw$ by

$$\frac{1}{n+1}\{\sin\tfrac{1}{2}w + \sin\tfrac{3}{2}w + \ldots + \sin(n+\tfrac{1}{2})w\} = \frac{1}{n+1}\frac{\sin^2\frac{1}{2}(n+1)w}{\sin\frac{1}{2}w}$$

Since $$\int_0^\pi \frac{\sin^2\frac{1}{2}(n+1)w}{(n+1)w\sin\frac{1}{2}w}dw \leqslant \frac{1}{2(n+1)}\int_0^\pi\left\{\frac{\sin\frac{1}{2}(n+1)w}{\sin\frac{1}{2}w}\right\}^2 dw$$

is uniformly bounded, there is no logarithm in the upper bound for the Cesàro mean of L_n, or that of S_n.

10. Theorem 2. *If $f(x)$ belongs to M^*, then the Fourier series of $f(x)$ converges to*

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\pi(y-x)}{y-x}f(y)\,dy, \tag{10·1}$$

uniformly in any finite interval of values of x.

Theorem 3. *If $f(x)$ belongs to M, then the series is summable $(C, 1)$ to $F(x)$, with the same uniformity.*

To prove Theorem 2, we have only to observe that

$$\sum_{-n}^{n} a_m\psi_m(x) = \int_{-\infty}^{\infty} f(y)\,S_n(x,y)\,dy,$$

that the integral is majorized, for $|x| \leqslant X$, by

$$B(X)\int_{-\infty}^{\infty}|f(y)|\frac{\log(2+|y|)}{2+|y|}dy,$$

and that $$S_n(x,y) \to \frac{\sin\pi(x-y)}{\pi(x-y)}$$

uniformly for $|x| \leqslant X$ and all y. The result then follows from a familiar theorem of Lebesgue.

Theorem 3 follows in the same way if we use clause (v) of Theorem 1 instead of clause (iv).

11. If $f(x)$ satisfies the integral equation

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\pi(y-x)}{y-x} f(y)\,dy, \tag{11·1}$$

then the sum of the series is $f(x)$. This integral equation seems to have been considered first by Bateman†.

If f is L^2 then, after § 4, it is necessary and sufficient that f should be a Paley-Wiener function. We now consider conditions for the truth of (11·1) when we are given only that $f(x)$ belongs to M. In this case (and only in this case‡) the integral in (11·1) is absolutely convergent for all x. We shall denote by B the sub-class of M formed by functions which satisfy (11·1).

THEOREM 4. *If f is B then*

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{xit}\,dG(t), \tag{11·2}$$

where

$$G(x) = i\int_{-\infty}^{\infty} \frac{e^{-xit}-1}{t} f(t)\,dt. \tag{11·3}$$

The integral in (11·2) is a Stieltjes integral as defined originally by Stieltjes, and§ (11·2) is equivalent to

$$f(x) = \frac{G(\pi)}{2\pi}e^{\pi i x} - \frac{G(-\pi)}{2\pi}e^{-\pi i x} - \frac{ix}{2\pi}\int_{-\pi}^{\pi} e^{xit}\,G(t)\,dt. \tag{11·4}$$

The integral (11·3) is absolutely convergent, and uniformly convergent in any finite interval of x, so that $G(x)$ is continuous.

If we substitute for $G(x)$, in (11·4), from (11·3), we obtain a sum of three terms.

The first term is $$\frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{f(u)}{u}\{e^{\pi i(x-u)} - e^{\pi i x}\}\,du$$

and the second is $$-\frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{f(u)}{u}\{e^{-\pi i(x-u)} - e^{-\pi i x}\}\,du,$$

so that the two together give

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(u)}{u}\{\sin\pi(u-x) + \sin\pi x\}\,du. \tag{11·5}$$

† Bateman (**1**). See also Hardy (**2**), Hardy and Titchmarsh (**3**), and Titchmarsh (**8**, **349**).

‡ Consider the special cases $x = 0$ and $x = \frac{1}{2}$, and observe that

$$|\sin\pi x| + |\cos\pi x| \geqslant \sin^2\pi x + \cos^2\pi x = 1.$$

§ See, for example, Pollard (**7**, **79**).

The third gives

$$\frac{x}{2\pi}\int_{-\pi}^{\pi} e^{xit}\,dt\int_{-\infty}^{\infty} f(u)\frac{e^{-tiu}-1}{u}\,du = \frac{x}{2\pi}\int_{-\infty}^{\infty}\frac{f(u)}{u}\,du\int_{-\pi}^{\pi}\{e^{(x-u)it}-e^{xit}\}\,dt$$

$$= \frac{x}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{u}\left\{\frac{\sin\pi(u-x)}{u-x}-\frac{\sin\pi x}{x}\right\}du; \tag{11·6}$$

and this, added to (11·5), gives

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{u}\sin\pi(u-x)\left(\frac{x}{u-x}+1\right)dx = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\frac{\sin\pi(u-x)}{u-x}\,du = f(x).$$

12. THEOREM 5. *$G(x)$ is constant outside $(-\pi,\pi)$; i.e.*

$$G(x) = G(-\pi)\quad (x<-\pi),\qquad G(x) = G(\pi)\quad (x>\pi).$$

We have
$$G(x) = i\lim_{T\to\infty}\int_{-T}^{T}\frac{e^{-xit}-1}{t}f(t)\,dt \tag{12·1}$$

$$= i\lim\int_{-T}^{T}\frac{e^{-xit}-1}{t}\,dt\int_{-\infty}^{\infty}\frac{\sin\pi(u-t)}{\pi(u-t)}f(u)\,du$$

$$= i\lim\int_{-\infty}^{\infty} f(u)\,\chi(u,x,T)\,du,$$

where
$$\chi(u,x,T) = \int_{-T}^{T}\frac{e^{-xit}-1}{t}\,\frac{\sin\pi(t-u)}{\pi(t-u)}\,dt \tag{12·2}$$

(the inversion of the order of integration being justified by absolute convergence). Let us assume provisionally that

$$\lim\int_{-\infty}^{\infty} f(u)\,\chi(u,x,T)\,du = \int_{-\infty}^{\infty} f(u)\,\chi(u,x)\,du, \tag{12·3}$$

where
$$\chi(u,x) = \lim\chi(u,x,T) = \int_{-\infty}^{\infty}\frac{e^{-xit}-1}{t}\,\frac{\sin\pi(t-u)}{\pi(t-u)}\,dt. \tag{12·4}$$

Then, after (5·3),

$$\chi(x,u) = \frac{e^{-\pi iu}-1}{u}\quad (x>\pi),\qquad \chi(x,u) = \frac{e^{\pi iu}-1}{u}\quad (x<-\pi).$$

Hence
$$G(x) = i\int_{-\infty}^{\infty}\frac{e^{-\pi iu}-1}{u}f(u)\,du = G(\pi)$$

if $x>\pi$; and similarly $G(x) = G(-\pi)$ if $x<-\pi$.

It is therefore only necessary to prove (12·3); and, since f belongs to M, it is sufficient to prove that

$$\chi(u,x,T) = O\left(\frac{1}{|u|}\right) \tag{12·5}$$

for large $|u|$, a fixed x outside $(-\pi,\pi)$, and uniformly in T. Let us suppose, for example, that $x>\pi$. Since (12·5) is trivial when $T\leqslant 1$, we may suppose that $T>1$, and it is enough to consider the integrals over the ranges $(-T,-1)$ and $(1,T)$: we select the latter.

Separating the real and imaginary parts, we have to prove that

$$I = \int_1^T \frac{\sin xt}{t} \frac{\sin \pi(t-u)}{t-u}\, dt = O\left(\frac{1}{|u|}\right), \tag{12·6}$$

and

$$J = \int_1^T \frac{1-\cos xt}{t} \frac{\sin \pi(t-u)}{t-u}\, dt = O\left(\frac{1}{|u|}\right). \tag{12·7}$$

We suppose $u > 0$ (the case $u < 0$ being a little simpler).

First
$$I = \frac{1}{u}\int_1^T \sin xt \sin \pi(t-u)\left(\frac{1}{t-u} - \frac{1}{t}\right) dt,$$

and it is sufficient to prove that

$$I_1 = \int_1^T \frac{\sin xt}{t} \sin \pi(t-u)\, dt = O(1), \quad I_2 = \int_1^T \sin xt \,\frac{\sin \pi(t-u)}{t-u}\, dt = O(1). \tag{12·8}$$

Now
$$I_1 = \cos \pi u \int_1^T \frac{\sin xt \sin \pi t}{t}\, dt - \sin \pi u \int_1^T \frac{\sin xt \cos \pi t}{t}\, dt.$$

The second term is uniformly bounded. The first is

$$\tfrac{1}{2}\cos \pi u \int_1^T \{\cos (x-\pi)t - \cos (x+\pi)t\} \frac{dt}{t}$$
$$= \tfrac{1}{2}\cos \pi u \left\{ \int_{x-\pi}^{x+\pi} \frac{\cos w}{w}\, dw - \int_{(x-\pi)T}^{(x+\pi)T} \frac{\cos w}{w}\, dw \right\}.$$

Each integral here is numerically less than

$$\log \frac{x+\pi}{x-\pi},$$

so that I_1 is uniformly bounded.

Next,
$$I_2 = \sin xu \int_{1-u}^{T-u} \frac{\cos xv \sin \pi v}{v}\, dv + \cos xu \int_{1-u}^{T-u} \frac{\sin xv \sin \pi v}{v}\, dv.$$

The first integral is plainly uniformly bounded. The second is

$$\tfrac{1}{2}\cos xu \int_{1-u}^{T-u} \frac{\cos (x-\pi)v - \cos (x+\pi)v}{v}\, dv$$
$$= -\tfrac{1}{2}\cos xu \left\{ \int_{(x-\pi)(1-u)}^{(x+\pi)(1-u)} \frac{1-\cos w}{w}\, dw - \int_{(x-\pi)(T-u)}^{(x+\pi)(T-u)} \frac{1-\cos w}{w}\, dw \right\},$$

and is also uniformly bounded; so that I_2, and therefore I, is uniformly bounded.

Similarly, in order to prove (12·7), it is sufficient to prove that

$$J_1 = \int_1^T \frac{1-\cos xt}{t} \sin \pi(t-u)\, dt = O(1), \quad J_2 = \int_1^T (1-\cos xt)\frac{\sin \pi(t-u)}{t-u}\, dt = O(1). \tag{12·9}$$

Now
$$J_1 = \cos \pi u \int_1^T \frac{(1-\cos xt)\sin \pi t}{t}\, dt - \sin \pi u \int_1^T \frac{(1-\cos xt)\cos \pi t}{t}\, dt.$$

The first term is uniformly bounded. The second is

$$-\tfrac{1}{2}\sin\pi u\left\{\int_1^T \frac{\cos\pi t-\cos(x+\pi)t}{t}\,dt+\int_1^T \frac{\cos\pi t-\cos(x-\pi)t}{t}\,dt\right\},$$

and may be accounted for in the same way as the second term in I_2. Finally,

$$J_2 = \int_{1-u}^{T-u}\{1-\cos x(u+v)\}\frac{\sin\pi v}{v}\,dv$$

$$= \int_{1-u}^{T-u}\frac{\sin\pi v}{v}\,dv-\cos xu\int_{1-u}^{T-u}\frac{\cos xv\sin\pi v}{v}\,dv+\sin xu\int_{1-u}^{T-u}\frac{\sin xv\sin\pi v}{v}\,dv,$$

and all the integrals here are uniformly bounded, so that J_2, and therefore J, is uniformly bounded. This completes the proof of Theorem 5.

13. It follows from Theorems 4 and 5 that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{xit}\,dG(t), \tag{13·1}$$

the integral being defined as $\lim \int_{-T}^{T}$. We have also

$$\frac{1}{2\pi}\int_{-T}^{T} e^{xit}\,dG(t) = \frac{G(\pi)}{2\pi}e^{xiT}-\frac{G(-\pi)}{2\pi}e^{-xiT}-\frac{xi}{2\pi}\int_{-T}^{T} e^{xit}\,G(t)\,dt.$$

The left-hand side tends to $f(x)$ when $T\to\infty$. The two first terms on the right do not tend to limits in the ordinary sense, but each of them has the $(C, 1)$ limit 0 except when $x = 0$†. Hence

$$f(x) = \frac{x}{2\pi i}\int_{-\infty}^{\infty} e^{xit}\,G(t)\,dt \quad (C,1) \tag{13·2}$$

if $x\neq 0$, this meaning that

$$f(x) = \frac{x}{2\pi i}\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\int_{-t}^{t} e^{xiu}\,G(u)\,du = \frac{x}{2\pi i}\lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)e^{xit}\,G(t)\,dt.$$

14. I prove next two theorems in which f is not assumed to belong to M.

THEOREM 6. *If* $f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^{xit}\,dG(t)$, *where* $G(t)$ *is continuous, then*

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\pi(y-x)}{y-x}f(y)\,dy \quad (C,1). \tag{14·1}$$

We write

$$f(y) = \frac{G(\pi)}{2\pi}e^{\pi iy}-\frac{G(-\pi)}{2\pi}e^{-\pi iy}-\frac{iy}{2\pi}\int_{-\pi}^{\pi} e^{yit}\,G(t)\,dt,$$

$$\frac{1}{\pi}\int_{-Y}^{Y}\left(1-\frac{|y|}{Y}\right)\frac{\sin\pi(y-x)}{y-x}f(y)\,dy = I_1+I_2+I_3, \tag{14·2}$$

† $\phi(T)\to l$ $(C, 1)$ means that $\dfrac{1}{T}\displaystyle\int_a^T \phi(u)\,du\to l$ for some fixed a, which it is natural to take as 0.

where $$I_1 = \frac{G(\pi)}{2\pi^2}\int_{-Y}^{Y}\left(1-\frac{|y|}{Y}\right)\frac{\sin\pi(y-x)}{y-x}e^{\pi iy}\,dy, \tag{14·3}$$

$$I_2 = -\frac{G(-\pi)}{2\pi^2}\int_{-Y}^{Y}\left(1-\frac{|y|}{Y}\right)\frac{\sin\pi(y-x)}{y-x}e^{-\pi iy}\,dy, \tag{14·4}$$

$$I_3 = -\frac{i}{2\pi^2}\int_{-Y}^{Y}\left(1-\frac{|y|}{Y}\right)\frac{\sin\pi(y-x)}{y-x}y\,dy\int_{-\pi}^{\pi}e^{yit}G(t)\,dt. \tag{14·5}$$

Then $$I_1 \to \frac{G(\pi)}{2\pi^2}\int_{-\infty}^{\infty}\frac{\sin\pi(y-x)}{y-x}e^{\pi iy}\,dy = \frac{G(\pi)}{4\pi}e^{\pi ix}, \tag{14·6}$$

and similarly $$I_2 \to -\frac{G(-\pi)}{4\pi}e^{-\pi ix}, \tag{14·7}$$

when $Y\to\infty$.

As regards I_3, the integrand (with respect to y) is

$$O\left(\frac{1}{Y}\cdot\frac{1}{Y}\cdot Y\,.\,1\right) = O\left(\frac{1}{Y}\right)$$

near $y = Y$, so that we can replace the limits by $-Y+x$ and $Y+x$. We may also replace $|y|$ by $|y-x|$, since this makes a difference

$$\int_{-Y+x}^{Y+x}O\left(\frac{1}{Y}\right)\left|\frac{\sin\pi(y-x)}{y-x}\right||y|\,o(1)\,dy$$

$$= O\left(\int_{x-1}^{x+1}\frac{1}{Y}dy\right)+o\left(\int_{-Y+x}^{Y+x}\frac{1}{Y}dy\right) = o(1).$$

That is to say, we may replace I_3 by

$$J_3 = -\frac{i}{2\pi^2}\int_{-Y+x}^{Y+x}\left(1-\frac{|y-x|}{Y}\right)\frac{\sin\pi(y-x)}{y-x}y\,dy\int_{-\pi}^{\pi}e^{yit}G(t)\,dt \tag{14·8}$$

$$= -\frac{i}{2\pi^2}\int_{-Y}^{Y}\left(1-\frac{|u|}{Y}\right)\frac{\sin\pi u}{u}(u+x)\,du\int_{-\pi}^{\pi}e^{(u+x)it}G(t)\,dt$$

$$= -\frac{i}{2\pi^2}\int_{-\pi}^{\pi}e^{xit}G(t)\,dt\int_{-Y}^{Y}\left(1-\frac{|u|}{Y}\right)(x+u)\frac{\sin\pi u}{u}e^{uit}\,du.$$

The inner integral here is

$$\int_{-Y}^{Y}\left(1-\frac{|u|}{Y}\right)\sin\pi u\, e^{uit}\,du + x\int_{-Y}^{Y}\left(1-\frac{|u|}{Y}\right)\frac{\sin\pi u}{u}e^{uit}\,du. \tag{14·9}$$

The first term is $$\frac{2i}{Y}\left\{\frac{\sin^2\frac{1}{2}(\pi-t)\,Y}{(\pi-t)^2}-\frac{\sin^2\frac{1}{2}(\pi+t)\,Y}{(\pi+t)^2}\right\};$$

and its contribution to J_3 is

$$\frac{1}{\pi^2Y}\int_{-\pi}^{\pi}e^{xit}G(t)\frac{\sin^2\frac{1}{2}(\pi-t)\,Y}{(\pi-t)^2}dt-\frac{1}{\pi^2Y}\int_{-\pi}^{\pi}e^{xit}G(t)\frac{\sin^2\frac{1}{2}(\pi+t)\,Y}{(\pi+t)^2}dt,$$

which tends to $$\frac{G(\pi)}{4\pi}e^{\pi ix}-\frac{G(-\pi)}{4\pi}e^{-\pi ix}.$$

The second term in (14·9) converges boundedly to

$$x\int_{-\infty}^{\infty}\frac{\sin \pi u}{u}e^{uit}\,du = \pi x$$

for $|t| < \pi$. Hence its contribution is

$$-\frac{xi}{2\pi}\int_{-\pi}^{\pi}e^{xit}\,G(t)\,dt,$$

and

$$I_3 \to \frac{G(\pi)}{4\pi}e^{\pi i x} - \frac{G(-\pi)}{4\pi}e^{-\pi i x} - \frac{xi}{2\pi}\int_{-\pi}^{\pi}e^{xit}\,G(t)\,dt. \tag{14·10}$$

Finally, from (14·2), (14·6), (14·7) and (14·10) it follows that

$$\frac{1}{\pi}\int_{-Y}^{Y}\left(1-\frac{|y|}{Y}\right)\frac{\sin \pi(y-x)}{y-x}f(y)\,dy \to \frac{G(\pi)}{2\pi}e^{\pi i x} - \frac{G(-\pi)}{2\pi}e^{-\pi i x} - \frac{xi}{2\pi}\int_{-\pi}^{\pi}e^{xit}\,G(t)\,dt = f(x).$$

THEOREM 7. *If $G(t)$ is continuous and satisfies a Fourier convergence criterion at $t = -\pi$ and $t = \pi$ (for example, if it is of bounded variation near these points), then*

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin \pi(y-x)}{y-x}f(y)\,dy, \tag{14·11}$$

the integral being a Cauchy integral.

The proof is an obvious modification of that of Theorem 6. We have only to omit the factor $1-|y|/Y$, and to appeal to the theory of Dirichlet's instead of that of Fejér's singular integral. When f is M, the integral in (14·1) or (14·11) is absolutely convergent.

Combining either of Theorems 6 and 7 with Theorem 4, we obtain

THEOREM 8. *If f is M then, in order that it should be B, it is necessary and sufficient that it should be expressible in the form (11·2), with a continuous $G(t)$.*

THE CARDINAL SERIES

15. The *cardinal series*† of $f(x)$ is

$$\frac{\sin \pi x}{\pi}\sum_{-\infty}^{\infty}(-1)^n\frac{f(n)}{x-n} = \frac{\sin \pi x}{\pi}\lim_{N\to\infty}\sum_{-N}^{N}(-1)^n\frac{f(n)}{x-n}$$
$$= \frac{\sin \pi x}{\pi}\left[\frac{f(0)}{x} + \sum_{1}^{\infty}(-1)^n\left\{\frac{f(n)}{x-n}+\frac{f(-n)}{x+n}\right\}\right]. \tag{15·1}$$

It reduces formally to $f(n)$ when x is an integer n. It may also be written as

$$\sum_{-\infty}^{\infty}f(n)\,\psi_n(x).$$

If f is M, and F is defined by (10·1), then

$$F(n) = \frac{(-1)^n}{\pi}\int_{-\infty}^{\infty}\frac{\sin \pi y}{y-n}f(y)\,dy = a_n,$$

and the Fourier series of $f(x)$ is $\Sigma F(n)\,\psi_n(x)$. If $f(x)$ is B, then $F(x) = f(x)$, and the

† The term was introduced by E. T. Whittaker (9).

Fourier series is the cardinal series. Hence, combining Theorem 8 with Theorems 2 and 3, we obtain

THEOREM 9. *If $f(x)$ is M^*, then* (11·2) *is a sufficient condition for the convergence of the cardinal series to $f(x)$ for all x. If $f(x)$ is M, it is sufficient for summability* $(C, 1)$.

16. There is no reference to the class M (or any other order condition on f) in Theorems 6 and 7, and it is natural to ask what we can prove about the convergence of the cardinal series when we assume only that f is of the form (11·2). J. M. Whittaker† has proved that the cardinal series is then summable $(C, 1)$ to $f(x)$, and the theorem which follows is much the same as his, though a little more precise.

I observe first that, if $f(x)$ is of the form (11·2), then

$$f(n) = \int_{-\infty}^{\infty} \psi_n(y) f(y)\, dy \quad (C, 1),$$

by (14·1), so that we can still, if we like, call $f(n)$ the Fourier coefficient and the cardinal series the Fourier series of $f(x)$.

THEOREM 10. *If $f(x)$ is of the form* (11·2), *then*

$$f(x) = \frac{\sin \pi x}{\pi}\left\{f'(0) + \frac{f(0)}{x} + \Sigma'(-1)^n\left(\frac{1}{x-n} + \frac{1}{n}\right)f(n)\right\},$$

the series, summed over all integral values of n except $n = 0$, being convergent. If the Fourier series of $G(t)$, for the interval $(-\pi, \pi)$, is convergent for $t = -\pi$ and $t = \pi$, then the cardinal series of $f(x)$ converges to $f(x)$; and it is summable $(C, 1)$ to $f(x)$ in any case.

Suppose that
$$\Sigma u_n e^{-nit}$$
is the Fourier series of G. Then

$$\Sigma(-1)^n u_n = \tfrac{1}{2}\{G(\pi) + G(-\pi)\} \quad (C, 1). \tag{16·1}$$

Also, from (11·4),
$$2\pi f(0) = G(\pi) - G(-\pi), \tag{16·2}$$

$$2\pi f'(0) = \pi i\{G(\pi) + G(-\pi)\} - i\int_{-\pi}^{\pi} G(t)\, dt = \pi i\{G(\pi) + G(-\pi) - 2u_0\}, \tag{16·3}$$

$$2\pi f(n) = (-1)^n\{G(\pi) - G(-\pi)\} - ni\int_{-\pi}^{\pi} e^{nit} G(t)\, dt = (-1)^n 2\pi f(0) - 2n\pi i u_n, \tag{16·4}$$

and so
$$u_n = \frac{i}{n}\{f(n) - (-1)^n f(0)\} \quad (n \neq 0). \tag{16·5}$$

If now we substitute the Fourier series for $G(t)$ in (11·4), remember that we may integrate it term by term after multiplication by e^{xit}, and use a dash to exclude $n = 0$ from summation, we obtain

$$2\pi f(x) = \cos \pi x\{G(\pi) - G(-\pi)\} + i \sin \pi x\{G(\pi) + G(-\pi)\} - xi\Sigma u_n \int_{-\pi}^{\pi} e^{(x-n)it}\, dt$$

$$= 2\pi \cos \pi x f(0) + i \sin \pi x\left\{\frac{2f'(0)}{i} + 2u_0\right\} - 2xi\Sigma u_n \frac{\sin(x-n)\pi}{x-n},$$

† See J. M. Whittaker (**10**), 67–8.

$$f(x) = \cos \pi x f(0) + \frac{\sin \pi x}{\pi} f'(0) - \frac{ix \sin \pi x}{\pi} \Sigma'(-1)^n \frac{u_n}{x-n}$$

$$= \cos \pi x f(0) + \frac{\sin \pi x}{\pi} f'(0) + \frac{\sin \pi x}{\pi} \Sigma'(-1)^n \left(\frac{1}{x-n} + \frac{1}{n}\right) \{f(n) - (-1)^n f(0)\}. \tag{16·6}$$

Here the coefficient of $f(0)$ is

$$\cos \pi x - \frac{\sin \pi x}{\pi} \Sigma' \left(\frac{1}{n} + \frac{1}{x-n}\right) = \cos \pi x + \frac{\sin \pi x}{\pi x} - \frac{\sin \pi x}{\pi} \pi \cot \pi x = \frac{\sin \pi x}{\pi x},$$

and so $$f(x) = \frac{\sin \pi x}{\pi x} \{f(0) + x f'(0)\} + \frac{\sin \pi x}{\pi} \left\{\Sigma'(-1)^n \left(\frac{1}{x-n} + \frac{1}{n}\right) f(n)\right\}, \tag{16·7}$$

the main result of the theorem.

Next, the Fourier series of $G(t)$ for $t = -\pi$ or $t = \pi$, viz.

$$u_0 + \Sigma'(-1)^n u_n, \tag{16·8}$$

is summable $(C, 1)$ to sum

$$\tfrac{1}{2}\{G(\pi) + G(-\pi)\} = \frac{f'(0)}{i} + u_0,$$

and so

$$\frac{f'(0)}{i} = \Sigma'(-1)^n u_n = i\Sigma' \frac{(-1)^n}{n} \{f(n) - (-1)^n f(0)\} = i\Sigma' \frac{(-1)^n}{n} f(n).$$

Substituting for $f'(0)$ in (16·7), we obtain

$$f(x) = \frac{\sin \pi x}{\pi} \left\{\frac{f(0)}{x} + \Sigma'(-1)^n \frac{f(n)}{x-n}\right\}, \tag{16·9}$$

the series being summable $(C, 1)$. This is the last result in Theorem 10 (J. M. Whittaker's theorem). Finally, if (16·8) is convergent, (16·9) is convergent.

17. When $$\Sigma' \left|\frac{f(n)}{n}\right| < \infty \tag{17·1}$$

the cardinal series is absolutely convergent. We may call the class of functions satisfying (17·1) the class M'. It is known that any function of M which satisfies (11·2) belongs to M'. More generally, any integral function of finite type which belongs to M belongs to M'†. The converse is not true; a function of the form (11·2), even with an absolutely continuous G, may belong to M' but not to M. For example, if

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{xit} \frac{\operatorname{sgn} t\, dt}{(\pi - |t|)\{\log(\pi - |t|)\}^2},$$

† See Plancherel and Pólya (6), p. 126. It follows that the cardinal series of any f of M is absolutely convergent, and this enables us to strengthen the last clause of Theorem 9: *if f is M, then* (11·2) *is a sufficient condition for the* (*absolute*) *convergence of the cardinal series, to $f(x)$, for all x.*

then
$$f(x)=\frac{\sin \pi x}{\pi \log x}+O\left\{\frac{1}{(\log x)^2}\right\}$$
for large positive x, so that $f(x)$ is M' but not M.

It would be natural to ask whether Theorem 8 remains true when M' is substituted for M; but there is no such clear-cut theorem.

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NOTES ON FOURIER SERIES (I): ON SINE SERIES WITH POSITIVE COEFFICIENTS

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1. In this note we prove five theorems concerning series of sines. We suppose that $f(\theta)$ is odd and integrable and that $S(\theta) = \Sigma b_n \sin n\theta$ is its Fourier series (F.s.). By

$$f(\theta) \to \tfrac{1}{2}\pi A \quad (C, \alpha), \tag{1.1}$$

where $\alpha > 0$, when $\theta \to +0$, we mean that

$$\frac{f_\alpha(\theta)}{\theta^\alpha} = \frac{\alpha}{\theta^\alpha} \int_0^\theta (\theta - t)^{\alpha-1} f(t)\, dt \to \tfrac{1}{2}\pi A\,; \tag{1.2}$$

and by

$$nb_n \to A \quad (C, \beta), \tag{1.3}$$

where $\beta > 0$, we mean that the Cesàro mean of nb_n, of order β, tends to A. In the most important case, when $\alpha = \beta = 1$, (1.2) and (1.3) become

$$\frac{1}{\theta} \Sigma \frac{b_n}{n} (1 - \cos n\theta) \to \tfrac{1}{2}\pi A \tag{1.4}$$

and

$$n^{-1}(b_1 + 2b_2 + \ldots + nb_n) \to A. \tag{1.5}$$

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1943, 2 (with W. W. Rogosinski) *Journal of the London Mathematical Society*, 18, 50–7.

THEOREM I. *If b_n is positive and decreasing, then a necessary and sufficient condition, either for the uniform convergence of S or for the continuity of f, is that $nb_n \to 0$.*

THEOREM II. *If b_n is positive and decreasing, and $A > 0$, then a necessary and sufficient condition that $f(\theta) \to \frac{1}{2}\pi A$ is that $nb_n \to A$.*

THEOREM III. *If $b_n \geqslant 0$ or, more generally, if*

$$nb_n \geqslant -1, \tag{1.6}$$

then (1.5) *is necessary and sufficient for* (1.4).

THEOREM IV. *If b_n satisfies* (1.6), *and f is continuous, then S is uniformly convergent.*

THEOREM V. *If b_n satisfies* (1.6), *then* (1.3), *for any $\beta \geqslant 1$, is necessary and sufficient for* (1.1), *for any $\alpha > 0$ (so that all such hypotheses are equivalent).*

We begin by proving Theorems I–IV. None of these theorems is new: we indicate their history in §6. But our proofs are a good deal simpler than any given before, and show interesting relations between the theorems.

2. *Proof of* I. (i) Suppose that $nb_n \to 0$, and choose N so that $nb_n < \epsilon$ for $n \geqslant N$. If $q > p \geqslant N$, $0 < \theta < \pi$, and $k = [\theta^{-1}]$, then

$$S_{p,q} = \sum_p^q b_n \sin n\theta = \sum_p^k + \sum_{k+1}^q = U + V,$$

say: either sum may be empty. Either U is empty, or

$$|U| \leqslant \theta \sum_p^k nb_n < \epsilon k\theta \leqslant \epsilon;$$

and either V is empty or (by "Abel's lemma")

$$|V| \leqslant b_{k+1} \operatorname{cosec} \tfrac{1}{2}\theta \leqslant \pi b_{k+1}/\theta \leqslant \pi(k+1)\, b_{k+1} < \pi\epsilon.$$

Thus in any case $|S_{p,q}| < (\pi+1)\,\epsilon$ and S is uniformly convergent.

(ii) If $f \to 0$ when $\theta \to 0$ then

$$\sum \frac{b_n}{n}(1-\cos n\theta) = \int_0^\theta f(t)\, dt = o(\theta).$$

But the series is at least

$$\sum_{\frac{1}{2}k}^{k} \frac{b_n}{n}(1-\cos n\theta) \geqslant \tfrac{1}{2}\theta^2 \sum_{\frac{1}{2}k}^{k} nb_n \geqslant \tfrac{1}{2}\theta^2(\tfrac{1}{2}k)^2 b_k \geqslant \tfrac{1}{9}b_k$$

for large k. Hence $kb_k = o(k\theta) = o(1)$.

Plainly the same argument shows that $nb_n = O(1)$ is necessary and sufficient for either the bounded convergence of S or the boundedness of f.

3. *Proof of* II. We may take $A = 1$.

(i) If $nb_n \to 1$ then $|nb_n - 1| < \epsilon^2$ for $n \geqslant N$ (and *a fortiori* $nb_n < 2$). We suppose that $q > p \geqslant N$ and $k = [(\epsilon\theta)^{-1}]$, and write

$$\sum_p^q c_n \sin n\theta = \sum_p^q \left(b_n - \frac{1}{n}\right) \sin n\theta = \sum_p^k + \sum_{k+1}^q = U + V.$$

Either U is empty, or

$$|U| < \epsilon^2 \sum_p^k \frac{n\theta}{n} \leqslant \epsilon^2 k\theta \leqslant \epsilon;$$

and either V is empty, or

$$|V| \leqslant \left(b_{k+1} + \frac{1}{k+1}\right) \operatorname{cosec} \tfrac{1}{2}\theta < \frac{3\pi}{(k+1)\theta} < 3\pi\epsilon.$$

Hence $\Sigma c_n \sin n\theta$ is uniformly convergent, and

$$\lim \Sigma\, b_n \sin n\theta = \lim \Sigma\, n^{-1} \sin n\theta = \tfrac{1}{2}\pi.$$

(ii) Since b_n decreases, S converges uniformly for $0 < \delta \leqslant \theta \leqslant \pi$. If also $f \to \frac{1}{2}\pi$, then f is bounded and, by the last remark of § 2, $b_n = O(n^{-1})$ and $c_n = O(n^{-1})$. Since $\Sigma c_n \sin n\theta$ is the F.s. of a continuous function, it is uniformly summable $(C, 1)$; and it follows* that it is uniformly convergent.

We can therefore choose λ and μ so that

(3.1) $$\tfrac{1}{4}\pi < \lambda < \mu < \tfrac{1}{2}\pi, \quad \tfrac{1}{2}\epsilon < \mu - \lambda < \epsilon,$$

and

(3.2) $$\left|\sum_p^q c_n \sin n\theta\right| < \epsilon^2$$

for $$\rho = [\lambda\theta^{-1}] \leqslant p < q \leqslant [\mu\theta^{-1}] = \sigma$$

* See § 6.

and small θ. Also

$$\sum_{\rho+1}^{\sigma}(nb_n-1)=\frac{1}{\theta}\sum_{\rho+1}^{\sigma}\left(c_n\sin n\theta\,.\,\frac{n\theta}{\sin n\theta}\right),$$

and $n\theta/(\sin n\theta)$ increases with n and does not exceed $\frac{1}{2}\pi$. It follows by partial summation that

$$\left|\sum_{\rho+1}^{\sigma}(nb_n-1)\right|\leqslant\frac{\pi\epsilon^2}{2\theta},\qquad \sum_{\rho+1}^{\sigma}nb_n\leqslant\sigma-\rho+\frac{\pi\epsilon^2}{2\theta}.$$

Hence, since b_n decreases,

$$(\sigma-\rho)\rho b_\sigma\leqslant\sigma-\rho+\frac{\pi\epsilon^2}{2\theta},\qquad \sigma b_\sigma\leqslant\frac{\sigma}{\rho}\left\{1+\frac{\pi\epsilon^2}{2(\sigma-\rho)\theta}\right\};$$

and hence

$$\overline{\lim}\,\sigma b_\sigma\leqslant\left(1+\frac{\mu-\lambda}{\lambda}\right)\left\{1+\frac{\pi\epsilon^2}{2(\mu-\lambda)}\right\}<\left(1+\frac{4\epsilon}{\pi}\right)(1+\pi\epsilon).$$

That is to say, $\overline{\lim}\,\sigma b_\sigma\leqslant 1$; and a similar argument shows that $\underline{\lim}\,\sigma b_\sigma\geqslant 1$.

4. *Proof of* III. We may suppose $b_n\geqslant 0$, replacing b_n by b_n+n^{-1} when b_n satisfies (1.6). We need a lemma.

LEMMA. *The hypotheses*

$$u_n=b_1+2b_2+\ldots+nb_n\sim An,\qquad v_n=\frac{b_n}{n}+\frac{b_{n+1}}{n+1}+\ldots\sim\frac{A}{n}$$

are equivalent; and $u_n=O(n)$ is equivalent to $v_n=O(n^{-1})$.

Here there is no restriction on b_n, and we may suppose $A=0$. If $u_n=o(n)$ then

$$\sum_{n}^{N}\frac{b_m}{m}=\sum_{n}^{N}\frac{u_m-u_{m-1}}{m^2}=o\left(\frac{1}{n}\right)+\sum_{n}^{N-1}o(m)\,O\left(\frac{1}{m^3}\right)=o\left(\frac{1}{n}\right),$$

uniformly in N, so that v_n exists and is $o(n^{-1})$. And if $v_n=o(n^{-1})$ then

$$u_n=\sum_{1}^{n}m^2(v_m-v_{m+1})=o(n)+\sum_{1}^{n}o\left(\frac{1}{m}\right)O(m)=o(n).$$

Passing to the proof of the theorem, we suppose first that $nb_n\to A\,(C,1)$, i.e. $u_n\sim An$. Then $\Sigma n^{-1}b_n<\infty$. Also

$$\Sigma\frac{b_n}{n}(1-\cos n\theta)=O\left(\theta^2\sum_{n\leqslant 1/\theta}nb_n\right)+O\left(\sum_{n\geqslant 1/\theta}\frac{b_n}{n}\right)=O(\theta).$$

Thus $$\phi(\theta) = \tfrac{1}{2}\cot\tfrac{1}{2}\theta\, \Sigma \frac{b_n}{n}(1-\cos n\theta) = \tfrac{1}{2}\cot\tfrac{1}{2}\theta f_1(\theta)$$

is odd and bounded. If $\phi(\theta) \sim \Sigma\beta_n \sin n\theta$, then

$$\beta_{n+1}-\beta_n = \frac{2}{\pi}\int_0^\pi \{\sin(n+1)\theta - \sin n\theta\}\phi(\theta)\,d\theta$$

$$= \frac{1}{\pi}\int_0^\pi \{\cos n\theta + \cos(n+1)\theta\} f_1(\theta)\,d\theta = -\tfrac{1}{2}\left(\frac{b_n}{n}+\frac{b_{n+1}}{n+1}\right);$$

and so $$\beta_n = \frac{b_n}{2n}+\frac{b_{n+1}}{n+1}+\frac{b_{n+2}}{n+2}+\ldots = v_n - \frac{b_n}{2n}.$$

On the other hand, if $\theta^{-1}f_1(\theta)\to\tfrac{1}{2}\pi A$, then again $\phi(\theta)$ is bounded, and

$$\theta\sum_{n\leqslant 1/\theta} nb_n = O\left\{\frac{1}{\theta}\sum_{n\leqslant 1/\theta}\frac{b_n}{n}(1-\cos n\theta)\right\} = O(1),$$

or $u_n = O(n)$. It follows that $\Sigma n^{-1}b_n < \infty$, and the F.s of $\phi(\theta)$ may be calculated as before.

The theorem now follows from Theorem II. For β_n decreases, and $\phi(\theta)$ has a F.s. satisfying the conditions of Theorem II. Also $\phi(\theta)\to\tfrac{1}{2}\pi A$ is equivalent to (1.4), and $n\beta_n\to A$ is equivalent to $nv_n\to A$ and therefore, by the lemma, to (1.5).

It should be observed that our proof is valid even if, as may happen, $\Sigma b_n \sin n\theta$ is not a F.s. Thus $\Sigma m^{-\frac{1}{2}}\sin 2^m\theta$ is not a F.s., though the conditions are satisfied with this choice of b_n. In such a case, of course, we must not state (1.4) in the form (1.1).

5. *Proof of* IV. Since f is continuous at 0, and b_n satisfies (1.6), $nb_n\to 0$ $(C, 1)$, and *a fortiori* $(C, 2)$.

If $s_n(\theta)$ and $\sigma_n(\theta)$ are the partial sum and $(C, 1)$ mean of S, then $\sigma_n(\theta)\to f(\theta)$ uniformly, by Fejér's theorem, and therefore

$$\frac{b_1\sin\theta+\ldots+nb_n\sin n\theta}{n+1} = s_n(\theta)-\sigma_n(\theta)\to 0 \quad (C, 1),$$

i.e. $$nb_n\sin n\theta\to 0 \quad (C, 2).$$

Hence $$w_n = \tfrac{1}{2}nb_n(1-\sin n\theta)\to 0 \quad (C, 2).$$

But $w_n \geqslant -1$, by (1.6), and therefore* $w_n\to 0$ $(C, 1)$. And hence

* See § 6.

$nb_n \sin n\theta \to 0$ $(C, 1)$, i.e. $s_n(\theta) - \sigma_n(\theta) \to 0$; and all this is true uniformly in θ. Hence $s_n(\theta) \to f(\theta)$ uniformly.

The argument proves rather more, viz. that *if* b_n *satisfies* (1.5) *and* (1.6), *then* S *converges at any point where it is summable* (*and so almost everywhere*), *and uniformly in any closed interval of continuity of* f.

6. *History of the theorems.* Theorem I is due to Chaundy and Jolliffe, *Proc. L.M.S.* (2), 15 (1916), 214–6, except for the clause about continuity, which was added by Jolliffe in *Proc. Camb. Phil. Soc.*, 19 (1921), 191–5. Theorem II was proved by Hardy, *Proc. L.M.S.* (2), 32 (1931), 441–8: Hardy uses Poisson's integral and the Hardy-Littlewood Tauberian theorem for power-series with positive coefficients. Theorem III was proved by Szász, *Acta Math.*, 61 (1933), 185–201, with similar weapons. The essentials of the theorem can be found in earlier work of Wiener: thus the integral analogue is Theorem 21 of his book *The Fourier Integral* (Cambridge, 1933); the substance of this is in an earlier paper of his in *Journal L.M.S.*, 2 (1927), 118–23; and the substance of the theorem itself is contained in Theorems XI–XI″ of his paper on "Tauberian theorems" in *Annals of Math.*, 33 (1932), 1–109 (though it is not included exactly in any of these theorems). These general theorems of Wiener are of course much more difficult.

Finally, Theorem IV was proved, when $b_n \geqslant 0$, by Paley, *Journal L.M.S.*, 7 (1932), 205–8. The extension was made by Szász, *l.c.*, and our proof is a simplification of his.

The "Tauberian" machinery which we have used is very slight. There is a Tauberian *argument* in § 3 (ii); apart from this, we have appealed only to two of the simplest Tauberian theorems. These are (i) *if* Σa_n *is summable* $(C, 1)$, *and* $na_n > -H$, *then* Σa_n *is convergent*, and (ii) *if* $A_n \to l$ $(C, 2)$, *and* $A_n > -H$, *then* $A_n \to l$ $(C, 1)$. Either of these is easily deducible from the other. We used the first (in the case in which na_n is bounded) in § 3, and the second in § 5.

7. *Proof of* V. We state the proof very shortly: we may suppose $b_n \geqslant 0$. We add to our Tauberian apparatus the most obvious generalization of (ii), viz. (*a*) *if* $A_n \to l$ (C, k) *for some* k, *and* $A_n > -H$, *then* $A_n \to l$ $(C, 1)$; and (*b*) *if* $f \to \frac{1}{2}\pi A$ (C, k) *for some* k, *and* $f = O(1)$ (C, α), *then* $f \to \frac{1}{2}\pi A$ (C, α') *for* $\alpha' > \alpha$. We denote the hypotheses (1.1) and (1.3) by F_α and B_β.

(1) Suppose that k is a positive integer. We saw in § 4 that F_1 is equivalent to B_1, and k repetitions of the argument show that F_k is

equivalent to B_k.* Hence, by (*a*), F_k is equivalent to B_1; and so F_α, for any $\alpha > 0$, implies B_1.

(2) Assume B_1. This implies F_1, and, if we can show that it implies

(7.1) $$f = O(1) \quad (C, \alpha)$$

for any $\alpha > 0$, it will follow from (*b*) that it implies $F_{\alpha'}$ for $\alpha' > \alpha$, and so F_α for $\alpha > 0$. We may suppose that $0 < \alpha < 1$.

If $0 < \eta < \theta$, then $(\theta - t)^{\alpha-1}$ is of bounded variation in $0 \leqslant t \leqslant \eta$, and so

(7.2) $$\int_0^\eta (\theta-t)^{\alpha-1} f(t)\,dt = \Sigma\, b_n \int_0^\eta (\theta-t)^{\alpha-1} \sin nt\,dt = \Sigma\, b_n \chi_n(\eta),$$

say. We prove that

(7.3) $$\chi_n(\eta) = O(n\theta^{\alpha+1}) = O(n^{-\alpha}) \quad (n\theta \leqslant 1), \qquad \chi_n(\eta) = O(n^{-\alpha}) \quad (n\theta \geqslant 1),$$

in each case uniformly in η. For the first,

$$\chi_n(\eta) = O\left\{n \int_0^\eta (\theta-t)^{\alpha-1} t\,dt\right\} = O\left(n \int_0^\theta\right) = O(n\theta^{\alpha+1}).$$

To prove the second, we write $\zeta = \min(\eta, \theta - n^{-1})$. Then

$$\chi_n(\eta) = \left(\int_0^\zeta + \int_\zeta^\eta\right)(\theta-t)^{\alpha-1} \sin nt\,dt = \chi_n^{(1)}(\eta) + \chi_n^{(2)}(\eta),$$

say: the second term is absent if $n(\theta-\eta) > 1$. Also $\chi_n^{(2)}$ is plainly $O(n^{-\alpha})$, and

$$\chi_n^{(1)}(\eta) = (\theta-\zeta)^{\alpha-1} \int_{\zeta'}^\zeta \sin nt\,dt = O(n^{1-\alpha} \cdot n^{-1}) = O(n^{-\alpha})$$

for a ζ' between 0 and ζ.

It now follows that $\Sigma\, b_n \chi_n(\eta)$ is majorized by

$$\Sigma\, n^{-\alpha} b_n = \Sigma\, n^{-\alpha-1}(u_n - u_{n-1}) = \Sigma\, u_n \Delta n^{-\alpha-1} = \Sigma\, O(n^{-\alpha-1}),$$

and is absolutely and uniformly convergent for $0 < \eta < \theta$; and from this it follows that

$$\frac{1}{\alpha} f_\alpha(\theta) = \Sigma\, b_n \chi_n(\theta) = \sum_{n \leqslant 1/\theta} b_n\, O(n\theta^{1+\alpha}) + \sum_{n > 1/\theta} b_n\, O(n^{-\alpha}).$$

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* Suppose, *e.g.*, $k = 2$, and write $f^{(1)}$ and $b_n^{(1)}$ for the ϕ and β_n of § 4, $f^{(2)}$ and $b_n^{(2)}$ for the ϕ and β_n derived similarly from $f^{(1)}$ and $b_n^{(1)}$. Then F_2 is equivalent to $f^{(1)} \to \frac{1}{2}\pi A$ $(C, 1)$, *i.e.* to $f^{(2)} \to \frac{1}{2}\pi A$, and therefore to $nb_n^{(2)} \to A$; and this, by the lemma, is equivalent to $nb_n^{(1)} \to A$ $(C, 1)$, *i.e.* to B_2.

The first term is $O(\theta^{1+\alpha}.\theta^{-1}) = O(\theta^{\alpha})$, and the second is

$$O\left\{\sum_{n>1/\theta} n^{-\alpha-1}(u_n - u_{n-1})\right\} = O(\theta^{\alpha+1}.\theta^{-1}) + O\left(\sum_{n>1/\theta} n.n^{-\alpha-2}\right) = O(\theta^{\alpha}).$$

This proves (7.1) and completes the proof of the theorem.

The result is "best possible" in the sense that we cannot take $\alpha = 0$ or $\beta < 1$.

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NOTES ON FOURIER SERIES (III): ASYMPTOTIC FORMULAE FOR THE SUMS OF CERTAIN TRIGONOMETRICAL SERIES

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1. Our object in this note is to find asymptotic formulae for the sums of the series

$$\mathbf{C} = \mathbf{C}(x) = \sum a_n \cos nx, \qquad \mathbf{S} = \mathbf{S}(x) = \sum b_n \sin nx, \tag{1.1}$$

where a_n and b_n behave approximately like a power $n^{-\alpha}$, when $x \to 0$. We shall suppose that a_n and b_n are of the form $n^{-\alpha}\phi_n$, where

$$\alpha = \beta + i\gamma, \tag{1.2}$$

and ϕ_n is a function, in one sense or another, of a 'slowly oscillating' type; and that $0 < \beta < 1$ or $0 < \beta \leqslant 1$ according as we are dealing with $\mathbf{C}$ or $\mathbf{S}$. The case $\beta = 0$ is exceptional for $\mathbf{S}$, and both $\beta = 0$ and $\beta = 1$ for $\mathbf{C}$: we reserve these cases for later consideration. We state our proofs primarily for $\mathbf{S}$, since in this case we can allow β to be 1. Our main results are

Theorem 1. *Suppose that*

$$b_n = n^{-\alpha}\phi_n, \tag{1.3}$$

where $\alpha = \beta + i\gamma$, $0 < \beta \leqslant 1$; *and that* ϕ_n *satisfies*

(A) $$\phi_1 + \phi_2 + \dots + \phi_n \sim n\phi_n,$$

and **one** *of the three conditions*

(B 1) ϕ_n *is real and* $|b_n|$ *decreases*;

(B 2) ϕ_n *is real and increasing*;

(B 3) $\phi_{n+1}/\phi_n = 1 + O(n^{-1})$.

Then $\mathbf{S}(x) = \sum b_n \sin nx$ *is convergent, and*

$$\mathbf{S}(x) \sim \xi b_\xi J = \xi^{1-\alpha}\phi_\xi J \tag{1.4}$$

when $x \to +0$, *where* $\xi = [x^{-1}]$,

$$J = \int_0^\infty t^{-\alpha} \sin t \, dt = \Gamma(1-\alpha)\cos \tfrac{1}{2}\alpha\pi \tag{1.5}$$

1945, 1 (with W. W. Rogosinski) *Quarterly Journal of Mathematics*, 16, 49–58.

when $\alpha \neq 1$, *and* $J = \frac{1}{2}\pi$ *when* $\alpha = 1$. *In particular, either of the conditions*

(C 1) ϕ_n *is real and* $n^{\delta}\phi_n$ *increasing*, $n^{-\delta}\phi_n$ *decreasing, for every positive* δ,

(C 2) $$\phi_{n+1}/\phi_n = 1+o(n^{-1}),$$

is in itself sufficient for (1.4).

It should be observed that a real ϕ_n which satisfies (A) is ultimately of fixed sign.

THEOREM 2. *There are corresponding results for* $C(x)$ *when* $0 < \beta < 1$, *with* J *replaced by the corresponding cosine integral.*

It is not obvious in either case, except when we adopt hypothesis (B 1), that the series are convergent.

2. It will be convenient to begin by proving the results in the special case $\phi_n = 1$. They are then familiar (at any rate for real α); but the usual proofs follow different lines, and it will elucidate our method and shorten our later analysis if we insert one here.

LEMMA 1. *If* γ *is real and fixed then*

$$\sum_{\mu}^{\nu} n^{-i\gamma}e^{nix} = O\left(\frac{1}{|x|}\right)$$

when $x \to 0$, *uniformly in* μ *and* ν.

We suppose that $x > 0$ and $\mu \leqslant \xi = [x^{-1}] < \nu$, and write

$$T = \sum_{\mu}^{\nu} n^{-i\gamma}e^{nix} = \sum_{\mu}^{\xi} + \sum_{\xi+1}^{\nu} = T_1+T_2.$$

Plainly $T_1 = O(x^{-1})$. Also

$$(1-e^{ix})T_2 = \sum_{\xi+1}^{\nu} n^{-i\gamma}\Delta e^{nix} = O(1)-\sum_{\xi+1}^{\nu} e^{nix}\Delta(n-1)^{-i\gamma} = O(1)-T_3,$$

$$(1-e^{ix})T_3 = \sum_{\xi+1}^{\nu} \Delta(n-1)^{-i\gamma}\Delta e^{nix} = O\left(\frac{1}{\xi}\right)-\sum_{\xi+1}^{\nu} e^{nix}\Delta^2(n-2)^{-i\gamma}$$

$$= O\left(\frac{1}{\xi}\right)+\sum_{\xi+1}^{\nu} O\left(\frac{1}{n^2}\right) = O\left(\frac{1}{\xi}\right) = O(x).$$

Hence $T_3 = O(1)$ and so $T_2 = O(x^{-1})$.

If $\xi < \mu$ or $\xi \geqslant \nu$, then the estimate for T_1 or T_2 alone suffices.

LEMMA 2. *Theorems* 1 *and* 2 *are true when* $\phi_n = 1$.

Here, and later, we argue as follows. Suppose that

$$\psi(x) = \chi(x)+p(x,c)+q(x,C)+r(x,c,C),$$

where c and C are positive parameters; that η_c and η_C are positive functions of c and C, which tend to 0 when $c \to 0$ or $C \to \infty$; that

$$|p| < \eta_c|\chi| \quad (0 < x \leqslant x_1), \qquad |q| < \eta_C|\chi| \quad (0 < x \leqslant x_2), \tag{2.1}$$

where $x_1 = x_1(c)$ and $x_2 = x_2(C)$; and that

$$r = o(|\chi|) \tag{2.2}$$

when c and C are fixed and $x \to 0$. Then

$$\psi(x) \sim \chi(x). \tag{2.3}$$

For plainly, given any positive ϵ, we have

$$|\psi-\chi| < (\eta_c+\eta_C)|\chi|+|r| < \epsilon|\chi|+|r|$$

for sufficiently small c, sufficiently large C, and $0 < x \leqslant x_3(c, C)$; and so

$$\overline{\lim}\left|\frac{\psi-\chi}{\chi}\right| \leqslant \epsilon.$$

We shall express (2.1) by writing

$$p = O(\eta_c|\chi|), \qquad q = O(\eta_C|\chi|). \tag{2.4}$$

The O's are 'uniform in c or C' in the sense that the constants which they imply are independent of c or C, but not in the sense that the ranges of x for which they hold are independent of c or C.

In the special case relevant here we write

$$\mathbf{S} = \Big(\sum_{n<c/x} + \sum_{c/x\leqslant n\leqslant C/x} + \sum_{n>C/x}\Big)n^{-\alpha}\sin nx = S_1+S_2+S_3. \tag{2.5}$$

Then
$$S_1 = O\Big(x\sum_{n<c/x} n^{1-\beta}\Big) = O(c^{2-\beta}x^{\beta-1}) = O(\eta_c x^{\beta-1}), \tag{2.6}$$

because $|\sin nx| \leqslant nx$; and

$$S_3 = O\{(C/x)^{-\beta}x^{-1}\} = O(C^{-\beta}x^{\beta-1}) = O(\eta_C x^{\beta-1}), \tag{2.7}$$

by partial summation and Lemma 1. Also, if $M = [c/x]$, $N = [C/x]$,

$$S_2 - \int_{c/x}^{C/x} t^{-\alpha}\sin tx\,dt = O(M^{-\beta})+O(N^{-\beta})+$$
$$+\sum_{M}^{N-1}\int_n^{n+1}\{n^{-\alpha}(\sin nx-\sin tx)+(n^{-\alpha}-t^{-\alpha})\sin tx\}\,dt. \tag{2.8}$$

The first two terms are $o(x^{\beta-1})$, and the last is

$$\sum_{M}^{N-1}\int_{n}^{n+1}\{O(xn^{-\beta})+O(n^{-\beta-1})\}\,dt = O\Big(x\sum_{M}^{N-1}n^{-\beta}\Big)+O(1) = o(x^{\beta-1}).$$

Hence (2.6) gives

$$S_2 = x^{\alpha-1}\int_{c}^{C} t^{-\alpha}\sin t\,dt+o(x^{\beta-1}) = x^{\alpha-1}\{J+O(\eta_c)+O(\eta_C)+o(1)\}, \tag{2.9}$$

for fixed c and C. Finally, from (2.5)–(2.9), and our remarks at the beginning of the proof, it follows that $\mathbf{S} \sim x^{\alpha-1}J$. The proof for $\mathbf{C}$ is similar, except that we use $|\cos nx| \leqslant 1$ instead of $|\sin nx| \leqslant nx$ and must suppose $\beta < 1$.

3. We need three more lemmas, in which $t \to \infty$ and $\phi(t)$ is a function of t continuous for $t \geqslant 1$.

LEMMA 3.† *If*

(D′) $\quad$ *$\phi(kt) \sim \phi(t)$, when $t \to \infty$, for every fixed positive k,*

then $\phi(kt) \sim \phi(t)$ uniformly in any finite interval $0 < k_1 \leqslant k \leqslant k_2$.

Let $$f(t) = \phi(e^t), \qquad \psi = \psi(t,h) = \frac{f(t+h)}{f(t)}.$$

Then $\psi \to 1$ for every fixed h, and we have to show that it does so uniformly in any finite interval of h, say $I(H) = (-H,H)$.

Since $\psi \to 0$ for every h of $I(2H)$, $\psi \to 0$ uniformly in a set K included in $I(2H)$ and of measure greater than $3H$,‡ so that $|\psi| < \epsilon$ for all h of K and $t > t_0(H,\epsilon)$. If h is a given number of $I(H)$, and h_1 and h_2 run through K, then $h-h_1$ runs through a set of measure greater than $3H$ and included in $I(3H)$; and this set and K, having measures whose sum exceeds $6H$, must have points in common; so that every h of $I(H)$ is expressible as $h = h_1+h_2$, where h_1 and h_2 belong to K. Also $|\psi(t,h_1)-1| < \epsilon$ for $t > t_0(H,\epsilon)$ and $|\psi(t+h_1,h_2)-1| < \epsilon$ for $t+h_1 > t_0(H,\epsilon)$; and therefore

$$|\psi(t,h)-1| = |\psi(t,h_1)\psi(t+h_1,h_2)-1| < 2\epsilon+\epsilon^2 < 3\epsilon$$

for $t > t_0(H,\epsilon)+2H$, which proves the lemma.

† We owe this lemma (in which ϕ need not be continuous) to Mr. Besicovitch.

‡ By 'Egoroff's theorem'.

LEMMA 4. *The condition*

$$\text{(A}')\qquad \phi_1(t) = \int\limits_1^t \phi(u)\,du \sim t\phi(t)$$

is equivalent to (D′).

(1) If (A′) is satisfied, then $\phi_1 \neq 0$ for large t, and

$$\frac{\phi_1'}{\phi_1} = \frac{1}{t}+o\left(\frac{1}{t}\right), \qquad \log\frac{\phi_1(kt)}{\phi_1(t)} = \log k+o(1),\dagger$$

$$\frac{\phi(kt)}{\phi(t)} \sim \frac{1}{k}\frac{\phi_1(kt)}{\phi_1(t)} = \frac{1}{k}e^{\log k+o(1)} \to 1.$$

(2) If (D′) is satisfied, $\delta > 0$, l is a positive integer, and $t = 2^l u$, then

$$|\phi(t)| \leqslant 2^{\frac{1}{2}\delta}|\phi(\tfrac{1}{2}t)| \leqslant \ldots \leqslant 2^{\frac{1}{2}l\delta}|\phi(u)| \tag{3.1}$$

for $u \geqslant u_0(\delta)$. Hence $|\phi(t)| \leqslant \{u^{-\frac{1}{2}\delta}|\phi(u)|\}t^{\frac{1}{2}\delta}$, and $t^{-\delta}|\phi(t)| \to 0$. Similarly $t^{\delta}|\phi(t)| \to \infty$ for any positive δ. In particular, $t|\phi(t)| \to \infty$.

In what follows $0 < \epsilon < \frac{1}{2}$ and ζ, $\zeta_1^{(1)}$, $\zeta_2^{(1)}$, $\zeta_2^{(2)}$,... are numbers, depending on various variables, whose moduli are less than ϵ. By Lemma 3, $\phi(u) \sim \phi(t)$ uniformly for $\frac{1}{2}t \leqslant u \leqslant t$. Hence

$$\int\limits_{\frac{1}{2}t}^{t} \phi(u)\,du = \phi(t)\int\limits_{\frac{1}{2}t}^{t}(1+\zeta)\,du = \tfrac{1}{2}t\phi(t)(1+\zeta_1^{(1)}) \quad (t \geqslant t_0(\epsilon)),$$

$$\int\limits_{\frac{1}{4}t}^{\frac{1}{2}t} \phi(u)\,du = \tfrac{1}{4}t\phi(\tfrac{1}{2}t)(1+\zeta_2^{(1)}) = \tfrac{1}{4}t\phi(t)(1+\zeta_2^{(1)})(1+\zeta_2^{(2)}) \quad (\tfrac{1}{2}t \geqslant t_0(\epsilon)),$$

$$\int\limits_{2^{-l}t}^{2^{-l+1}t} \phi(u)\,du = 2^{-l}t\phi(t)(1+\zeta_l^{(1)})(1+\zeta_l^{(2)})\ldots(1+\zeta_l^{(l)}) \quad (2^{-l+1}t \geqslant t_0(\epsilon)).$$

Hence, if $2^{-l}t = \tau$, where $\frac{1}{2}t_0 \leqslant \tau \leqslant t_0$, we have

$$\phi_1(t)-\phi(\tau) = t\phi(t)\sum_{h=1}\frac{(1+\zeta_h^{(1)})\ldots(1+\zeta_h^{(h)})}{2^h},$$

$$\frac{\phi_1(t)}{t\phi(t)}-\sum_1^l\frac{1}{2^h} = \frac{\phi_1(\tau)}{t\phi(t)}+\sum_{h=1}^{l}\frac{(1+\zeta_h^{(1)})\ldots(1+\zeta_h^{(h)})-1}{2^h}. \tag{3.2}$$

The last term has a modulus less than

$$\sum_{h=1}^{\infty}\frac{(1+\epsilon)^h-1}{2^h} < \epsilon\sum_{h=1}^{\infty}h\left(\frac{1+\epsilon}{2}\right)^h = \frac{2\epsilon(1+\epsilon)}{(1-\epsilon)^2} < 12\epsilon.$$

† With the principal values of the logarithms.

Also $l \to \infty$ when $t \to \infty$, and $t|\phi(t)| \to \infty$. Hence it follows from (3.2) that

$$\overline{\lim}\left|\frac{\phi_1(t)}{t\phi(t)}-1\right| < 12\epsilon$$

or $\phi_1 \sim t\phi$.

Lemma 5. *Either* (A′) *or* (D′) *involves the following consequences*:

(i) $t^{-\delta}|\phi| \to 0$, $t^{\delta}|\phi| \to \infty$, *for any positive* δ;

(ii) *if* $\phi_1^* = \int_1^t |\phi(u)|\,du$, *then* $\phi_1^* \sim t|\phi| \sim |\phi_1|$;

(iii) $\sum_{n\leqslant t} n^{\delta-1}\phi(n) \sim \int_1^t u^{\delta-1}\phi(u)\,du \sim \frac{t^\delta}{\delta}\phi(t)$,

for any positive δ, *with similar relations in which* ϕ *is replaced by* $|\phi|$;

(iv) $\sum_{n>t} n^{-\delta-1}\phi(n) \sim \int_t^\infty u^{-\delta-1}\phi(u)\,du \sim \frac{t^{-\delta}}{\delta}\phi(t)$,

with a similar gloss.†

Of these assertions, (i) has been proved already. For (ii), $|\phi(kt)| \sim |\phi(t)|$ and so $\phi_1^* \sim t|\phi| \sim |\phi_1|$. To prove (iii) we have

$$\begin{aligned}\int_1^t u^{\delta-1}\phi\,du &= t^{\delta-1}\phi_1+(1-\delta)\int_1^t u^{\delta-2}\phi_1\,du,\\ \int_1^t u^{\delta-1}|\phi|\,du &= t^{\delta-1}\phi_1^*+(1-\delta)\int_1^t u^{\delta-2}\phi_1^*\,du.\end{aligned} \tag{3.3}$$

Since $\phi_1^* \sim t|\phi|$ and $t^{\delta-1}\phi_1^* \to \infty$, the second of (3.3) gives

$$\delta\int_1^t u^{\delta-1}|\phi|\,du \sim t^{\delta-1}\phi_1^* \sim t^\delta|\phi|.$$

But then

$$\int_1^t u^{\delta-2}\phi_1\,du = \int_1^t u^{\delta-1}\phi\,du + o\left(\int_1^t u^{\delta-1}|\phi|\,du\right) = \int_1^t u^{\delta-1}\phi\,du + o(t^\delta|\phi|),$$

and the first of (3.3) gives $\delta\int_1^t u^{\delta-1}\phi\,du \sim t^{\delta-1}\phi_1 \sim t^\delta\phi$.

† We prove a little more than we actually need, for the sake of symmetry.

We have thus proved the assertions about integrals in (iii). Next

$$\left|\sum_{n\leqslant t} n^{\delta-1}\phi(n)-\int\limits_1^t u^{\delta-1}\phi\,du\right|$$
$$\leqslant \sum_{n=1}^{[t]}\int\limits_n^{n+1}|n^{\delta-1}\phi(n)-u^{\delta-1}\phi(u)|\,du+\left|\int\limits_t^{[t]+1}u^{\delta-1}\phi(u)\,du\right|.$$

Here $n^{\delta-1}-u^{\delta-1}=O(n^{\delta-2})$, $\phi(u)\sim\phi(n)$, and the last integral is plainly $o(t^{\delta}|\phi|)$. Hence the sum is

$$o\left(\sum_{n=1}^{[t]}\int\limits_n^{n+1}u^{\delta-1}|\phi|\,du\right)=o\left(\int\limits_1^{t+1}u^{\delta-1}|\phi|\,du\right)=o(t^{\delta}|\phi|),$$

and
$$\sum_{n\leqslant t}n^{\delta-1}\phi(n)\sim\int\limits_1^t u^{\delta-1}\phi\,du\sim\frac{t^{\delta}}{\delta}\phi(t).$$

The sum with $|\phi(n)|$ may be dealt with similarly, and this completes the proof of (iii). That of (iv) follows the same lines and need not be set out in detail.

4. We can now prove Theorems 1 and 2. It is convenient for technical reasons to suppose that b_n and ϕ_n are the values, for $t=n$, of continuous functions $b(t)$ and $\phi(t)$, the conditions on these functions corresponding to (A)–(C) of the theorems being (A′) $\phi_1\sim t\phi$; (B 1′) ϕ is real and $|b|$ decreases; (B 2′) ϕ is real and increasing; (B 3′) $\phi'=O(t^{-1}|\phi|)$; (C 1′) ϕ is real, $t^{\delta}\phi$ increases, $t^{-\delta}\phi$ decreases; (C 2′) $\phi'=o(t^{-1}|\phi|)$. These conditions are in fact satisfied by the $b(t)$ or $\phi(t)$ obtained from b_n or ϕ_n by linear interpolation.†

After Lemma 4 we may replace (A′) by $\phi(kt)\sim\phi(t)$, or (A) by $\phi_{[kn]}\sim\phi_n$.

We decompose **S** (assuming its convergence provisionally) as in (2.5), and we have to show that

$$S_1=O\left\{\frac{\eta_c}{x}\left|b\left(\frac{1}{x}\right)\right|\right\},\tag{4.1}$$

$$S_3=O\left\{\frac{\eta_C}{x}\left|b\left(\frac{1}{x}\right)\right|\right\},\tag{4.2}$$

$$S_2=\{J+O(\eta_c)+O(\eta_C)+o(1)\}\frac{1}{x}b\left(\frac{1}{x}\right).\tag{4.3}$$

† We must interpolate b_n linearly in case (B 1), ϕ_n in case (B 2) or (C 1): otherwise either interpolation is effective. In either case $b(t)$ or $\phi(t)$ will have angles at the integers, but we may interpret (B 3′) or (C 2′) as referring to the backward and forward derivatives (or 'smooth out' the angles if we prefer to).

First,

$$S_1 = \sum_{n<c/x} n^{-\alpha}\phi(n)\sin nx = O\Big\{x \sum_{n<c/x} n^{1-\beta}|\phi(n)|\Big\} = O\Big\{c^{2-\beta}x^{\beta-1}\Big|\phi\Big(\frac{c}{x}\Big)\Big|\Big\},$$

by (iii) of Lemma 5. Since $\phi(c/x) \sim \phi(1/x)$, this gives (4.1).†

Next

$$S_2 = \sum_{c/x}^{C/x} n^{-\alpha}\phi(n)\sin nx = \phi\Big(\frac{1}{x}\Big)\sum_{c/x}^{C/x} n^{-\alpha}\sin nx + o\Big\{\Big|\phi\Big(\frac{1}{x}\Big)\Big|\sum_{c/x}^{C/x} n^{-\beta}\Big\}.$$

The last term is $o\{x^{\beta-1}|\phi(1/x)|\}$; and so (4.3) follows from (2.9).

So far we have used only hypothesis (A′). In discussing S_3 we must assume one of (B 1′), (B 2′), and (B 3′).

(*a*) If (A′) and (B 1′) are satisfied then, after Lemma 5 (i), $|b_n| \to 0$, and S_3 is certainly convergent. And, if $N = [C/x]$, then

$$\begin{aligned} S_3 = \sum_{n>C/x} b_n \sin nx = b_{N+1}\, O\Big(\frac{1}{x}\Big) &= O\Big\{\frac{1}{x} b\Big(\frac{C}{x}\Big)\Big\} \\ &= O\Big\{\frac{C^{-\beta}}{x} b\Big(\frac{1}{x}\Big)\Big\} = O\Big\{\frac{\eta_C}{x} b\Big(\frac{1}{x}\Big)\Big\}. \end{aligned}$$

(*b*) The convergence of S_3 is less obvious when (B 1′) is replaced by (B 2′). We then write

$$s_l = \sum_{2^l C/x < n \leqslant 2^{l+1}C/x} n^{-\beta-i\gamma}\phi(n)\sin nx \quad (l = 0, 1, 2,\ldots).$$

Summing partially,‡ and using Lemma 1 and (3.1), we see that

$$s_l = O\Big\{\Big(\frac{2^l C}{x}\Big)^{-\beta}\phi\Big(\frac{2^{l+1}C}{x}\Big)\frac{1}{x}\Big\} = O\Big\{C^{-\beta}2^{l(\delta-\beta)}x^{\beta-1}\phi\Big(\frac{1}{x}\Big)\Big\}$$

for any positive δ; and the same estimate is valid for any partial sum of s_l. We may take $\delta = \frac{1}{2}\beta$, and then it becomes plain that $\sum s_l$, S_3, and **S** are convergent, and that S_3 is of the form (4.2).

(*c*) Finally, suppose that (B 3′) is satisfied; that $N = [C/x]$; and that

$$\chi_n = \sum_n^{\infty} m^{-\alpha}\sin mx,$$

so that $\chi_n = O(n^{-\beta}x^{-1})$, after Lemma 1. Then

$$\sum_N^{N'} b_n \sin nx = \sum_N^{N'} \phi_n(\chi_n - \chi_{n+1}) = \phi_N\chi_N - \phi_{N'}\chi_{N'+1} - \sum_N^{N'-1}\chi_{n+1}\Delta\phi_n.$$

† $|S_1| < Hc^{2-\beta}x^{\beta-1}|\phi(c/x)|$, where H is independent of c, for $0 < x < x_0(c)$, and $\phi(c/x) \sim \phi(1/x)$ for every c, imply $|S_1| < 2Hc^{2-\beta}x^{\beta-1}|\phi(1/x)|$ for $0 < x < x_1(c)$. We shall argue thus repeatedly.

‡ Twice, once on $n^{-\alpha}\sin nx$ and once on $n^{-i\gamma}\sin nx$.

Now $\phi_n \chi_n = O(n^{\delta-\beta}x^{-1})$, for any positive δ, and $\Delta\phi_n = O(n^{-1}|\phi_n|)$, by (B 3′). It follows that S_3 is convergent, and that

$$S_3 = \phi_N \chi_N - \sum_N^\infty \chi_{n+1}\Delta\phi_n = O\left(\frac{N^{-\beta}}{x}|\phi_N|\right)+O\left(\frac{1}{x}\sum_N^\infty n^{-1-\beta}|\phi_n|\right).$$

The second term here may be absorbed in the first, by (iv) of Lemma 5; and so

$$S_3 = O\left\{C^{-\beta}x^{\beta-1}\left|\phi\left(\frac{C}{x}\right)\right|\right\} = O\left\{\frac{\eta_C}{x}\left|b\left(\frac{1}{x}\right)\right|\right\}.$$

5. This completes the proof of Theorem 1, so far as hypotheses (A) and (B) are concerned. As regards hypotheses (C) or (C′), it is plain that either of them implies one of (B) or (B′). It is therefore sufficient to prove that either (C 1′) or (C 2′) implies (A′).

(*a*) If (C 1′) is satisfied, then

$$\phi_1 = \int_1^t u^\delta\phi.u^{-\delta}\,du \leqslant t^\delta\phi\frac{t^{1-\delta}}{1-\delta} = \frac{t\phi}{1-\delta}$$

for every positive δ. Hence

$$\varlimsup\frac{\phi_1}{t\phi} \leqslant \frac{1}{1-\delta}, \qquad \varlimsup\frac{\phi_1}{t\phi} \leqslant 1;$$

and similarly the lower limit is at least 1.

(*b*) If (C 2′) is satisfied, then

$$\phi_1 = t\phi - \int_1^t u\phi'\,du = t\phi + o\left(\int_1^t |\phi|\,du\right) = t\phi + o(\phi_1^*),$$

$$\phi_1^* = t|\phi| - \int_1^t u|\phi|'\,du = t|\phi| + o(\phi_1^*),$$

since $|\phi|' \leqslant |\phi'|$. It follows by comparison that $\phi_1^* \sim t|\phi|$ and $\phi_1 \sim t\phi$.

This completes the proof of Theorem 1. The proof of Theorem 2 is the same except for the discussion of S_1. Here

$$S_1 = \sum_{n<c/x} n^{-\alpha}\phi(n)\cos nx = O\left\{\sum_{n<c/x} n^{-\beta}|\phi(n)|\right\} = O\left\{c^{1-\beta}x^{\beta-1}\left|\phi\left(\frac{1}{x}\right)\right|\right\}:$$

we must suppose $\beta < 1$.† A typical case of either theorem is that in which a_n or b_n is of the form $n^{-\alpha}\psi_n e^{i\chi_n}$, where $\alpha = \beta + i\gamma$, $0 < \beta < 1$, and ψ_n and χ_n are 'L-functions', in the sense of Hardy's

† $\delta > 0$ in (iii) of Lemma 5.

Orders of infinity, of the ranges $n^{-\delta} \prec \psi_n \prec n^{\delta}$ and $1 \prec \chi_n \prec \log n$. Thus a_n or b_n might be

$$n^{-\alpha}(\log n)^A(\log\log n)^B e^{Ci(\log n)^\rho} \quad (0 < \rho < 1).$$

6. There are naturally similar theorems for integrals

$$\int_0^\infty a(t)\cos xt\, dt, \qquad \int_0^\infty b(t)\sin xt\, dt,$$

which the reader can formulate for himself. Here, however, there is a dual problem, since x may tend to ∞ instead of 0, in which case it is the behaviour of $a(t)$ or $b(t)$ for *small* t that is important. This case corresponds to the problem of finding asymptotic formulae for the Fourier constants of a function with a singularity of assigned nature at the origin.

There are a good many known theorems which are special cases of Theorems 1 and 2 or their integral analogues. Different writers have tended to consider slightly different problems, and the literature is fragmentary. We may refer to Bromwich, *Infinite Series*, 494; Hardy, *Proc. London Math. Soc.* (2), 32 (1931), 441–8; Haslam-Jones, *Journal London Math. Soc.* 2 (1927), 151–4; Titchmarsh, ibid., 1 (1926), 35–7, and *Fourier Integrals*, 172–4. Theorems with wider conditions but less precise results have been proved by Salem, *Comptes rendus*, 186 (1928), 1804–6, and Young, *Proc. London Math. Soc.* (2), 12 (1913), 433–52: for these see Zygmund, *Trigonometrical Series*, 112–16. The theorems proved by Hardy, *Messenger of Math.* 58 (1929), 130–5, and Young, *Proc. Royal Soc.* (A), 93 (1917), 42–55, are of a rather different character.

[*Added Feb. 1944.*] Lemma 1 is a case of the following more general theorem, in which

$$A_n = a_0+a_1+\dots+a_n, \qquad s_n(z) = a_0+a_1 z+\dots+a_n z^n, \qquad |z| \leqslant 1,$$

H is an absolute constant, and $H(\beta)$ a function of β only.

If $0 < \beta < 1$, $a_n = o(1)$, $\Delta a_n = O(n^{\beta-2})$, *then*

$$|s_n(z)| < H(\beta)|1-z|^{-\beta}.$$

The conditions are not sufficient when $\beta = 0$ *or* $\beta = 1$. *But, if* $A_n = O(1)$, $\Delta a_n = O(n^{-2})$, *then* $|s_n(z)| < H$; *and if* $a_n = O(1)$, $\Delta^2 a_n = O(n^{-2})$, *then* $|s_n(z)| < H|1-z|^{-1}$.

This theorem includes (besides Lemma 1) a number of more special theorems proved by Hardy, *Quarterly Journal*, 44 (1917), 147–60; Landau, *Ergebnisse der Funktionentheorie* (1929), 68–9; M. Riesz, *Acta Univ. Hungaricae*, 1 (1923), 104–13; Szegö, *Math. Zeitschrift*, 25 (1926), 172–87.

COMMENT

p. 49. The cases $\beta = 0$ and $\beta = 1$ under the hypothesis (C1) of Theorem 1 have been discussed by Zygmund (Z I, pp. 187–90).

(*e*) Other Papers on Trigonometric Series

INTRODUCTION TO OTHER PAPERS ON TRIGONOMETRIC SERIES

Of the papers in this group, perhaps the most striking is 1939, 3, where Hardy shows that, within certain limits, a function orthogonal with respect to its own zeros is necessarily a Bessel function. This paper displays well Hardy's great familiarity with the properties of special functions.

Other results proved in the papers of this group are:

(i) An important theorem on term-by-term integration of integrals involving Fourier series (1922, 10);

(ii) Extensions of Parseval's theorem to non-conjugate classes L^p, L^q (1927, 4);

(iii) The arithmetic means of the Fourier coefficients of a function of L^p are the Fourier coefficients of a function of L^p (1929, 7).

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

LV.

On the integration of Fourier series.

1. PROF. W. H. YOUNG, in the course of his researches in the theory of Fourier series, has discovered a very beautiful theorem which is of great importance for the evaluation of definite integrals which contain a periodic factor. The theorem may be stated as follows:

THEOREM. *Suppose* (i) *that* $f(x)$ *is summable and periodic, with period* 2π; (ii) *that* $g(x)$ *is of bounded variation in the*

1922, 10 *Messenger of Mathematics*, 51, 186–92.

interval $(0, \infty)$; *and* (iii) *that the integral*

$$(1) \qquad \int_0^\infty |g(x)|\,dx$$

is convergent. Then the value of the integral

$$(2) \qquad \int_0^\infty f(x)\,g(x)\,dx$$

may be calculated by substituting for $f(x)$ *its Fourier series*

$$\tfrac{1}{2}a_0 + \Sigma\,(a_n \cos nx + b_n \sin nx)$$

and integrating formally term by term; so that

$$(3) \quad \int_0^\infty f(x)\,g(x)\,dx = \tfrac{1}{2}a_0 \int_0^\infty g(x)\,dx + \Sigma\left\{a_n \int_0^\infty g(x)\cos nx\,dx + b_n \int_0^\infty g(x)\sin nx\,dx\right\}.$$

In particular this is so if (ii*a*) $g(x)$ *is positive and decreases steadily as* x *increases, and* (iii*a*) *the integral*

$$(4) \qquad \int_0^\infty g(x)\,dx$$

is convergent.

If $a_0 = 0$, *the condition* (iii) *or* (iii*a*) *may be replaced by the less exacting condition that* (iii*b*) $g(x)$ *tends to zero when* x *tends to infinity.*

Although everything stated here has been proved by Young, his various publications bearing on the matter* do not, so far as I am aware, contain any quite definite and explicit statement of the theorem as a whole, which is the result of the collation of a number of different passages. And the proof, as presented in his writings, is in any case somewhat intricate. In the first place it involves the assumption of 'Parseval's Theorem' in a very general form. Secondly it depends, in part at any rate, on a general theorem concerning

* W. H. Young: (1) 'On the integration of Fourier series', *Proc. London Math. Soc.*, ser. 2, vol. ix., 1910, pp. 449–462; (2) 'On the theory of the application of expansions to definite integrals', *ibid.*, pp. 463–485; (3) 'On integration with respect to a function of bounded variation', *ibid.*, vol. xiii., 1913, pp. 109–150; (4) 'On the Fourier constants of a function', *Proc. Royal Soc.* (A), vol. lxxxv., 1911, pp. 14–24.

In order to obtain the theorem as I have stated it, it is necessary to compare Theorem 6 of (2) and Theorem 2 of (1), together with the extension of this latter theorem to an infinite range of integration [(1), pp. 454–455]. The complete result is fundamental in (4): see in particular pp. 17–18. The simplification in the proof of Theorem 6 of (2), due to the generalised theory of integration, is indicated on pp. 147–148 of (3): the argument here supersedes the 'delicate and lengthy' argument of (2), pp. 475–481.

the integration of series, the proof of which presents considerable difficulty. Of this theorem he has offered two proofs. The first is, as he remarks himself, 'delicate and lengthy'; while the second, which is very much simpler, depends upon his general theory of integration with respect to a monotonic function (or function of bounded variation).

It seems worth while, therefore, to include in these notes a proof which is more direct and presupposes a good deal less.

2. I shall require three lemmas.

(1) *If* (i) $s_m(x)$ *is, for every positive integral value of* m, *a measurable function of* x; (ii) $s_m(x)$ *is bounded for* $a \leq x \leq b$, $m = 1, 2, 3, \ldots$; (iii) $s_m(x)$ *tends to a limit* $s(x)$ *when* $m \to \infty$, *for all, or almost all, values of* x; *and* (iv) $f(x)$ *is summable: then* $s(x)f(x)$ *is summable and*

$$\lim_{m\to\infty} \int_a^b s_m(x) f(x)\,dx = \int_a^b s(x) f(x)\,dx.$$

This is a well-known theorem due to Lebesgue and Vitali.* I follow Young in saying that, when conditions (ii) and (iii) are satisfied, $s_n(x)$ *converges boundedly to* $s(x)$.

(2) *If* $s(x)$ *is of bounded variation in the interval* $(0, 2\pi)$, *then the sum of the first* n *terms of its Fourier series converges boundedly to* $s(x)$.†

This is an immediate consequence of the ordinary theory of Dirichlet's integral, when developed (in Jordan's manner) by means of the Second Theorem of the Mean. A formal proof is given by Young.‡

(3) *If* $g(x)$ *is summable and of bounded variation in the infinite interval* $(0, \infty)$§, *then the series*

$$g(x) + g(x + 2\pi) + g(x + 4\pi) + \ldots$$

is convergent for every positive value of x, *and its sum* $G(x)$ *is summable and of bounded variation in the interval* $(0, 2\pi)$.

$$\text{Let } g(x + 2n\pi) = u_n(x), \quad \int_{x+2n\pi}^{x+2(n+1)\pi} g(t)\,dt = v_n(x),$$

where $0 \leq x \leq 2\pi$. The series $\Sigma v_n(x)$ is (absolutely and uniformly) convergent. Also

* See de la Vallée-Poussin, *Cours d'Analyse*, vol. i. (third edition, 1914), p. 264 (Theorem II.); or Young's paper (2), p. 468 (Theorem 2).

† The limit function is

$$\tfrac{1}{2}\{s(x-0) + s(x+0)\},$$

which differs from $s(x)$ at most at an enumerable set of points.

‡ *l.c.* (2), p. 453.

§ *i.e.*, if the conditions (ii) and (iii) of the main theorem are satisfied.

$$u_n(x) - v_n(x) = \int_{x+2n\pi}^{x+2(n+1)\pi} \{g(x+2n\pi) - g(t)\}\, dt$$

is plainly not greater in absolute value than $2\pi V_n$, where V_n is the total variation of $g(t)$ in the interval

$$x + 2n\pi, \quad x + 2(n+1)\pi.$$

Hence $$\Sigma\{u_n(x) - v_n(x)\},$$

and therefore $\Sigma u_n(x)$, is (absolutely and uniformly) convergent.* Thus $G(x)$ is defined for $0 \leq x \leq 2\pi$. Also

$$|G(x)| \leq |g(x)| + |g(x+2n\pi)| + \dots,$$

so that $G(x)$ is summable and

$$\int_0^{2\pi} |G(x)|\, dx \leq \int_0^\infty |g(x)|\, dx.$$

Finally, if x_1 and x_2 are any two points of the interval $(0, 2\pi)$, we have

$$|G(x_1) - G(x_2)| \leq \Sigma\, |g(x_1 + 2nx) - g(x_2 + 2n\pi)|.$$

Hence if we form one of the sums by means of which the variation of $G(x)$ in $(0, 2\pi)$ is defined, it is less than or equal to a corresponding sum formed for $g(x)$ and the infinite interval $(0, \infty)$. Thus the variation of $G(x)$ in $(0, 2\pi)$ does not exceed that of $g(x)$ in $(0, \infty)$.

3. We can now prove the main theorem. Suppose first that the conditions (i), (ii), and (iii) are satisfied. Then

$$\begin{aligned} \tfrac{1}{2}a_0 \int_0^\infty g(x)\, dx + \sum_1^m \left(a_n \int_0^\infty g(x) \cos nx\, dx + b_n \int_0^\infty g(x) \sin nx\, dx\right) \\ = \frac{1}{2\pi} \int_0^\infty g(x)\, dx \int_0^{2\pi} \frac{\sin(m+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)} f(t)\, dt \\ = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt \int_0^\infty \frac{\sin(m+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)} g(x)\, dx, \end{aligned}$$

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* It is plain that, by a trifling modification of this argument, we can prove the following proposition: *if $g(x)$ is of bounded variation in $(0, \infty)$, then the difference where a is a constant, tends to a finite limit when $n \to \infty$; so that the series and integral*

$$\Sigma g(n+a) - \int^{n+a} g(x)\, dx,$$

$$\Sigma g(n+a), \quad \int^\infty g(x)\, dx$$

converge or diverge together. This is a generalisation of the classical 'Cauchy-Maclaurin' test for the convergence of series. If $g(x)$ is an integral, its variation is

$$\int_0^\infty |g'(x)|\, dx.$$

The theorem then reduces to one proved by Bromwich, 'The relation between the convergence of series and that of integrals', *Proc. Lond. Math. Soc.*, ser. 2, vol. vi., pp. 327–338.

the inversion of the order of integration being plainly legitimate. But

$$G_m(t) = \frac{1}{2\pi}\int_0^\infty \frac{\sin(m+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}\, g(x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin(m+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}\, G(x)\,dx$$

is the sum of the first $m+1$ terms of the Fourier series of $G(t)$, and so, by Lemmas 2 and 3, converges boundedly to $G(t)$. Hence, by Lemma 1,

$$\int_0^{2\pi} f(t)\, G_m(t)\,dt \rightarrow \int_0^{2\pi} f(t)\, G(t)\,dt = \int_0^\infty f(t)\, g(t)\,dt$$

when $m \rightarrow \infty$. This proves the first part of our theorem.*

4. We have still to consider the second part of the theorem, in which

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = 0,$$

but the integral (1) is not convergent. This case is easily reduced to dependence on that which we have already discussed.

Suppose that the conditions (i), (ii), and (iii *b*) of the theorem are satisfied, and let

$$\gamma(x) = g(2m\pi) \qquad \{2m\pi \leq x < (2m+1)\pi\}$$

and
$$\bar{g}(x) = \gamma(x) - g(x).$$

It is plain that $\gamma(x)$, and therefore $\bar{g}(x)$, is of bounded variation in $(0, \infty)$. Also

$$\int_{2m\pi}^{2(m+1)\pi} |\bar{g}(x)|\,dx = \int_{2m\pi}^{2(m+1)\pi} |g(2m\pi) - g(x)|\,dx \leq 2\pi V_m,$$

where V_n has the same meaning as in the proof of Lemma 3 (§ 2). Hence the integral

$$\int_0^\infty |\bar{g}(x)|\,dx$$

is convergent.

* It might be thought that it would be simpler to begin by considering the case in which $g(x)$ is monotonic. If so, $G(x)$ is plainly summable and monotonic in $(0, 2\pi)$; and the proof is materially simplified.

There is however a difficulty. If $g(x)$ is summable and of bounded variation in $(0, \infty)$, it must tend to zero; and $g(x) = h(x) - k(x)$, where $h(x)$ and $k(x)$ are positive functions which tend steadily to zero. But the summability of $g(x)$ does not necessarily involve that of $h(x)$ and $k(x)$. The point may be illustrated by the example of the function

$$g(x) = \frac{1}{x} - \frac{1}{n+1} \quad (n \leq x < n+1;\ n = 1, 2, \ldots).$$

Thus $\bar{g}(x)$ satisfies the conditions imposed upon $g(x)$ in the preceding analysis, and so (3) holds when $\gamma(x)-g(x)$ is written for $g(x)$. But

$$\int_0^\infty f(x)\,\gamma(x)\,dx = \sum_0^\infty \int_{2m\pi}^{2(m+1)\pi} f(x)\,\gamma(x)\,dx$$
$$= \sum_0^\infty g(2m\pi)\int_0^{2\pi} f(x)\,dx = 0,$$

and similarly

$$\int_0^\infty \gamma(x)\cos nx\,dx = 0, \quad \int_0^\infty \gamma(x)\sin nx\,dx = 0.$$

Hence (3) also holds when $\gamma(x)$ is substituted for $g(x)$, since every integral which occurs in it vanishes. And hence, finally, (3) holds as it stands.

5. The theorem of § 1 enables us to evaluate, in the form of infinite series, the integrals considered in Notes IV. and IX. If

$$f(x) = \phi(\sin^2 x)$$

we have $\quad f(x) \sim \tfrac{1}{2}a_0 + \Sigma\,(a_{2m}\cos 2mx + b_{2m}\sin 2mx).$

If $g(x)$ is positive and tends steadily to zero, and the integral (4) is convergent, then

$$(5)\quad \int_0^\infty \phi(\sin^2 x)\,g(x)\,dx = \tfrac{1}{2}a_0\int_0^\infty g(x)\,dx$$
$$+\Sigma\left\{a_{2m}\int_0^\infty g(x)\cos 2mx\,dx + b_{2m}\int_0^\infty g(x)\sin 2mx\,dx\right\}.$$

If (4) is divergent, then

$$(6)\quad \int_0^\infty \{\phi(\sin^2 x) - \tfrac{1}{2}a_0\}\,g(x)\,dx$$
$$= \Sigma\left\{a_{2m}\int_0^\infty g(x)\cos 2mx\,dx + b_{2m}\int_0^\infty g(x)\sin 2mx\,dx\right\}.$$

One of the most interesting applications of the theorem, which is signalised by Young, is to the case in which

$$g(x) = x^{p-1} \quad (0 < p < 1).$$

In this case $g(x)$ has an infinity at the origin, and the analysis requires modification. We may begin by taking

$$\bar{g}(x) = 0\ (0 < x < c), \quad g(x) = x^{p-1}\ (x \geq c),$$

and applying the theorem to $\bar{g}(x)$. We thus justify term-by-term integration over the range (c, ∞). In order to justify

integration over $(0, c)$, we have to impose an additional condition on $f(x)$ in the neighbourhood of the origin. If we suppose that $f(x)$ is of bounded variation in $(0, c)$, its Fourier series converges boundedly in that interval. We may therefore, by Lemma 1, multiply by the summable function x^{p-1} and integrate term-by-term. Combining our results, we see that term-by-term integration over the whole interval $(0, \infty)$ is permissible. We thus obtain Young's formula

$$(7) \quad \int_0^\infty \{f(x) - \tfrac{1}{2}a_0\} x^{p-1} dx = \Gamma(p) \sum_1^\infty \frac{a_n \cos \frac{1}{2}p\pi + b_n \sin \frac{1}{2}p\pi}{n^p}:$$

$f(x)$ being any periodic and summable function which has bounded variation in an interval $(0, c)$.

6. As an example of the use of theorems of this character, I add the following very simple deduction of the functional equation of Riemann's Zeta-function.

Suppose that

$$f(x) = \sum_0^\infty \frac{\sin(2m+1)x}{2m+1},$$

so that

$$f(x) = \tfrac{1}{4}\pi \quad \{2k\pi < x < (2k+1)\pi\},$$

$$f(x) = -\tfrac{1}{4}\pi \quad \{(2k+1)\pi < x < 2(k+1)\}.$$

Then (7) becomes

$$\tfrac{1}{4}\pi \sum_0^\infty (-1)^k \int_{k\pi}^{(k+1)\pi} x^{p-1} dx = \Gamma(p) \sin \tfrac{1}{2}p\pi \sum_0^\infty \frac{1}{(2m+1)^{1+p}}.$$

The series on the right-hand side converges to

$$(1 - 2^{1+p})\, \zeta(1+p)$$

if $R(p) > 0$. That on the left-hand side is

$$\frac{\pi^{1+p}}{4p} \sum_0^\infty (-1)^k \{(k+1)^p - k^p\}.$$

It is convergent if $R(p) < 1$. Further, if $R(p) < 0$, it is equal to

$$\frac{\pi^{1+p}}{2p}(1^p - 2^p + 3^p - \ldots) = \frac{\pi^{p+1}}{2p}(1 - 2^{p+1})\, \zeta(-p).$$

Writing $1 + p = s$, we obtain

$$\zeta(1-s) = 2(2\pi)^{-s} \cos \tfrac{1}{2}s\pi\, \Gamma(s)\, \zeta(s),$$

the functional equation.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

[Extracted from *Messenger of Mathematics*, no. 616, vol. lii., August, 1922.]

LVI.

On Fourier's series and Fourier's integral.

1. There are three familiar representations of an arbitrary function associated with Fourier's name, viz.

$$(1)\qquad f(x)=\frac{1}{\pi}\sum_{n=0}^{\infty}\epsilon_n\int_0^{2\pi}f(t)\cos n(t-x)\,dt$$

(where ϵ_n is $\frac{1}{2}$ if $n=0$ and 1 otherwise),

$$(2)\qquad f(x)=\lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\frac{\sin\lambda(t-x)}{t-x}\,dt,$$

$$(3)\qquad f(x)=\frac{1}{\pi}\int_0^{\infty}dy\int_{-\infty}^{\infty}f(t)\cos y(t-x)\,dt.$$

These are (1) Fourier's *series*, (2) Fourier's *single integral*,* and (3) Fourier's *double integral*. It is with the first two only that I am concerned in this note.

The conditions under which (1) and (2) are valid are, so far as the behaviour of $f(t)$ in the neighbourhood of $t=x$ is concerned, identical. This is of course well known; but there is a simple formal relation between the two formulæ which I have not seen established generally.

2. Suppose first that $f(x)$ is a trigonometrical polynomial

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\nu}(a_n\cos nx+b_n\sin nx)=\tfrac{1}{2}A_0+\sum_{n=1}^{\nu}A_n,$$

and that λ is positive. A simple calculation shows that

$$(4)\qquad \frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\frac{\sin\lambda(t-x)}{t-x}\,dt=\sum_{n\leq\lambda}\eta_n A_n,$$

where $\eta_n=\frac{1}{2}$ if $n=0$ or $n=\lambda$ and $\eta_n=1$ otherwise. The integral is defined in the ordinary manner when λ is non-

* I follow Prof. Hobson's nomenclature: see his *Theory of functions of a real variable* (first ed.), p. 760.

integral; but when λ is an integer it must be defined as a principal value

$$(5) \qquad \lim_{T\to\infty} \int_{-T}^{T},$$

as it would otherwise be divergent.

Suppose that λ is non-integral. Then *the value of Fourier's integral is the sum of the terms of Fourier's series whose rank is less than* λ. It is this result which I wish to generalise.

3. THEOREM 1. *The formula* (4) *is true for every periodic and integrable* function* $f(x)$, *provided that the integral is interpreted as a principal value when* λ *is an integer.*

We may suppose that $0 \leq x \leq 2\pi$, since each side of (4) is periodic in x.

Suppose first that $N < \lambda < N+1$, where N is an integer. Then, by the ordinary formulæ for Fourier coefficients,

$$(6) \qquad \sum_{n<\lambda} \eta_n A_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin (N+\frac{1}{2})(t-x)}{\sin \frac{1}{2}(t-x)} f(t)\, dt.$$

On the other hand

$$(7) \qquad \int_{-\infty}^{\infty} f(t) \frac{\sin \lambda (t-x)}{t-x}\, dt = \sum_{-\infty}^{\infty} \int_{2k\pi}^{2(k+1)\pi} f(t) \frac{\sin \lambda (t-x)}{t-x}\, dt$$

$$= \sum_{-\infty}^{\infty} \int_0^{2\pi} f(t) \frac{\sin \lambda (t-x+2k\pi)}{t-x+2k\pi}\, dt,$$

if this series is convergent. Let us assume for a moment that the order of summation and integration may be reversed. We have†

$$(8) \qquad \psi(\lambda) = \sum_{-\infty}^{\infty} \frac{\sin \lambda (t-x+2k\pi)}{t-x+2k\pi} = \frac{\sin (N+\frac{1}{2})(t-x)}{2 \sin \frac{1}{2}(t-x)}.$$

There are certain terms in the series, namely those for which k is -1, 0, and 1, whose definition fails for particular values of t and x. It is to be understood that such a term is then to be replaced by its limiting value λ.

The formula (4) follows at once from (6), (7), and (8), and it remains only to justify the inversion.

It is sufficient for this purpose to show that the series (8) is boundedly convergent, that is to say that

$$(9) \qquad \left| \sum_{-K'}^{K} \frac{\sin \lambda (t-x+2k\pi)}{t-x+2k\pi} \right| < A,$$

* In the sense of Lebesgue.

† See, for example, Bromwich, *Infinite Series*, p. 257 (ex. 19).

where A is a constant, for $0 \leqq t \leqq 2\pi$, $0 \leqq x \leqq 2\pi$, and all positive integral values of K and K'.

As each term of the series is individually bounded, we may ignore the terms for which k is -1, 0, or 1. We have then

$$\left| \frac{1}{t-x+2k\pi} - \frac{1}{2k\pi} \right| = \left| \frac{t-x}{(t-x+2k\pi)\,2k\pi} \right| < \frac{A}{k^2},$$

$$\left| \sum_2^K \frac{\sin\lambda(t-x+2k\pi)}{t-x+2k\pi} \right| < A + \left| \sum_2^K \frac{\sin\lambda(t-x+2k\pi)}{2k\pi} \right|$$

$$= A + \frac{1}{2\pi}\left| \cos\lambda(t-x) \sum_2^K \frac{\sin 2\lambda k\pi}{k} \right| + \frac{1}{2\pi}\left| \sin\lambda(t-x) \sum_2^K \frac{\cos 2\lambda k\pi}{k} \right|$$

$$< A;$$

and the same argument may be applied to the sum from $-K'$ to -2. We thus deduce (9), which completes the proof of the theorem when λ is non-integral.

If λ is an integer N, we have

$$\sum_{n \leqq \lambda} \eta_n A_n = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{\sin(N+\frac{1}{2})(t-x)}{\sin\frac{1}{2}(t-x)} - \cos N(t-x) \right\} f(t)\,dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \sin N(t-x) \cot\tfrac{1}{2}(t-x) f(t)\,dt,$$

and

$$\lim_{K\to\infty} \sum_{-K}^{K} \frac{\sin N(t-x+2k\pi)}{t-x+2k\pi} = \sin N(t-x) \lim \sum_{-K}^{K} \frac{1}{t-x+2k\pi}$$

$$= \tfrac{1}{2} \sin N(t-x) \cot\tfrac{1}{2}(t-x);$$

so that our formal analysis still holds, the integral being interpreted as a principal value, and the series in the special manner indicated above. Also, if the terms for which $k=-1$, 0, 1 are omitted from the summations as before, we have

$$\left| \sum_{-K}^{K} \frac{\sin N(t-x+2k\pi)}{t-x+2k\pi} \right| \leqq \left| \sum_{-K}^{K} \frac{1}{t-x+2k\pi} \right|$$

$$= \left| 2(t-x) \sum_2^K \frac{1}{(t-x)^2 + 4k^2\pi^2} \right| < A,$$

so that the inversion is still legitimate. It is essential here that our special definitions of the series and integral should be adhered to, neither the proof nor the result being valid without them.

4. A similar argument establishes

THEOREM 2. *If*

$$\tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx)$$

is the Fourier series of a periodic and integrable function $f(x)$, and λ is positive and non-integral, then

$$(10)\quad \int_0^\infty f(x)\frac{\sin\lambda x}{x}\,dx = \tfrac{1}{2}\pi\left(\tfrac{1}{2}a_0 + \sum_{1\leq n\leq\lambda} a_n\right) + \tfrac{1}{2}\sum_1^\infty b_n \log\left|\frac{n-\lambda}{n+\lambda}\right|.$$

5. It appears then that, if $f(x)$ is any function integrable over $(0, 2\pi)$, and $F(x)$ is the function defined over $(-\infty, \infty)$ by periodic continuation, the problem of expressing $f(x)$ by a Fourier's series is identical with that of expressing $F(x)$ by a Fourier's single integral. We cannot extend this equivalence to Fourier's double integral (3), without some generalisation of the definition of an infinite integral, since

$$\int_{-\infty}^{\infty} F(t)\cos y\,(t-x)\,dt$$

is not convergent.

There are similar expressions for the Rieszian means of the Fourier's series of $f(x)$. Thus

$$(11)\quad \frac{2}{\pi}\int_{-\infty}^{\infty} f(t)\left\{\frac{\sin\frac{1}{2}\lambda(x-t)}{x-t}\right\}^2 dt = \tfrac{1}{2}A_0\lambda + \sum_{1\leq n\leq\lambda}(\lambda-n)A_n$$

is a formula for the Rieszian (or Cesàro) mean of order 1. This formula has however been established already by Young, who has given similar formulæ for the Rieszian means of any positive order.*

It should be noted that there is a serious difference between (4) and (11), or any of the formulæ given by Young. These latter formulæ are direct deductions from the general theorems which I considered in Note 55, since, for example, the function

$$g(x) = \left(\frac{\sin\frac{1}{2}\lambda x}{x}\right)^2$$

has bounded variation, and a convergent integral, over the whole interval $(-\infty, \infty)$. Neither of these conditions is satisfied when

$$g(x) = \frac{\sin\lambda x}{x},$$

so that Theorem 1 is not deducible from the theorems of Note 55.

* See his papers 'Über eine Summationsmethode für die Fouriersche Reihe' (*Leipziger Berichte*, 43 (1911), pp. 369–387) and 'On infinite integrals involving a generalisation of the sine and cosine functions' (*Quarterly Journal*, 43 (1912), pp. 161–177). Young restricts λ to be an integer.

6. There is a theorem concerning the allied series

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n$$

corresponding to Theorem 1.

THEOREM 3. *If $f(x)$ is any periodic and integrable function, and ΣB_n is the series allied to the Fourier series of $f(x)$, then*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{1-\cos\lambda(t-x)}{t-x}\,dt = \sum_{1 \leqq n \leqq \lambda} \eta_n B_n,$$

where η_n is $\frac{1}{2}$ if $n=\lambda$ and 1 otherwise. The integral is an ordinary integral if $a_n = 0$ and λ is not an integer; otherwise it is a principal value.

It is unnecessary to give the details of the proof, which will present no difficulty to anyone who has followed the proof of Theorem 1.

NOTES ON THE THEORY OF SERIES (V): ON PARSEVAL'S THEOREM

G. H. Hardy *and* J. E. Littlewood.

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1. Suppose that $f(\theta)$ is a real, periodic, and integrable function of θ, and that

$$(1.1) \qquad \tfrac{1}{2}a_0 + \sum_1^{\infty}(a_n \cos n\theta + b_n \sin n\theta), \quad \sum_1^{\infty}(b_n \cos n\theta - a_n \sin n\theta)$$

are its Fourier series and the conjugate or allied series. We say that f belongs to L^p, where $p > 1$, if $|f|^p$ is integrable. A fundamental theorem of M. Riesz* asserts that the allied series is then also a Fourier series, and that the function $-g(\theta)$ of which it is the Fourier series, and which we call the function conjugate to $f(\theta)$, itself belongs to L^p. The complex function

$$(1.2) \qquad H(z) = \Sigma c_n z^n = \Sigma(a_n - ib_n) r^n e^{ni\theta}$$

is then an analytic function of z regular for $r = |z| < 1$, and $h(\theta) = f(\theta) - ig(\theta)$ is the "boundary function" of $H(z)$, that is to say the function defined, for almost all θ, by the values assumed by $H(z)$ when z approaches the unit circle radially. Also

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\, e^{-ni\theta}\, d\theta;$$

and

$$(1.3) \qquad \int_{-\pi}^{\pi} |H(re^{i\theta})|^p \, d\theta \quad (r < 1)$$

is bounded, and indeed increases steadily to $\int |h|^p \, d\theta$ when $r \to 1$.

* M. Riesz, 5.

1927, 4 (with J. E. Littlewood) *Proceedings of the London Mathematical Society* (2), 26, 287–94.

It is naturally often convenient to consider the two functions f and g together, in the form $h = f - ig$. We write

$$(1.4) \qquad h(\theta) \sim \sum_{0}^{\infty} c_n e^{ni\theta}.$$

Here h is not an arbitrary complex function, its real and imaginary parts being determined by one another.

We may, on the other hand, start from a quite arbitrary complex and integrable h. Such a function will not usually be the boundary function of an analytic function. If

$$(1.5) \qquad c_n = \tfrac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\, e^{-ni\theta}\, d\theta,$$

we write

$$(1.6) \qquad h(\theta) \sim \sum_{-\infty}^{\infty} c_n e^{ni\theta},$$

and call this series the Fourier series of h. It will be convenient to suppose throughout, as plainly we may without real loss of generality, that $c_0 = 0$. In these circumstances $h = f - ig$, where f and g are real functions whose Fourier series are

$$(1.7) \qquad \sum_{1}^{\infty} (\alpha_n \cos n\theta + \beta_n \sin n\theta), \qquad \sum_{1}^{\infty} (\delta_n \cos n\theta - \gamma_n \sin n\theta),$$

with

$$2\alpha_n = a_n + a_{-n}, \quad 2\beta_n = b_n - b_{-n}, \quad 2\gamma_n = a_n - a_{-n}, \quad 2\delta_n = b_n + b_{-n}.$$

These series are conjugate only when $\alpha_n = \gamma_n$, $\beta_n = \delta_n$, *i.e.* when $a_n = b_n = 0$ for all negative n.

We shall call (1.6) a *general* Fourier series, and a Fourier series of the type (1.4), or a similar series extended only over negative values of n, a *Fourier* (positive or negative) *power series*. It follows from Riesz's theorem that, if the series (1.7) are the (ordinary) Fourier series of functions of L^p, where $p > 1$, then so are the series

$$\sum_{1}^{\infty} (\gamma_n \cos n\theta + \delta_n \sin n\theta), \qquad \sum_{1}^{\infty} (\beta_n \cos n\theta - \alpha_n \sin n\theta),$$

and therefore, by addition, the series (1.1). In other words, Riesz's theorem may be stated, from our present point of view, as follows: *if* $\sum c_n e^{ni\theta}$ *is the general Fourier series of a function of* L^p, *then the two*

halves of the series corresponding to positive and negative values of n *are the Fourier power series* (positive or negative) *of functions of* L^p.

The upshot of Riesz's theorem is then, to put the matter roughly, that, so long as we are concerned with functions of a Lebesgue class greater than 1, it is indifferent whether we consider Fourier power series or general Fourier series. There are, however, many parts of our recent researches where the distinction is essential. For example, if $\sum c_n e^{ni}$ is a Fourier power series, its integrated series is absolutely convergent, while the corresponding theorem for general Fourier series is false.

2. The classical form of "Parseval's theorem" may be stated as follows. If

$$f \sim \sum_{-\infty}^{\infty} a_n e^{ni\theta}, \qquad g \sim \sum_{-\infty}^{\infty} b_n e^{ni\theta} \tag{2.1}$$

are functions of L^2, then

$$\sum a_n b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, g(-\theta)\, d\theta. \tag{2.2}$$

The convergence of the series is absolute, since $\sum |a_n|^2$ and $\sum |b_n|^2$ are convergent.

Riesz, in the investigations which we have quoted, gives a very beautiful extension of Parseval's theorem. He supposes that f and g belong respectively to L^p and $L^{p'}$, p and p' being "conjugate indices" satisfying

$$p > 1, \quad p' = \frac{p}{p-1} > 1, \quad \frac{1}{p}+\frac{1}{p'} = 1 \tag{2.3}$$

(a notation which we adopt throughout, and for letters other than p), and he proves that (2.2) is still true under these conditions*. There is, however, an essential difference between the general case and the special case $p = 2$, in that the series (2.2) is not generally *absolutely* convergent.

Our recent researches have led us to consider Parseval's theorem in a still more general form. We suppose that f and g belong to L^p and L^q, p and q being greater than 1 but no longer conjugate. The series

* It had previously been proved by Young (7) that (2.2) holds when the series is summed $(C, 1)$.

which is then relevant is not (2 . 2) but

(2 . 4) $$\Sigma |n|^{-\mu} e^{-\frac{1}{2}\mu\pi i \operatorname{sgn} n} a_n b_n,$$

where

(2 . 41) $$\mu = \frac{1}{p} + \frac{1}{q} - 1.$$

We may suppose that $p \leqslant q$, and there are then four possibilities as regards the magnitudes of p and q, viz.

(a) $p \leqslant q \leqslant 2 \,(\leqslant p')$, (b) $(p <)\, 2 < q \leqslant p'$,

(c) $(p \leqslant)\, 2 \leqslant p' < q$, (d) $(p' <)\, 2 < p \leqslant q$.

Theorems which we have proved recently, together with Riesz's theorem, enable us to settle the problem in all cases.

3. THEOREM 1. *In case (a) the series (2 . 4) is absolutely convergent. In case (b) it is convergent, but not necessarily absolutely convergent. In cases (c) and (d) it is not necessarily convergent. In cases (a) and (b) the number μ is the best possible; the series may cease to converge when μ is replaced by any smaller number.*

We use, besides Riesz's theorem, two of our own. The first* is

THEOREM a. *If f belongs to L^p, and $p \leqslant q \leqslant p'$, then*

(3 . 1) $$\Sigma |n^{-\mu} a_n|^q$$

is convergent.

The second requires a word of preliminary explanation. Following Weyl†, we define $f_a(\theta)$, the integral of $f(\theta)$ of order a, where $0 < a < 1$, by

$$f_a(\theta) = \frac{1}{\Gamma(a)} \int_{-\infty}^{\theta} f(t)(\theta - t)^{a-1}\, dt.$$

The Fourier series of $f_a(\theta)$ is

$$\Sigma |n|^{-a} e^{-\frac{1}{2}a\pi i \operatorname{sgn} n} a_n e^{ni\theta}.$$

* Hardy and Littlewood, 3 (Theorem 10).

† Weyl, 6.

The theorem which we require is*

THEOREM β. *If f belongs to L^p, and $0 < a < 1/p$, then f_a belongs to $L^{p/(1-pa)}$.*

With the help of these theorems and Riesz's theorem, it is very easy to prove Theorem 1. The simplicity of the proof is naturally somewhat illusory, since the theorems on which it depends are difficult.

If one of (*a*) or (*b*) holds, then $p \leqslant q \leqslant p'$, and conversely. Suppose first that $p \leqslant q \leqslant 2$ [case (*a*)]. Since g belongs to L^q, the series $\Sigma |b_n|^{q'}$ is convergent, by Hausdorff's theorem†. But, by Hölder's inequality,

$$\Sigma |n^{-\mu} a_n b_n| \leqslant (\Sigma |n^{-\mu} a_n|^q)^{1/q} (\Sigma |b_n|^{q'})^{1/q'},$$

and therefore, by Theorem a, the series on the left is convergent.

Next, suppose that we are in either case (*a*) or case (*b*). Since $q > 1$, $\mu < 1/p$, and we may take $a = \mu$ in Theorem β. Hence f_μ belongs to $L^{q'}$. The convergence of (2.4) then follows from Riesz's theorem.

The remainder of the proof requires only the production of appropriate "Gegenbeispiele". We observe first that if

$$(3.2) \qquad f = \sum_1^\infty n^{-1-\delta+1/p} e^{ni\theta}, \qquad g = \sum_1^\infty n^{-1-\delta+1/q} e^{ni\theta} \quad (\delta > 0),$$

then f belongs to L^p and g to L^q. The series $\Sigma n^{-\nu} a_n b_n$ is then $\Sigma n^{-1-2\delta+\mu-\nu}$, which diverges if $\nu < \mu$ and δ is sufficiently small. There is therefore in no case any possibility of replacing μ by a smaller index.

Next we take

$$(3.3) \qquad f = g = \sum_1^\infty 2^{-\delta n} \cos 2^n \theta \quad (\delta > 0),$$

so that f and g are continuous. The series (2.4) is then $\Sigma 2^{-(\mu+2\delta) n}$, which diverges if $\mu < 0$ and δ is sufficiently small. We can therefore only infer convergence if $q \leqslant p'$, and so $p \leqslant p'$ or $p \leqslant 2$. This shows that (*a*) and (*b*) are the only cases in which the series is necessarily convergent.

Finally we choose f as in (3.2), but take

$$g = \sum_1^\infty n^{-\frac{1}{2}-\delta} e^{in \log n + ni\theta} \quad (\delta > 0).$$

* Hardy and Littlewood, **2** (Theorem 3). No proof is given there, but a proof is included in a memoir which we are preparing for publication.

† Hausdorff, **4**.

Then g is continuous*, while f, as before, belongs to L^p. The series of moduli is now

$$\Sigma n^{-\frac{1}{2}-2\delta-1/q},$$

and diverges if $q > 2$ and δ is sufficiently small. Thus we cannot assert *absolute* convergence in case (*b*).

The sum of the series, in all the cases in which Theorem 1 asserts its convergence, is given by the formula

$$\Sigma |n|^{-\mu} e^{-\frac{1}{2}\mu\pi i \operatorname{sgn} n} a_n b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\mu(\theta)\, g(-\theta)\, d\theta$$

(or the corresponding formula in which f and g are interchanged). This requires no further proof, as it follows directly from Riesz's theorem, at the point at which we appeal to this theorem above.

4. We may observe that the first paragraph of our proof proves more than is actually incorporated in Theorem 1, and the position may become clearer if we embody the additional information which we have obtained in an explicit theorem, though this theorem really adds nothing to Theorem α.

THEOREM 2. *The series* (2 . 4) *is absolutely convergent whenever f belongs to L^p, where $p \leqslant 2$, and the series $\Sigma |b_n|^{q'}$, where $p \leqslant q \leqslant p'$, is convergent.*

In case (*a*) this asserts more than Theorem 1, since $\Sigma |b_n|^{q'}$ is convergent whenever g belongs to L^q, while the converse proposition is false. In case (*b*) the hypotheses are *stronger* than those of Theorem 1†, and the conclusion (to absolute convergence) also stronger.

One further question remains. Since the hypotheses are now unsymmetrical, the assumption that $p \leqslant q$ entails a loss of generality, and it may be asked whether we have really exhausted the question. It is however easy to see that the hypothesis $p \leqslant q \leqslant p'$ is essential to Theorem 2. The conclusion involves the convergence of (3 . 1). Take now

$$f = \sum_{2}^{\infty} n^{-1+1/p}(\log n)^{-a} e^{ni\theta} \quad (a > 0).$$

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* Hardy and Littlewood, **1.**

† The converse inference being valid and not the direct one.

Then f is regular except for $\theta = 0$, where it behaves like a multiple of

$$|\theta|^{-1/p}\left(\log\frac{1}{|\theta|}\right)^{-a}.$$

Thus f belongs to L^p if $pa > 1$. The series (3 . 1), on the other hand, is convergent only when $qa > 1$. If $q < p$, the first hypothesis does not involve the second. We must therefore have $p \leqslant q$. Finally, the example (3 . 3) still shows that we must have $q \leqslant p'$.

5. So far we have supposed p and q to be greater than 1. Our theorems may, however, be extended, with appropriate reservations, to all positive values of p and q, and here the distinctions of § 1 become important. We have to appeal at one point to a theorem* which we have not stated before.

Suppose that $H(z)$ is the function (1 . 2), that $0 < p \leqslant 1$, and that the integral (1 . 3) is bounded. We shall then say that $H(z)$ belongs to the (complex) class L^p. It is still true that $H(re^{i\theta})$ tends almost always to a limit function $h(\theta)$, but $h(\theta)$ is not necessarily integrable.

We shall now require the following theorems.

THEOREM γ†. *If $H(z)$ belongs to L^p, where $p \leqslant 1$, then*

$$c_n = o\{n^{-(p-1)/p}\}, \tag{5 . 1}$$

and the series

$$\Sigma n^{p-2}|c_n|^p, \qquad \Sigma n^{-1/p}|c_n| \tag{5 . 2}$$

are convergent.

THEOREM δ‡. *If $0 < p \leqslant 1$, $(1-p)/p < a < 1/p$, H belongs to L^p, and*

$$H_a = \sum_1^\infty n^{-a} c_n z^n,$$

then H_a belongs to $L^{p/(1-pa)}$.

The last theorem is an analogue, for $p \leqslant 1$, of Theorem β. Since h is no longer integrable, our former definition of the "fractional integral" is inapplicable.

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* Theorem δ below.

† Hardy and Littlewood, **2** (Theorem 11) and **3** (Theorem 16).

‡ We have not stated this theorem before. A proof will be given in our memoir referred to in § 3 (p. 291, f.n. *).

THEOREM 3. *If $\Sigma a_n z_n$ and $\Sigma b_n z^n$ are functions of the classes L^p and L^q, then the series* (2.4) *is absolutely convergent whenever $0 < p \leqslant q \leqslant 2$, and convergent whenever $0 < p \leqslant 2 < q \leqslant p'$. If $p \leqslant 1$, the last inequality $q \leqslant p'$ is to be omitted.*

We may suppose $p \leqslant 1$, since otherwise there is nothing new to prove. If $q \leqslant 1$, then

$$\Sigma n^{-\mu} | a_n b_n | = \Sigma \{ n^{-1/p} | a_n | . n^{(q-1)/q} | b_n | \}$$

is convergent by Theorem γ. If $1 < q \leqslant 2$ it is sufficient, as in § 3, to prove the convergence of

$$(3.1) \qquad \Sigma (n^{-\mu} | a_n |)^q = \Sigma \{ n^{p-2} | a_n |^p . (n^{(p-1)/p} | a_n |)^{q-p} \};$$

and this is again convergent by Theorem γ. Finally, if $2 < q \leqslant p'$, f_μ belongs to $L^{q'}$, by Theorem δ, and the conclusion follows from Riesz's theorem as in § 3.

For the sake of completeness we add

THEOREM 4. *The series* (2.4) *is absolutely convergent whenever $0 < p \leqslant 1 \leqslant q$, f belongs to L^p, and $\Sigma | b_n |^{q'}$ is convergent.*

This has been proved above when $q = 1$. If $q > 1$, the argument above shows that (3.1) is convergent, and the conclusion then follows by Hölder's inequality, as in § 3.

Memoirs referred to.

1. G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation: A remarkable trigonometrical series", *Proc. Nat. Acad. of Sciences*, 2 (1916), 583-585 [for a correction, see *ibid.*, 3 (1917), 87].

2. G. H. Hardy and J. E. Littlewood. "Some properties of fractional integrals", *Proc. London Math. Soc.* (2), 24, xxxvii–xli (*Records* for 12 March, 1925).

3. G. H. Hardy and J. E. Littlewood, "New properties of Fourier constants", *Math. Annalen*, 97 (1926), 159-209.

4. F. Hausdorff, "Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen", *Math. Zeitschrift*, 16 (1923), 163-167.

5. M. Riesz, "Sur les séries trigonométriques conjuguées", *Comptes rendus*, 28 April, 1924.

6. H. Weyl, "Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung", *Vierteljahrsschrift d. naturf. Ges. in Zürich*, 62 (1917), 296–302.

7. W. H. and G. C. Young, "On the theorem of Riesz-Fischer", *Quart. J. of Math.*, 44 (1913), 49–88.

CORRECTIONS

p. **289,** *line* **7.** **Read** $\sum c_n e^{ni\theta}$.

p. **291,** *line* **6** *from below.* **Read** $\sum 2^{-(\mu+2\delta)n}$.

A POINT IN THE THEORY OF CONJUGATE FUNCTIONS

G. H. HARDY *and* J. E. LITTLEWOOD†.

[*Extracted from the Journal of the London Mathematical Society, Vol.* 4, *Part* 4.]

1. Suppose that $f(x)$ is periodic and belongs to the Lebesgue class L^p, where $p > 1$, and that

$$f(x) \sim \Sigma(a_n \cos nx + b_n \sin nx) = \Sigma A_n = A: \tag{1.1}$$

we may suppose, without real loss of generality, that the Fourier series has no constant term. Then the conjugates of $f(x)$ also belong to L^p‡. If we select that conjugate $g(x)$ whose Fourier series also has no constant term, then

$$-g(x) \sim \Sigma(b_n \cos nx - a_n \sin nx) = \Sigma B_n = B. \tag{1.2}$$

The formal relations between $f(x)$ and $g(x)$ are expressed by the equations

$$-g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot \tfrac{1}{2}(t-x) f(t)\, dt, \tag{1.31}$$

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot \tfrac{1}{2}(t-x)\, g(t)\, dt, \tag{1.32}$$

in which the integrals are Cauchy principal values. Each formula holds for almost all x. More precisely, it is known, after the researches of Fatou and Plessner§, that (1.31) holds whenever (*a*) $f(t)$ is continuous for $t = x$ or, more generally,

$$\int_0^t \{f(x+u) - f(x-u)\}\, du = o(t), \tag{1.41}$$

† Received 3 December, 1928; read 13 December, 1928.

‡ M. Riesz, **5**; see also Titchmarsh, **6**.

§ Fatou, **1**; Plessner, **4**.

1929, 4 (with J. E. Littlewood) *Journal of the London Mathematical Society*, 4, 242–5.

and (*b*) the series B is summable by Abel's limit. Similarly, of course, (1.32) holds whenever (a') $g(t)$ is continuous for $t = x$ or, more generally,

$$(1.42) \qquad \int_0^t \{g(x+u)-g(x-u)\}\, du = o(t),$$

and (b') A is summable by Abel's limit.

If we substitute from (1.31) into (1.32), we obtain

$$(1.5) \qquad f(x) = -\frac{1}{4\pi^2}\int_{-\pi}^{\pi} \cot \tfrac{1}{2}(t-x)\, dt \int_{-\pi}^{\pi} \cot \tfrac{1}{2}(u-t) f(u)\, du.$$

We may describe this formula as "Hilbert's integral formula"; it bears the same relation to (1.31) and (1.32) that Fourier's integral formula bears to the symmetrical formulae of the theory of Fourier transforms. (1.5) is true whenever the conditions (a') and (b') stated above are satisfied, for then (1.32) is, and the inner integral is almost always, equal to $g(t)$.

2. There is, however, a point which seems not to have been noticed. Of the conditions (a') and (b'), the first bears on g and the second on f. Both conditions are satisfied, for example, when *both* $f(t)$ and $g(t)$ are continuous for $t = x$, but this does not yield a criterion for the truth of (1.5) in terms of the behaviour of $f(t)$ alone. No one, so far as we know, has given such a criterion†; and it is plainly a criterion of this character that is required if the theory of (1.5) is to run parallel to that of Fourier's integral.

There may therefore be some interest in the following theorem:—

THEOREM 1. *It is sufficient for the validity of* (1.5) *that* $f(t)$ *should be continuous for* $t = x$ *and belong to* L^p, *where* $p > 1$.

We prove, in fact,

THEOREM 2. *It is sufficient for the validity of* (1.5) *that* $f(t)$ *should be continuous for* $t = x$ *and that* $g(t)$ *should belong to* L.

In Theorem 2 an explicit reference to g reappears; but the result is, none the less, a direct generalization of Theorem 1, since g certainly belongs to L^p, and *a fortiori* to L, if f belongs to L^p.

Since $f(t)$ is continuous for $t = x$, A is summable by Abel's limit (indeed, by Cesàro means). We have therefore only to verify (1.42).

† In terms of modern theories of integration. Criteria for the truth of (1.5), and of more general formulae, are given in paper 2, where references to the older literature of the formulae (1.3) and (1.5) will be found; but in these the behaviour of $f(t)$ is very severely restricted.

Now $$g(x+u)-g(x-u) \sim 2\Sigma A_n \sin nu,$$

and so

$$G = \int_0^t \{g(x+u)-g(x-u)\}\,du = 2\Sigma A_n \frac{1-\cos nt}{n} \tag{2.1}$$

$$= \frac{2}{\pi}\Sigma \frac{1-\cos nt}{n}\int_{-\pi}^{\pi} f(x+w)\cos nw\,dw.$$

We may invert the order of integration and summation, at any rate when t is sufficiently small. To prove this it is enough to show that

$$J_n = \int_{-\pi}^{\pi} f(x+w)\,R_n\,dw = o(1) \tag{2.2}$$

when $n \to \infty$, where

$$R_n = \sum_n^{\infty} \frac{1-\cos mt}{m}\cos mw.$$

Now for a suitable $\eta < \pi$ we have $|f(x+w)| < K$ in $|w| \leqslant \eta$. Suppose $|t| < \frac{1}{2}\eta$. Then

$$|J_n| \leqslant \int_{-\eta}^{\eta} |fR_n|\,dw + \left(\int_{-\pi}^{-\eta} + \int_{\eta}^{\pi}\right)|fR_n|\,dw = I_1 + I_2,$$

say. For I_1 we have

$$I_1 \leqslant K\int_{-\eta}^{\eta} |R_n|\,dw \leqslant K\left(2\eta\int_{-\pi}^{\pi} |R_n|^2\,dw\right)^{\frac{1}{2}}$$

$$= K\left\{2\pi\eta\sum_n^{\infty}\left(\frac{1-\cos mt}{m}\right)^2\right\}^{\frac{1}{2}} = o(1).$$

In I_2 we have $|w| \geqslant \eta$, $|w \pm t| \geqslant \frac{1}{2}\eta$, and so

$$\left|\sum_n^N (1-\cos mt)\cos mw\right| < \frac{A}{\eta},$$

where A is an absolute constant. Hence

$$|R_n| \leqslant \frac{1}{n}\operatorname*{Max}_{(N)}\left|\sum_n^N\right| < \frac{A}{n\eta},$$

$$I_2 \leqslant \frac{A}{n\eta}\int_{-\pi}^{\pi} |f(x+w)|\,dw = o(1).$$

This completes the proof of (2.2).

If now we perform the interchange in (2.1) and sum, we obtain

$$G = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+w)\log\left|\frac{\sin\frac{1}{2}(w+t)\sin\frac{1}{2}(w-t)}{\sin^2\frac{1}{2}w}\right|dw.$$

We have to prove that $G = o(t)$ when $t = x$ is a point of continuity of $f(t)$.

We may suppose, without real loss of generality, that $x = 0$ and (since G is unaltered by the addition of a constant to f) that $f(x) = 0$. Further, we may treat the parts of G in which w is positive and negative separately and by the same method. We may therefore consider

$$G^* = \int_0^\pi f(w) \log \left| \frac{\sin \frac{1}{2}(w+t) \sin \frac{1}{2}(w-t)}{\sin^2 \frac{1}{2} w} \right| dw = \int_0^{2t} + \int_{2t}^{\pi} = G_1 + G_2.$$

In G_2

$$\frac{\sin \frac{1}{2}(w+t) \sin \frac{1}{2}(w-t)}{\sin^2 \frac{1}{2} w} = 1 - \frac{1-\cos t}{1-\cos w} = 1 + O\left(\frac{t^2}{w^2}\right),$$

so that
$$G_2 = \int_{2t}^{\pi} o\left(\frac{t^2}{w^2}\right) dw = o\left(t^2 . \frac{1}{t}\right) = o(t).$$

In G_1 the ratio of each sine to its argument is bounded, and we may replace G_1 by

$$G_3 = \int_0^{2t} f(w) \log \left| \frac{w^2-t^2}{w^2} \right| dw$$

with error $O(t)\, o(1) = o(t)$. Finally we have

$$G_3 = \int_0^{2t} o(1) \left| \log \left| \frac{w^2-t^2}{w^2} \right| \right| dw = o\left(t \int_0^\infty \left| \log \left| 1 - \frac{1}{y^2} \right| \right| dy\right) = o(t).$$

This completes the proof of Theorem 2.

We may add finally that, by a recent theorem of Zygmund†, g belongs to L whenever $f \log |f|$ belongs to L. Theorem 1 may therefore be generalised by adopting this in place of the more restrictive hypothesis that f belongs to L^p.

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5. M. Riesz, "Sur les fonctions conjuguées", *Math. Zeitschrift*, 27 (1927), 218-244.
6. E. C. Titchmarsh, "Reciprocal formulae involving series and integrals", *Math. Zeitschrift*, 25 (1926), 321-347.
7. A. Zygmund, "Sur les fonctions conjuguées", *Fundamenta mathematicae*, 13 (1929), 284-303.

† Zygmund, 7.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy*.

LXVI.

The arithmetic mean of a Fourier constant.

1. It will be convenient in what follows to confine our attention to even functions, with period 2π, whose mean value over a period is zero. The Fourier series of such functions are pure cosine series without constant term.

The necessary and sufficient condition that $a_1, a_2, a_3, \ldots$ should be the Fourier constants of a function belonging to the Lebesgue class L^2 is that Σa_n^2 should be convergent. If this is so, and if

$$A_n = a_1 + a_2 + \ldots + a_n,$$

1929, 7 *Messenger of Mathematics*, 58, 50–2.

then $\Sigma n^{-2}A_n^2$ is convergent*; and so

$$A_1,\ \tfrac{1}{2}A_2,\ \tfrac{1}{3}A_3,\ \ldots$$

are also the Fourier constants of a function of L^2. It is natural to suppose that this result may be extended to functions of any Lebesgue class. The proof just given, however, cannot be generalised.

2. THEOREM. *If $a_1, a_2, a_3, \ldots$ are the Fourier constants of a function of L^p, where $p \geq 1$, then $A_1, \frac{1}{2}A_2, \frac{1}{3}A_3, \ldots$ are also the Fourier constants of a function of L^p.*

We have

$$a_n = \frac{2}{\pi}\int_0^\pi \cos nx f(x)\,dx,$$

$f(x)$ being the function of which a_n is the Fourier constant, and so†

$$A_n^* = a_1 + a_2 + \ldots + a_{n-1} + \tfrac{1}{2}a_n = \frac{1}{\pi}\int_0^\pi \sin nx \cot \tfrac{1}{2}x f(x)\,dx.$$

It is obviously enough to prove that A_n^* has the property postulated for A_n, since

$$\frac{A_n - A_n^*}{n} = \frac{a_n}{2n} = o\left(\frac{1}{n}\right)$$

is, by the theorems of Young and Hausdorff, the Fourier constant of a function of every Lebesgue class L^p.

We write

$$g(x) = \int_x^\pi \cot \tfrac{1}{2}u f(u)\,du,$$

and we prove first

(i) that $g(x)$ is integrable in $(0, \pi)$;

(ii) that $xg(x) \to 0$ when $x \to 0$.

Of these assertions, the first follows from the inequalities

$$\int_0^\pi |g(x)|\,dx \leq 2\int_0^\pi dx \int_x^\pi \frac{|f(u)|}{u}\,du$$

$$= 2\int_0^\pi \frac{|f(u)|}{u}\,du \int_0^u dx = 2\int_0^\pi |f(u)|\,du.$$

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* Many different proofs and generalisations of this theorem, which I stated first in Note 41 of this series (vol xliv., 1915, 163–166: see also Notes 51, vol. xlviii., 1919, 107–112, and 60, vol. liv., 1925, 150–156), have now been given. The simplest is by Prof. Elliott, *Journal London Math. Soc.*, vol. i., 1926, 93–96.

† Remembering that the integral is zero when $n=0$.

To prove the second we observe that

$$|xg(x)| \leq 2x \int_x^\pi \frac{|f(u)|}{u} du \leq 2 \int_x^\delta |f(u)| du + 2x \int_\delta^\pi \frac{|f(u)|}{u} du$$

for any δ between x and π, and that the right hand side may be made as small as we please by choice first of δ and then of x.

We have now

$$A_n^* = -\frac{1}{\pi} \int_0^\pi \sin nx\, g'(x)\, dx = -\frac{1}{\pi} \lim \int_\epsilon^\pi$$

$$= \frac{1}{\pi} \lim \left[\sin n\epsilon\, g(\epsilon) + n \int_\epsilon^\pi \cos nx\, g(x)\, dx\right];$$

and so, after (i) and (ii) above,

$$\frac{A_n^*}{n} = \frac{1}{\pi} \int_0^\pi \cos nx\, g(x)\, dx,$$

the integral being a Lebesgue integral. This proves the theorem when $p=1$.

3. It remains only to prove that g belongs to L^p, where $p>1$, if f does so; and it is plainly sufficient to prove this for

$$h(x) = \int_x^\pi \frac{f(u)}{u} du$$

(since $g-2h$ is continuous).

Let $$p' = \frac{p}{p-1}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

The necessary and sufficient condition that h should belong to L^p is that hk should be integrable for every k of $L^{p'}$. But, if k belongs to $L^{p'}$, then so does

$$K(x) = \frac{1}{x} \int_0^x |k(u)|\, du.*$$

Accordingly

$$\int_0^\pi |hk|\, dx \leq \int_0^\pi |k(x)|\, dx \int_x^\pi \frac{|f(u)|}{u} du = \int_0^\pi \frac{|f(u)|}{u} du \int_0^u |k(x)|\, dx$$

$$= \int_0^\pi |f(u)|\, K(u)\, du \leq \left(\int_0^\pi |f|^p du\right)^{1/p} \left(\int_0^\pi K^{p'} du\right)^{1/p'}$$

exists for every k of $L^{p'}$. This completes the proof of the theorem.

* By the principal theorem of Note 60, referred to above.

COMMENTS

A 'dual' of the theorem of this paper has been obtained by R. Bellman, *Bull. Amer. Math. Soc.* 50 (1944), 741–4, namely:

Let g be an even periodic function of L^p, where $p > 1$, with Fourier series $\sum_1^\infty b_n \cos n\theta$, *and let* $B_n = \sum_n^\infty b_k/k$ $(n = 1, 2,...)$. *Then*

(i) $\sum_1^\infty B_n \cos n\theta$ *is the Fourier series of a function* $G \in L^p$,

(ii) *if f is a function of $L^{p'}$ with Fourier series* $\sum_1^\infty a_n \cos n\theta$, *and F is the function (of $L^{p'}$) with Fourier series* $\sum_1^\infty A_n \cos n\theta$, *where* $A_n = n^{-1} \sum_1^n a_m$ $(n = 1, 2,...)$, *we have*

$$\int_0^\pi fG\,d\theta + \int_0^\pi Fg\,d\theta = 0.$$

The result (i) was obtained independently by G. I. Sunouchi, *Proc. Imp. Acad. Tokyo*, 20 (1944), 542–4. The corresponding result for sine series is due to T. Kawata, ibid. 218–22. The cases $p = 1$ and $p = \infty$ were subsequently discussed by C.-T. Loo, *Amer. J. of Math.* 71 (1949), 269–82.

SUMMATION OF A SERIES OF POLYNOMIALS OF LAGUERRE

G. H. Hardy*.

[*Extracted from the Journal of the London Mathematical Society, Vol.* 7, *Part* 2.]

1. It is known that, if $H_n(x)$ is the polynomial of Hermite, and

$$\phi_n(x) = \frac{e^{-\frac{1}{2}x^2}}{2^{\frac{1}{2}n}(n\,!)^{\frac{1}{2}}\pi^{\frac{1}{4}}} H_n(x) = (-1)^n \frac{e^{\frac{1}{2}x^2}}{2^{\frac{1}{2}n}(n\,!)^{\frac{1}{2}}\pi^{\frac{1}{4}}} \left(\frac{d}{dx}\right)^n e^{-x^2}$$

the corresponding orthogonal function for the interval $(-\infty, \infty)$, then

$$(1) \qquad \sum_0^\infty (-t)^n \phi_n(x)\,\phi_n(y) = \frac{\pi}{\sqrt{(1-t^2)}} \exp\left(-\frac{x^2+2txy+y^2}{1-t^2}\right)$$

for $|t| < 1$†.

It may be worth while to record the corresponding formula involving the polynomial of Laguerre. If

$$L_n(x) = e^x \left(\frac{d}{dx}\right)^n (e^{-x}x^n)$$

is Laguerre's polynomial, and

$$\psi_n(x) = \frac{e^{-\frac{1}{2}x}}{n!} L_n(x)$$

the corresponding orthogonal function for the interval $(0, \infty)$, then

$$(2) \qquad \sum_0^\infty (-t)^n \psi_n(x)\,\psi_n(y) = \frac{1}{1+t} \exp\left\{-\tfrac{1}{2}(x+y)\frac{1-t}{1+t}\right\} J_0\left\{\frac{2\sqrt{(xyt)}}{1+t}\right\}$$

(again for $|t| < 1$). More generally, if

$$\chi_n(x) = \{n\,!\,\Gamma(n+1+2a)\}^{-\frac{1}{2}} \frac{e^{\frac{1}{2}x}}{x^a} \left(\frac{d}{dx}\right)^n (e^{-x}x^{n+2a})‡,$$

then

$$(3) \qquad \sum_0^\infty (-t)^n \chi_n(x)\,\chi_n(y) = \frac{t^{-a}}{1+t} \exp\left\{-\tfrac{1}{2}(x+y)\frac{1-t}{1+t}\right\} J_{2a}\left\{\frac{2\sqrt{(xyt)}}{1+t}\right\}.$$

* Received and read 10 March, 1932.

† Prof. N. Wiener, from whose lectures on Fourier transforms I learnt the formula (1), informs me that a substantially equivalent formula has been given by Müntz, "Über die Potenzsummation einer Entwicklung nach Hermiteschen Polynomen", *Math. Zeitschrift*, 31 (1930), 350–355; and refers me also to an earlier paper of his own, "Hermitian polynomials and Fourier analysis", *Journal Mass. Inst. Technology*, 8 (1929), 70–73. See also J. Geronimus, "On the polynomials of Legendre and Hermite", *Tohôku Math. Journal*, 34 (1931), 295–296, where other formulae of similar type are proved. Geronimus attributes (1) to Kapteyn, but without reference.

‡ These orthogonal functions are used by Weyl, "Singuläre Integralgleichungen", *Math. Annalen*, 66 (1909), 273–324 (especially 317–324).

1932, 8 *Journal of the London Mathematical Society*, 7, 138–9; and addendum, ibid. 192.

2. It will be sufficient to indicate shortly the proof of (2). We find from the definition of ψ_n that

$$\int_0^\infty e^{-sx}\psi_n(x)\,dx = \frac{(s-\frac{1}{2})^n}{(s+\frac{1}{2})^{n+1}}, \tag{4}$$

if $s > -\frac{1}{2}$, and hence, by a familiar inversion formula, that

$$\psi_n(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx}\frac{(s-\frac{1}{2})^n}{(s+\frac{1}{2})^{n+1}}\,ds \quad (x>0,\ c>0).$$

Hence we deduce

$$\begin{aligned}\Psi(x,t) &= \sum_0^\infty (-t)^n\psi_n(x) \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{sx}\,ds}{s(1+t)+\frac{1}{2}(1-t)} = \frac{1}{1+t}\exp\left(-\tfrac{1}{2}x\,\frac{1-t}{1+t}\right),\end{aligned} \tag{5}$$

the well known formula for the generating function of the Laguerre polynomials*.

If now $\qquad \Omega = \Omega(x, y, t) = \sum_0^\infty (-t)^n\psi_n(x)\psi_n(y),$

we have

$$\begin{aligned}\int_0^\infty e^{-sx}\,\Omega(x,y,t)\,dx &= \sum_0^\infty (-t)^n\psi_n(y)\int_0^\infty e^{-sx}\psi_n(x)\,dx \\ &= \frac{1}{s+\frac{1}{2}}\sum_0^\infty\left(-t\,\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^n\psi_n(y) = \frac{1}{s+\frac{1}{2}}\,\Psi\left(y,\ t\,\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)\end{aligned}$$

by (4) and (5). If we invert again, and substitute for Ψ from (5), we find that

$$\Omega = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{sx}}{s+\frac{1}{2}+t(s-\frac{1}{2})}\exp\left\{-\tfrac{1}{2}y\,\frac{s+\frac{1}{2}-t(s-\frac{1}{2})}{s+\frac{1}{2}+t(s-\frac{1}{2})}\right\}ds.$$

If here we make the substitution

$$s+\tfrac{1}{2}\,\frac{1-t}{1+t} = w,$$

we obtain

$$\frac{1}{1+t}\exp\left\{-\tfrac{1}{2}(x+y)\,\frac{1-t}{1+t}\right\}\cdot\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\exp\left(xw-\frac{yt}{(1+t)^2w}\right)\frac{dw}{w},$$

with $k>0$; and this is (2)†. The proof of (3) follows the same lines, but the formulae are slightly more complicated.

The ψ_n are eigenfunctions for the kernel $\frac{1}{2}J_0\{\sqrt{(xy)}\}$, with eigenvalues $(-1)^n$, while the χ_n are eigenfunctions for $\frac{1}{2}(xy)^{-a}J_{2a}\{\sqrt{(xy)}\}$.

* See, for example, Courant-Hilbert, *Methoden der Math. Phys.*, ed. 2 (1931), 79.

† See G. N. Watson, *Bessel functions*, 177.

ADDENDUM: G. H. HARDY.

SUMMATION OF A SERIES OF POLYNOMIALS OF LAGUERRE*.

[*Extracted from the Journal of the London Mathematical Society, Vol.* 7, *Part* 3.]

Formula (1) is stated incorrectly, a factor

$$\pi^{-\frac{1}{2}} e^{\frac{1}{2}x^2+\frac{1}{2}y^2}$$

having been omitted. It should read

$$\sum_0^\infty (-t)^n \phi_n(x)\, \phi_n(y) = \frac{1}{\surd\pi\, \surd(1-t^2)} \exp\left\{-\frac{(x^2+y^2)(1+t^2)+4xyt}{2(1-t^2)}\right\}.$$

Prof. E. Hille informs me that this formula (for which I referred to Geronimus, Kapteyn, Muntz, and Wiener) was probably first found by Mehler [*Journal für Math.*, 66 (1866), 161–176 (173-176)], "and has been rediscovered by almost everybody who has worked in this field."

Prof. Hille also refers me, in connection with (1) and my formulae (2) and (3), to

1. S. Wigert, *Arkiv för Mat.*, 15 (1921), no. 27.
2. E. Hille (3 notes), *Proc. Nat. Acad. Sc.*, 12 (1926), 261–269, 348–352.
3. ———, *Annals of Math.* (2), 27 (1926), 427–464.
4. ———, *Math. Zeitschrift*, 32 (1930), 422–425.

To these references I add

5. E. Kogbetliantz, *Comptes rendus*, 25 April, 1932.

The last paper, which is practically simultaneous with my own, contains (2) and (3), in a slightly different notation. But (2) is present implicitly, though never actually stated, in Wigert (1), and (3) similarly in Hille (2). The formulae are those on which the theory of the Abel or Poisson summability of Laguerre developments depend.

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* *Journal London Math. Soc.*, 7 (1932), 138–139.

NOTES ON SPECIAL SYSTEMS OF ORTHOGONAL FUNCTIONS (II): ON FUNCTIONS ORTHOGONAL WITH RESPECT TO THEIR OWN ZEROS

G. H. HARDY*.

[*Extracted from the Journal of the London Mathematical Society, Vol. 14, 1939.*]

1. It is familiar that

$$\int_0^1 \sin m\pi t \sin n\pi t \, dt = 0 \tag{1.1}$$

if $m \neq n$, and, more generally, that†

$$\int_0^1 t J_\nu(\lambda_m t) J_\nu(\lambda_n t) \, dt = 0 \tag{1.2}$$

if $J_\nu(t)$ is Bessel's function of order ν, λ_m and λ_n are zeros of $J_\nu(t)$, $\nu > -1$, and $m \neq n$. We may express this by saying that the functions

$$\sin t, \quad t^{\frac{1}{2}} J_\nu(t) \tag{1.3}$$

are orthogonal with respect to their own zeros‡ in the interval (0, 1).

The equation (1.2) is the foundation of the theory of "Bessel-Fourier" series. It is natural to ask whether there are other functions, of the same general character as the Bessel functions, which possess the same property. I prove here that, within certain limits, this is not so; within these limits, the property is characteristic of the Bessel functions, and no generalization of the theory of Bessel-Fourier series is possible.

I am indebted to Dr. W. Rogosinski for suggestions which have enabled me to enlarge the scope of my original analysis considerably.

2. I consider first functions with real zeros situated symmetrically about the origin.

THEOREM 1. *If*

$$f(z) = z^\rho F(z) = z^\rho \prod_1^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right),$$

where $\rho > -\frac{1}{2}, \quad \lambda_n > 0, \quad \Sigma \lambda_n^{-2} < \infty,$

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* Received 10 December, 1938; read 15 December, 1938.

† Watson (8), 576.

‡ I borrow the phrase from J. M. Whittaker (9) 71.

so that $F(z)$ *is an integral function of order less than* 2, *or of order* 2 *and minimal type*; *and* $f(z)$ *is orthogonal with respect to its zeros*; *then*

$$f(z) = Az^{\frac{1}{2}} J_{\rho-\frac{1}{2}}(cz),$$

where A *and* c *are constants.*

The proof depends on two lemmas which have an independent interest.

If
$$\int_0^1 \{f(\lambda_n t)\}^2\, dt = A_n,$$

then the system

$$\phi_n(t) = A_n^{-\frac{1}{2}} f(\lambda_n t)$$

is orthogonal and normal in $(0, 1)$. Theorem 2 (which does not depend upon orthogonality) shows that the system is necessarily complete.

THEOREM 2. *If* $f(z)$ *satisfies the conditions of Theorem* 1 (*apart from orthogonality*), $g(t)$ *is integrable, and*

$$\int_0^1 g(t) f(\lambda_n t)\, dt = 0$$

for every n, *then* $g(t) = 0$.

We write $z = re^{i\theta}$, and

$$h(z) = \int_0^1 g(t) f(zt)\, dt.$$

It is plain that

$$h(z) = z^{\rho} H(z),$$

where $H(z)$ is integral.

I suppose first that $F(z)$ is of order less than 2, when $H(z)$ is also of order less than 2. Since $h(\lambda_n) = 0$ for every n,

$$\chi(z) = \frac{h(z)}{f(z)} = \frac{H(z)}{F(z)}$$

is also integral, and of order less than 2*.

If z lies on one of the lines l bisecting the angles between the axes, then $z^2 = \pm ir^2$ and

$$\psi(z, t) = \left|\frac{F(zt)}{F(z)}\right| = \prod_1^{\infty} \left|\frac{1 \pm (it^2 r^2/\lambda_n^2)}{1 \pm (ir^2/\lambda_n^2)}\right| = \prod_1^{\infty} \left\{\frac{1+(t^4 r^4/\lambda_n^4)}{1+(r^4/\lambda_n^4)}\right\}^{\frac{1}{2}}.$$

* See Titchmarsh (7), 255.

No factor here exceeds 1, and the limit of each factor, when $r \to \infty$, is t^2. Hence $\psi \leqslant 1$ for all r and t in question, and $\psi \to 0$, when $r \to \infty$, for every fixed $t < 1$; and therefore

$$|\chi(z)| = \left| \int_0^1 g(t) \frac{F(zt)}{F(z)} dt \right| \leqslant \int_0^1 |g(t)| \psi(z, t) dt$$

is bounded, and tends to zero, along the lines l. It now follows, by the simplest of the Phragmén-Lindelöf theorems concerning angles*, that $\chi(z)$ is bounded for all z, and is therefore a constant, which must be 0.

Thus
$$\int_0^1 g(t) f(zt)\, dt = 0$$

for all z. But
$$f(z) = z^{\rho} \sum_0^{\infty} a_{2n} z^{2n},$$

where $a_{2n} \neq 0$ for any n. Hence

$$\int_0^1 g(t)\, t^{\rho+2n}\, dt = 0$$

for every n, and therefore $g(t) \equiv 0$†.

The proof is substantially the same when $F(z)$ is of order 2 and minimal type. In this case $H(z)$ and $\chi(z)$ also are of order 2 and minimal type, and we must use the more delicate Phragmén-Lindelöf theorem in which the angle has its critical value‡.

3. The Fourier series of $f(zt)$, with respect to the system $\phi_n(t)$, is

$$f(zt) \sim \Sigma a_n(z) \phi_n(t), \tag{3.1}$$

where

$$a_n(z) = \int_0^1 f(zt) \phi_n(t)\, dt = A_n^{-\frac{1}{2}} \int_0^1 f(zt) f(\lambda_n t)\, dt; \tag{3.2}$$

and Parseval's theorem, which is true for any complete system, gives

$$p(z, \zeta) = \int_0^1 f(zt) f(\zeta t)\, dt = \sum_1^{\infty} a_n(z) a_n(\zeta). \tag{3.3}$$

* See, for example, Titchmarsh (7), 177, or Pólya and Szegö (6), 145.

† By "Lerch's theorem"; see, for example, Hobson (3), 22.

‡ Titchmarsh (7), 178; Pólya and Szegö (6), 149–334.

Theorem 3 gives the value of $a_n(z)$.

THEOREM 3. *If the conditions of Theorem* 1 *are satisfied, and* $z \neq \lambda_n$, *then*

$$\int_0^1 f(zt)\, f(\lambda_n t)\, dt = \frac{2A_n \lambda_n}{f'(\lambda_n)} \frac{f(z)}{z^2-\lambda_n^2}. \tag{3.4}$$

The proof is much the same as that of Theorem 2 (though here orthogonality is essential). Supposing first that $F(z)$ is of order less than 2, we write

$$\int_0^1 f(zt)\, f(\lambda_n t)\, dt = h(z), \qquad \frac{f(z)}{z^2-\lambda_n^2} = f_n(z),$$

$$g(z) = \frac{h(z)}{f_n(z)}, \qquad G(z) = \frac{g(z)}{(z+1)^2}.$$

Then $g(z)$ is an integral function of order less than 2; $G(z)$ is regular, and of order less than 2, in the half-plane $x > 0$; and, as in § 2,

$$G(z) = \frac{z^2-\lambda_n^2}{(z+1)^2} \int_0^1 \frac{f(zt)}{f(z)} f(\lambda_n t)\, dt$$

is bounded, and tends to 0, along the lines $\theta = \pm\frac{1}{4}\pi$. Hence

$$g(z) = O(|z|^2) \tag{3.5}$$

in the quadrant between the lines; and it may be shown similarly that (3.5) is true in the other three quadrants of the plane.

It follows that $g(z)$ is a quadratic and that

$$h(z) = g(z)\, f_n(z) = \frac{\alpha z^2+\beta}{z^2-\lambda_n^2} f(z).$$

But $G(z)$ tends to zero along the line $\theta = \frac{1}{4}\pi$, so that

$$g(z) = o(|z|^2), \quad \alpha = 0, \quad h(z) = \frac{\beta f(z)}{z^2-\lambda_n^2}.$$

Finally, determining the constant by making $z \to \lambda_n$, we obtain (3.4).

The modifications in the proof required when $F(z)$ is of order 2 and minimal type are the same as before.

4. It follows from (3.1), (3.3), and (3.4) that

$$p(z, \zeta) = \int_0^1 f(zt)\, f(\zeta t)\, dt = 4f(z)\, f(\zeta) \sum_1^\infty \frac{A_n \lambda_n^2}{\{f'(\lambda_n)\}^2} \frac{1}{(z^2-\lambda_n^2)(\zeta^2-\lambda_n^2)} \tag{4.1}$$

$$= -f(z)\, f(\zeta) \frac{q(z)-q(\zeta)}{z^2-\zeta^2},$$

where

$$q(z) = 4\sum_{1}^{\infty} \frac{A_n \lambda_n^2}{\{f'(\lambda_n)\}^2} \left(\frac{1}{z^2-\lambda_n^2} + \frac{1}{\lambda_n^2}\right). \tag{4.2}$$

The integral equation (4.1) will enable us to determine $f(z)$. The form of $q(z)$ will be irrelevant, except in so far as it shows that

$$q(0) = 0. \tag{4.3}$$

It follows from (4.1), by making $\zeta \to 0$, that

$$\int_0^1 t^\rho f(zt)\,dt = -\frac{f(z)\,q(z)}{z^2}$$

i.e.

$$\int_0^z u^\rho f(u)\,du = -z^{\rho-1} f(z)\,q(z). \tag{4.4}$$

Hence
$$\int_0^z u^{2\rho}\,du \sim -\tfrac{1}{2} z^{2\rho+1} q''(0)$$

for small z, and so

$$q''(0) = -\frac{2}{2\rho+1}. \tag{4.5}$$

Next, writing (4.1) in the form

$$\int_0^1 t^{2\rho} F(zt)\,F(\zeta t)\,dt = -F(z)\,F(\zeta)\,\frac{q(z)-q(\zeta)}{z^2-\zeta^2},$$

differentiating twice with respect to ζ, putting $\zeta = 0$, and using (4.3) and (4.5), we obtain

$$F''(0)\int_0^1 t^{\rho+2} f(zt)\,dt = -f(z)\left\{\frac{F''(0)\,q(z)}{z^2} + \frac{2}{(2\rho+1)z^2} + \frac{2q(z)}{z^4}\right\},$$

i.e.

$$a\int_0^z u^{\rho+2} f(u)\,du = -z^{\rho-1} f(z)\left\{(az^2+2)\,q(z) + \frac{2}{2\rho+1}\,z^2\right\}, \tag{4.6}$$

where
$$a = F''(0) = -2\sum \frac{1}{\lambda_n^2}.$$

Finally, eliminating $q(z)$ between (4.4) and (4.6), we obtain

$$a\int_0^z u^{\rho+2} f(u)\,du = (az^2+2)\int_0^z u^\rho f(u)\,du - \frac{2}{2\rho+1}\,z^{\rho+1} f(z), \tag{4.7}$$

an integral equation for $f(z)$.

Differentiating (4.7), and dividing by $2z$, we obtain

$$a\int_0^z u^\rho f(u)\,du+\frac{\rho}{2\rho+1}z^{\rho-1}f(z)-\frac{1}{2\rho+1}z^\rho f'(z)=0.$$

A second differentiation gives

$$f''(z)=\left\{a(2\rho+1)+\frac{\rho(\rho-1)}{z^2}\right\}f(z).$$

The general solution of this equation* is

$$f(z)=z^{\frac{1}{2}}\{AJ_{\rho-\frac{1}{2}}(cz)+BY_{\rho-\frac{1}{2}}(cz)\},$$

where $c^2=-a(2\rho+1)$. Plainly $B=0$, and this proves Theorem 1.

5. If we drop the condition of symmetry about the origin, we obtain a more restricted result.

THEOREM 4. *If*

$$f(z)=z^\rho F(z),$$

where $\rho>-\frac{1}{2}$ and $F(z)$ is an integral function, with real but not necessarily positive zeros, and of order less than 1, or of order 1 and minimal type; if $F(0)\neq 0$ and $f(z)$ is orthogonal with respect to its zeros; then

$$f(z)=AJ_{2\rho}(cz^{\frac{1}{2}}),$$

where A and c are constants.

There are two cases†. Either (i) $\Sigma\lambda_n^{-1}$ is absolutely convergent, and

$$f(z)=z^\rho\prod_1^\infty\left(1-\frac{z}{\lambda_n}\right); \tag{5.1}$$

this will certainly be so if $F(z)$ is of order less than 1. Or (ii) $\Sigma\lambda_n^{-1}$ is convergent, but not absolutely, and

$$f(z)=z^\rho e^{Cz}\prod_1^\infty\left\{\left(1-\frac{z}{\lambda_n}\right)e^{z/\lambda_n}\right\}, \tag{5.2}$$

* See Watson (**8**), 97.
† See Bieberbach (**1**), 437; Lindelöf (**4**).

where

$$C = -\Sigma \frac{1}{\lambda_n}. \tag{5.3}$$

I take the latter case, as the more general.

We can prove, much as in §§ 2–3, (*a*) that the system $\phi_n(t)$ is complete, and (*b*) that

$$\int_0^1 f(zt) f(\lambda_n t)\, dt = \frac{A_n}{f'(\lambda_n)} \frac{f(z)}{z-\lambda_n}.$$

The imaginary axis, on which

$$|e^{Cz}| = 1, \quad |e^{z/\lambda_n}| = 1,$$

plays the part of the lines l of § 2, and we must use the more delicate form of the Phragmén-Lindelöf theorem. The argument at the end of § 2 also requires a little reconsideration. We have

$$\int_0^1 g(b)\, t^{\rho+n}\, dt = 0 \tag{5.4}$$

if $F(z) = \Sigma a_n z^n$ and $a_n \neq 0$. Now no two consecutive a_n vanish*. Hence (5.4) is true for a set of n of density $\frac{1}{2}$, and *a fortiori* for a set $n = n_i$ with $\Sigma n_i^{-1} = \infty$. The conclusion now follows from Müntz's generalization of Lerch's theorem†.

We thus obtain the formula

$$\int_0^1 f(zt) f(\zeta t)\, dt = -f(z) f(\zeta) \frac{q(z)-q(\zeta)}{z-\zeta}, \tag{5.5}$$

where now
$$q(z) = \Sigma \frac{A_n}{\{f'(\lambda_n)\}^2} \left(\frac{1}{z-\lambda_n} + \frac{1}{\lambda_n}\right).$$

From (5.5), by an argument similar to that of § 4, we deduce

$$z^2 f''(z) + z f'(z) - \{(2\rho+1)\alpha z + \rho^2\} f(z) = 0,$$

where $\alpha = F'(0) = C$. The solution of this equation is

$$f(z) = A J_{2\rho}(cz^{\frac{1}{2}}) + B Y_{2\rho}(cz^{\frac{1}{2}}),$$

where $c^2 = -4\alpha(2\rho+1)$, and it is plain that $B = 0$. This proves Theorem 4.

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* See Borel (**2**), 35.

† Müntz (**5**).

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7. E. C. Titchmarsh, *Theory of functions* (1932).
8. G. N. Watson, *Bessel functions* (1922).
9. J. M. Whittaker, *Interpolatory function theory* (1935).

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CORRECTIONS

p. 49. An equation similar to p. 39, line 8 should replace the displayed equation (5.4), and (5.4) should be inserted two lines further on. For $g(b)$ in (5.4) read $g(t)$.

2. MEAN VALUES OF POWER SERIES

INTRODUCTION TO PAPERS ON MEAN VALUES OF POWER SERIES

About the beginning of this century, a number of authors had discussed the behaviour of the 'maximum modulus'

$$\mu(f;\rho) = \max_{\theta} |f(\rho e^{i\theta})|$$

for a function f regular in the unit disc. In 1915, 4, following a suggestion of Bohr and Landau, Hardy considered the mean value

$$\mu_p(f;\rho) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\rho e^{i\theta})|^p \, d\theta \qquad (p > 0),$$

and showed that this mean value behaves in a similar manner to $\mu(f;\rho)$. In 1923, F. Riesz made a systematic study of the class of functions f regular in the unit disc for which the mean value $\mu_p(f;\rho)$ is bounded. This class of functions, which Riesz called the Hardy class H^p, has since proved a fruitful field of investigation, requiring the most subtle techniques of real-variable and complex-variable theory. In virtue of M. Riesz's theorem on conjugate functions, the classes H^p for $p > 1$ form an important tool in the study of Fourier series of functions of the corresponding classes L^p. Further, the classes H^p for $p \leqslant 1$ provide a natural setting for the extension of results concerning Fourier series of functions of class L^r with $r > 1$. In the development of this theory, the work of Hardy and Littlewood forms an outstanding contribution.

More recently, the definition of the classes H^p has been extended to functions regular in more general domains, and also to functions defined on abstract spaces. Accounts of these more general theories can be found in K. Hoffman, *Banach Spaces of Analytic Functions* (Englewood Cliffs, 1962), and W. Rudin, *Fourier Analysis on Groups* (New York, 1962). The now classical theory is fully discussed in the two volumes of Zygmund's *Trigonometric Series* (Cambridge, 1959); and, as in the preceding sections of this volume, we refer to these two volumes as Z I and Z II.

This present section contains all the papers of Hardy and Littlewood in which the principal results lie in this field. The keys to much of this work of Hardy and Littlewood were provided by their theorem on fractional integrals of real functions, proved in 1928, 5 (included in this section), and their maximal theorems, proved in 1930, 1 (Volume II, section 3). Using these results, the two authors gave in 1932, 4 a systematic treatment of the fractional integrals and derivatives of functions of the class H^p for

$p > 0$. Other papers included here deal with the behaviour of the mean value $\mu_p(f; \rho)$ when the rate of growth of the mean value of the real part of f is specified (1931, 2) and with the Cesàro means of functions of class H^p for $p < 1$ (1934, 1).

The final two papers of this section, 1937, 3 and 1941, 1, again written jointly by Hardy and Littlewood, are primarily concerned with the mean values of the convolution of two power series. The theorems obtained in these two papers provide a wide generalization of the authors' earlier results on fractional integrals. The paper 1941, 1 contains also a unified treatment of much of the earlier work on the H^p classes.

It should be mentioned here that a number of papers included in other sections contain contributions to this field. Thus in 1926, 7 (this volume, section 1 (*c*)), Hardy and Littlewood extended one of their coefficient inequalities to the class H^p for $p \leqslant 1$, and in 1927, 4 (this volume, section 1 (*e*)) they used this result,.together with then unpublished results on fractional integrals, to extend Parseval's theorem. Their paper 1928, 6 (this volume, section 1 (*a*)) contains a number of theorems on the classes H^p, and other results concerning mean values of power series can be found in 1928, 11 (Volume II, section 3), 1932, 5 and 6 (this volume, section 1 (*c*)), and 1935, 5 and 1936, 2 (this volume, section 1 (*b*)).

T. M. F.

THE MEAN VALUE OF THE MODULUS OF AN ANALYTIC FUNCTION

By G. H. Hardy.

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1. Suppose that $f(x)$ is an analytic function of the complex variable x, regular for $|x| < \rho$, and that $M(r)$ denotes, as usual, the maximum of $|f(x)|$ on the circle $|x| = r < \rho$. Then it is known that $M(r)$ possesses the following properties:—

(i) $M(r)$ is a steadily increasing function of r;

(ii) $\log M(r)$ is a convex function of $\log r$, so that

$$\log M(r) \leqslant \frac{\log (r_2/r)}{\log (r_2/r_1)} \log M(r_1) + \frac{\log (r/r_1)}{\log (r_2/r_1)} \log M(r_2),$$

if

$$0 < r_1 \leqslant r \leqslant r_2 < \rho.$$

Further, when $f(x)$ is an integral function, so that $\rho = \infty$, it is known that

(iii) $M(r)$ tends to infinity with (r), and, unless $f(x)$ is a polynomial, more rapidly than any power of r.*

It was suggested to me by Dr. H. Bohr and Prof. E. Landau, rather more than a year ago, that the property (i) is possessed also by the *mean* value of $|f(x)|$ on the circle $|x| = r$, *i.e.*, by the function

$$\mu(r) = \frac{1}{2\pi} \int_0^{\pi} |f(re^{i\theta})| \, d\theta.$$

* The theorems (i) and (iii) are classical. Theorem (ii) was discovered independently by Blumenthal (*Jahresbericht der Deutschen Math.-Vereinigung*, Vol. 16, p. 97), Faber (*Math. Annalen*, Vol. 63, p. 549), and Hadamard (*Bulletin de la Soc. Math. de France*, Vol. 24, p. 186). The first statement of the theorem was due to Hadamard and the first proof to Blumenthal. The theorem is a corollary of one concerning the associated radii of convergence of a power series in two variables, due to Fabry (*Comptes Rendus*, Vol. 134, p. 1190), and Hartogs (*Math. Annalen*, Vol. 62, p. 1).

In the attempt to prove this I have been led to prove a good deal more, in particular that the function $\mu(r)$, and the more general function

$$\mu_\delta(r) = \frac{1}{2\pi}\int_0^\pi |f(re^{i\theta})|^\delta \, d\theta,$$

where δ is any positive number, possesses *all* the properties (i)–(iii) characteristic of $M(r)$. It should be observed that this is obvious when $\delta = 1$ and $\sqrt{\{f(x)\}}$ is one-valued for $r < \rho$; for then we have

$$\sqrt{\{f(x)\}} = b_0 + b_1 x + b_2 x^2 + \ldots,$$

say, and

$$\mu(r) = |b_0|^2 + |b_1|^2 r^2 + |b_2|^2 r^4 + \ldots .$$

2. The argument of the following paragraphs depends on two lemmas concerning conjugate functions*.

Suppose that

$$x = \xi + i\eta,$$

and that

$$X = \Xi + i\mathrm{H}$$

is a function of x regular for all values of x under consideration. Then Ξ and H are real conjugate functions of ξ and η.

Let ψ be a real function of Ξ and H, and so of ξ and η, with continuous second derivatives. Then the lemmas in question are expressed by the formulæ

$$\text{(A)} \qquad \frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = \left(\frac{\partial^2\psi}{\partial\Xi^2} + \frac{\partial^2\psi}{\partial\mathrm{H}^2}\right) M^2,$$

$$\text{(B)} \qquad \left(\frac{\partial\psi}{\partial\xi}\right)^2 + \left(\frac{\partial\psi}{\partial\eta}\right)^2 = \left\{\left(\frac{\partial\psi}{\partial\Xi}\right)^2 + \left(\frac{\partial\psi}{\partial\mathrm{H}}\right)^2\right\} M^2,$$

where $M = \left|\frac{dX}{dx}\right| = \sqrt{\left\{\left(\frac{\partial\Xi}{\partial\xi}\right)^2 + \left(\frac{\partial\mathrm{H}}{\partial\xi}\right)^2\right\}} = \sqrt{\left\{\left(\frac{\partial\Xi}{\partial\eta}\right)^2 + \left(\frac{\partial\mathrm{H}}{\partial\eta}\right)^2\right\}}.$

* The use of these lemmas was suggested to me by Dr. Bromwich, at a time when the paper contained only a part of its present contents. The whole argument has been reconstructed in consequence of this suggestion, and is much more concise and elegant than it was before. I am also indebted to Dr. Bromwich and to a referee for a number of minor suggestions. The lemmas themselves are given in Clerk-Maxwell's *Electricity and Magnetism*, Vol. 1, p. 289, and Dr. Bromwich informs me that they are due to Lamé (" Mémoire sur les Lois de l'Équilibre du Fluide Éthéré", *Journal de l'École Polytechnique*, Vol. 3, cahier 23).

The formula (A) and (B) may be proved as follows. From the equations

$$\frac{\partial\psi}{\partial\xi}=\frac{\partial\psi}{\partial\Xi}\frac{\partial\Xi}{\partial\xi}+\frac{\partial\psi}{\partial\mathrm{H}}\frac{\partial\mathrm{H}}{\partial\xi},\quad \ldots,\quad \ldots,\quad \ldots,$$

$$\frac{dX}{dx}=\frac{\partial X}{\partial\xi}=-i\frac{\partial X}{\partial\eta}=\frac{\partial\Xi}{\partial\xi}+i\frac{\partial\mathrm{H}}{\partial\xi}=-i\frac{\partial\Xi}{\partial\eta}+\frac{\partial\mathrm{H}}{\partial\eta},$$

it is easy to deduce that

(1) $$\frac{\partial\psi}{\partial\xi}-i\frac{\partial\psi}{\partial\eta}=\left(\frac{\partial\psi}{\partial\Xi}-i\frac{\partial\psi}{\partial\mathrm{H}}\right)\mu,$$

(2) $$\frac{\partial\psi}{\partial\xi}+i\frac{\partial\psi}{\partial\eta}=\left(\frac{\partial\psi}{\partial\Xi}+i\frac{\partial\psi}{\partial\mathrm{H}}\right)\bar{\mu},$$

where $\mu=\dfrac{dX}{dx}$ and $\bar{\mu}$ is the conjugate of μ. The formula (B) follows at once by multiplication. To prove (A) we operate on (1) with the operator

$$\frac{\partial}{\partial\xi}+i\frac{\partial}{\partial\eta},$$

and apply (2), observing that $$\left(\frac{\partial}{\partial\xi}+i\frac{\partial}{\partial\eta}\right)\mu=0.$$

3. Suppose now that $X=f(x)$ is regular for $|x|<\rho$, and that D is an annular region, defined by inequalities of the form

$$0<r_1\leqslant r=|x|\leqslant r_2<\rho,$$

and including no zeros of $f(x)$.

Let
$$\log x=\log r+i\theta=\zeta=\rho+i\theta,$$
$$\log X=\log R+i\Theta=\mathrm{Z}=\mathrm{P}+i\Theta,$$

where $$r>0,\quad R>0,\quad -\pi<\theta\leqslant\pi,\quad -\pi<\Theta\leqslant\pi.$$

Then P and Θ are conjugate functions of ρ and θ, with second derivatives continuous for all values of ρ and θ which correspond to values of x in D.

Let us take $$\psi=F(R)=\phi(\mathrm{P}),$$

where $F(R)$ is a function with a continuous second differential coefficient. Applying Lemma A, we obtain

(1) $$\frac{\partial^2\phi}{\partial\rho^2}+\frac{\partial^2\phi}{\partial\theta^2}=\frac{\partial^2\phi}{\partial\mathrm{P}^2}M^2,$$

where

(2) $$M^2=\left|\frac{d\mathrm{Z}}{d\zeta}\right|^2=\left(\frac{\partial\mathrm{P}}{\partial\rho}\right)^2+\left(\frac{\partial\Theta}{\partial\rho}\right)^2.$$

Let us now suppose that $\log\phi(\mathrm{P})$ is a positive and convex function

of P, so that
$$\frac{\partial^2}{\partial \mathrm{P}^2}\log\phi(\mathrm{P}) \geqslant 0,$$
or
$$\phi\frac{\partial^2\phi}{\partial \mathrm{P}^2} \geqslant \left(\frac{\partial\phi}{\partial \mathrm{P}}\right)^2;$$
and let

(3)
$$\nu(\rho) = \frac{1}{2\pi}\int_0^{2\pi}\phi(\mathrm{P})\,d\theta.$$
Then
$$\nu'(\rho) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial\phi}{\partial \mathrm{P}}\frac{\partial \mathrm{P}}{\partial\rho}\,d\theta,$$
$$|\nu'(\rho)| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\partial\phi}{\partial \mathrm{P}}\right|\left|\frac{\partial \mathrm{P}}{\partial\rho}\right|d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi}\sqrt{\left\{\phi\frac{\partial^2\phi}{\partial \mathrm{P}^2}\right\}}\left|\frac{\partial \mathrm{P}}{\partial\rho}\right|d\theta,$$
and so, by Schwarz's inequality,

(4)
$$\{\nu'(\rho)\}^2 \leqslant \frac{1}{4\pi^2}\int_0^{2\pi}\phi\,d\theta\int_0^{2\pi}\frac{\partial^2\phi}{\partial \mathrm{P}^2}\left|\frac{\partial \mathrm{P}}{\partial\rho}\right|^2d\theta \leqslant \frac{\nu(\rho)}{2\pi}\int_0^{2\pi}\frac{\partial^2\phi}{\partial \mathrm{P}^2}M^2\,d\theta.$$
But
$$\nu''(\rho) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial^2\phi}{\partial\rho^2}\,d\theta$$
and
$$\frac{\partial^2\phi}{\partial\rho^2}+\frac{\partial^2\phi}{\partial\theta^2} = \frac{\partial^2\phi}{\partial \mathrm{P}^2}M^2,$$
by (1). Hence

(5)
$$\nu''(\rho) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial^2\phi}{\partial \mathrm{P}^2}M^2\,d\theta - \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial^2\phi}{\partial\theta^2}\,d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial^2\phi}{\partial \mathrm{P}^2}M^2\,d\theta,$$
since ϕ is a function of P or of R only, and R is periodic in θ. From (4) and (5) it follows that

(6)
$$\nu(\rho)\,\nu''(\rho) \geqslant \{\nu'(\rho)\}^2,$$
or that $\log\nu$ is a convex function of ρ.

We have thus proved

THEOREM I.—*If* $\log\{\phi(\log R)\}$ *is a convex function of* $\log R$, *then*
$$\log\nu(\log r) = \log\left\{\frac{1}{2\pi}\int_0^{2\pi}\phi(\log R)\,d\theta\right\}$$
is, throughout any interval of values of r *which includes no zeros of* $f(x)$, *a convex function of* $\log r$.

In particular, we may take

$$F(R) = \phi(\mathrm{P}) = e^{\delta \mathrm{P}} = R^{\delta},$$

in which case $\phi\phi'' = \phi'^2$. It follows that $\log \mu_\delta(r)$, and in particular $\log \mu(r)$, is a convex function of $\log r$, throughout any interval of values of r which includes no zeros of $f(x)$. This case is indeed the critical case of Theorem I, the condition that $\phi(\mathrm{P})$ should be a convex function of P being only just satisfied.

4. With Theorem I we may associate another theorem, in which less is postulated and less proved.

THEOREM II.—*If* $\phi(\log R)$ *is a convex function of* $\log R$, *then* $\nu(\log r)$ *is a convex function of* $\log r$.

For $\nu''(\rho)$ is positive, by (5) of § 3. The critical case of Theorem II is that in which $\phi(\log R) = \log R$. In this case we have, by a well known theorem of Jensen*,

$$\nu(\log r) = \frac{1}{2\pi}\int_0^{2\pi} \log R\, d\theta = \log \left| \frac{cr^n}{a_{m+1}a_{m+2}\ldots a_n} \right|,$$

where
$$f(x) = cx^m + \ldots,$$

and $a_{m+1}, a_{m+2}, \ldots, a_n$ are the zeros of $f(x)$, other than the origin, whose moduli are not greater than r. In this case $\nu(\log r)$ is a linear function of $\log r$ throughout any interval of values of r which includes no zeros of $f(x)$.

5. In order to proceed further with our investigations concerning $\mu_\delta(r)$, we must examine the behaviour of $\mu_\delta(r)$ for the exceptional values of r which correspond to zeros of $f(x)$, and for $r = 0$. I shall prove that

$$r\frac{d\mu_\delta(r)}{dr}$$

is continuous without exception.

Let
$$x_0 = \rho e^{i\phi} \quad (\rho > 0)$$

be a zero of $f(x)$. We have to prove that

$$\frac{d\mu_\delta(r)}{dr}$$

* *Acta Mathematica*, Vol. 22, p. 359.

is continuous throughout an interval of values of r of the type

$$\rho-\eta \leqslant r \leqslant \rho+\eta.$$

I shall suppose, for simplicity, that x_0 is the only zero of modulus ρ. The proof is substantially the same when there are several such zeros. I shall prove that the integral

$$\frac{d\mu_\delta(r)}{dr} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial R^\delta}{\partial r}\, d\theta$$

is uniformly convergent throughout the interval $\rho-\eta \leqslant r \leqslant \rho+\eta$, if η is small enough.

We have
$$f(x) = (x-x_0)^m f_1(x),$$

where m is a positive integer, and $f_1(x)$ has no zeros whose modulus lies between $\rho-\eta$ and $\rho+\eta$, so that $|f_1(x)|$ lies between positive constants H_1 and H_2.

Now, taking $\psi = F(R)$ in Lemma B, we have

$$\left(\frac{\partial F}{\partial \xi}\right)^2 + \left(\frac{\partial F}{\partial \eta}\right)^2 = \left\{\left(\frac{\partial F}{\partial \Xi}\right)^2 + \left(\frac{\partial F}{\partial \mathrm{H}}\right)^2\right\} \left|\frac{df}{dx}\right|^2.$$

In particular, if
$$F(R) = R = \sqrt{(\Xi^2+\mathrm{H}^2)},$$

we have
$$\left(\frac{\partial R}{\partial \xi}\right)^2 + \left(\frac{\partial R}{\partial \eta}\right)^2 = \left|\frac{df}{dx}\right|^2,$$

and
$$\left|\frac{\partial R}{\partial r}\right| = \left|\cos\theta \frac{\partial R}{\partial \xi} + \sin\theta \frac{\partial R}{\partial \eta}\right| \leqslant \sqrt{\left\{\left(\frac{\partial R}{\partial \xi}\right)^2 + \left(\frac{\partial R}{\partial \eta}\right)^2\right\}} = \left|\frac{df}{dx}\right|.$$

But
$$\frac{df}{dx} = m(x-x_0)^{m-1} f_1(x) + (x-x_0)^m \frac{df_1}{dx},$$

and so
$$\left|\frac{df}{dx}\right| < K|x-x_0|^{m-1},$$

where K is a constant. Hence

(5)
$$\left|\frac{\partial R}{\partial r}\right| < K|x-x_0|^{m-1}.$$

Also

(6)
$$R^{\delta-1} < H_2^{\delta-1}|x-x_0|^{m(\delta-1)}$$

if $\delta > 1$, and

(6′)
$$R^{\delta-1} < H_1^{\delta-1}|x-x_0|^{m(\delta-1)},$$

if $\delta < 1$. From (5) and (6) or (6′) it follows that

$$\left| R^{\delta-1} \frac{\partial R}{\partial r} \right| < K_1 \, |x - x_0|^{m\delta - 1}, \tag{7}$$

where K_1 is a constant. If $m\delta - 1 \geqslant 0$, we have

$$\left| R^{\delta-1} \frac{\partial R}{\partial r} \right| < K_2,$$

where K_2 is a constant, and then the integral

$$\int_0^{2\pi} R^{\delta-1} \frac{\partial R}{\partial r} \, d\theta$$

is obviously uniformly convergent. If, on the other hand, $m\delta - 1 < 0$, we have

$$|x - x_0| = \sqrt{(r^2 + \rho^2 - 2r\rho \cos \omega)},$$

where $\omega = \theta - \phi$, and so

$$|x - x_0| > K_3 \, |\sin \tfrac{1}{2}\omega|,$$

where K_3 is a constant. The uniform convergence of the integral then follows at once when we compare it with

$$\int_0^{2\pi} |\sin \tfrac{1}{2}\omega|^{m\delta - 1} \, d\omega.$$

6. We have thus proved that $\log \mu_\delta(r)$ is a convex function of $\log r$ for all positive values of r save certain exceptional values, and that

$$\frac{d \log \mu_\delta(r)}{d \log r}$$

is continuous even for these values of r. It follows that $\log \mu_\delta(r)$ is a convex function of $\log r$ for all positive values of r without exception*. *A fortiori* is $\mu_\delta(r)$ a convex function of $\log r$, and

$$r \frac{d\mu_\delta(r)}{dr},$$

an increasing function of r.

It remains to consider the behaviour of $r \dfrac{d\mu_\delta(r)}{dr}$ as $r \to 0$. Suppose that the origin is a zero of $f(x)$ of order m. Then

$$R^\delta = r^{m\delta} R_1^\delta,$$

* A series of continuous convex arcs, fitted together so as to have the same tangents at the points of junction, forms a single convex curve.

where R_1 is positive and has continuous derivatives. Hence

$$r\frac{d\mu_\delta(r)}{dr} = m\delta\, r^{m\delta}\int_0^{2\pi} R_1^\delta\, d\theta + r^{m\delta+1}\int_0^{2\pi} \frac{\partial R_1^\delta}{\partial r}\, d\theta,$$

which plainly tends to zero as $r \to 0$.

Thus $r\dfrac{d\mu_\delta(r)}{dr}$ is continuous and steadily increasing for all positive values of r, and tends to zero as $r \to 0$. It follows that

$$r\frac{d\mu_\delta(r)}{dr} \geqslant 0$$

for all positive values of r.

We have thus proved

THEOREM III.—*The integral*

$$\mu_\delta(r) = \frac{1}{2\pi}\int_0^{\pi} R^\delta\, d\theta \quad (\delta > 0)$$

is a positive, continuous, and steadily increasing function of r. *The same is true of*

$$r\frac{d\mu_\delta(r)}{dr}.$$

And $\log \mu_\delta(r)$, *and a fortiori* $\mu_\delta(r)$ *itself, is a convex function of* $\log r$.

7. The last theorem contains *inter alia* the answer to the question raised by Bohr and Landau. It should, however, be observed that the most appropriate measure of the "average increase" of $f(x)$ is not the mean value of R, or of any power of R, but of $\log R$; for the former means are not adequately affected by the occurrence of zeros of $f(x)$, or of arcs on which R is small.

8. It remains to discuss the analogues for $\mu_\delta(r)$ of the property (iii) of § 1.

We may suppose without loss of generality that $f(x)$ has infinitely many zeros. If it has not, it is of the form

$$P(x)\, e^{g(x)},$$

where $P(x)$ is a polynomial and $g(x)$ an integral function. Now

$$e^{\frac{1}{2}\delta g(x)} = b_0 + b_1 x + b_2 x^2 + \ldots,$$

say; and

$$\mu_\delta^{(1)}(r) = \frac{1}{2\pi}\int_0^{2\pi} |e^{g(x)}|^\delta \, d\theta = |b_0|^2 + |b_1|^2 r^2 + |b_2|^2 r^4 + \ldots,$$

certainly tends to infinity more rapidly than any power of r. It follows immediately that the same is true of $\mu_\delta(r)$.

Suppose, then, that $f(x)$ has an infinity of zeros, and that $r_{m+1}, r_{m+2}, \ldots, r_n$ are the moduli of those, other than the origin, whose moduli do not exceed r. Then, if $g(\theta)$ is any continuous function of θ, we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{g(\theta)}\, d\theta \geqslant e^{\frac{1}{2\pi}\int_0^{2\pi} g(\theta)\, d\theta};$$

and so
$$\mu_\delta(r) = \frac{1}{2\pi}\int_0^{2\pi} R^\delta \, d\theta \geqslant e^{\frac{\delta}{2\pi}\int_0^{2\pi}\log R\, d\theta} = \left| \frac{cr^n}{r_{m+1} r_{m+2} \ldots r_n} \right|^\delta,$$

by Jensen's theorem. It follows at once that $\mu_\delta(r)$ tends to infinity with r more rapidly than any power of r. We can indeed go further, and establish relations between the rate of increase of r_n, considered as a function of n, and $\mu_\delta(r)$, considered as a function of r, in every way analogous to those given by Jensen's theorem for $M(r)$.* For example, if the "real order" of $f(x)$ is ρ, we have

$$\mu_\delta(r) > e^{r^{\rho-\epsilon}}$$

for every positive ϵ and values of r surpassing all limit.

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* Lindelöf, *Acta Societatis Fennicae*, Vol. 31, No. 1; see also Borel, *Leçons sur les fonctions méromorphes*, p. 105.

CORRECTIONS

p. 269, *line* 6 *from below*. For (r) read r.

p. 269, *last line*, and *p.* 270, *line* 3 and *p.* 276, *line* 10. For $\int_0^\pi$ read $\int_0^{2\pi}$.

p. 271, § 3, *lines* 7, 8. If Θ is restricted to the range $-\pi < \Theta \leqslant \pi$ in the annulus, then it will not necessarily be continuous, as may be seen by considering $f(x) = x^m$, where $m > 1$. However, this does not affect subsequent work.

p. 274, *line* 9 *from below*. For $\frac{df}{dr}$ read $\frac{df}{dx}$.

pp. 274–5. The relations (5)–(7) should be renumbered (7)–(9).

COMMENTS

This paper initiates the study of the Hardy classes H^p, and was also the beginning of some important work on mean values. An elegant proof of Hardy's results for the special case $\delta = 1$ was given by G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis* (Berlin, 1925), vol. 1, Aufgabe 306. More recent proofs make use of the theory of subharmonic functions (see F. Riesz, *Acta Math.* 48 (1926), 329–43, and J. E. Littlewood, *Lectures on the theory of functions* (Oxford, 1944), 152–62).

p. 272. Hardy's proof of his identity (5) shows that this identity is valid whenever ϕ has a continuous second derivative and f has no zeros on the circle $|z| = r$. If we restate Hardy's identity (5) in terms of F and r (and note that, by (2),

$$M^2 = r^2|f'(re^{i\theta})|^2/|f(re^{i\theta})|^2),$$

we obtain the following result.

Let F have a continuous second derivative, and let

$$G(R) = \left(R\frac{d}{dR}\right)^2 F(R).$$

Let also f be regular in the unit disc, and let

$$\nu(r) = \frac{1}{2\pi}\int_0^{2\pi} F(|f(re^{i\theta})|)\,d\theta. \tag{1}$$

Then if $f(z)$ has no zeros on the circle $|z| = r$,

$$\left(r\frac{d}{dr}\right)^2 \nu(r) = \frac{r^2}{2\pi}\int_0^{2\pi} G(|f(re^{i\theta})|)|f(re^{i\theta})|^{-2}|f'(re^{i\theta})|^2\,d\theta. \tag{2}$$

If now H is a function with a continuous derivative, and we define F by

$$F(R) = \int H(R)\,d\log R, \tag{3}$$

then $G(R) = RH'(R)$, and the identity (2) becomes

$$\left(r\frac{d}{dr}\right)^2 \nu(r) = \frac{r^2}{2\pi}\int_0^{2\pi} H'(|f(re^{i\theta})|)|f(re^{i\theta})|^{-1}|f'(re^{i\theta})|^2\,d\theta \tag{4}$$

(provided as before that $f(z)$ has no zeros on the circle $|z| = r$).

In §§ 5–6, Hardy proved also that, in the special case in which $F(R) = R^\delta$, the function $r\nu'(r)$ is continuous for $0 \leqslant r < 1$ (even when f has zeros). Hence in this case we can divide both sides of (4) by r and integrate from 0 to ρ, where $0 \leqslant \rho < 1$. We thus obtain the identity

$$\rho\frac{d\mu_\delta(\rho)}{d\rho} = \frac{\delta^2}{2\pi}\int_0^{\rho}\int_0^{2\pi} |f(re^{i\theta})|^{\delta-2}|f'(re^{i\theta})|^2 r\,d\theta dr, \tag{5}$$

valid for any f regular in the unit disc.

The identity (5), which is implicit in the present paper, was rediscovered by P. Stein, *J. London Math. Soc.* 8 (1933), 242–7, and was used by him to obtain a new proof of M. Riesz's theorem on conjugate functions. Subsequently D. C. Spencer, *Amer. J. of Math.* 65 (1943), 147–60, obtained an integrated form of the identity (4), namely that if H is absolutely continuous, F and ν are defined by (3) and (1), and $f(z)$ has no zeros on the circle $|z| = \rho$, then

$$\rho\nu'(\rho) = \frac{1}{2\pi}\int_0^{\rho}\int_0^{2\pi} H'(|f(re^{i\theta})|)|f(re^{i\theta})|^{-1}|f'(re^{i\theta})|^2 r\,d\theta dr + H(0)n(\rho), \tag{6}$$

where $n(\rho)$ is the number of zeros of $f(z)$ for $|z| < \rho$. A simple proof of (6) in the case where H' is continuous has been given by T. M. Flett, *J. London Math. Soc.* 29 (1954), 115–18.

In connection with this paper see also the footnote on p. 406 of 1928, 2.

Some properties of fractional integrals

G. H. HARDY *and* J. E. LITTLEWOOD.

Extracted from Records of Proceedings at the Meeting of the London Mathematical Society, March, 1925.

1. The idea of an integral or derivative, of arbitrary non-integral order, was introduced into analysis by Liouville and Riemann.* Such integrals and derivatives may be, and have been by different writers, defined in a variety of manners, and different systems of definitions may be the most useful in different fields of analysis. The most usual definition is as follows. Suppose that we are considering $f(x)$ in an interval (A, B), that $a \leqslant A$, that $f(x)$ is integrable† in (a, B), and $0 < \alpha < 1$, $A \leqslant x \leqslant B$. Then we define $f_\alpha(x)$, the α-th integral, or integral of order α, of $f(x)$, by the equation

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt. \tag{1.1}$$

It is easy to see that $f_\alpha(x)$ exists for almost all x in (A, B), and is integrable. The definition may then be extended formally to values of α outside $(0, 1)$ by ordinary integration or differentiation.

The definition involves an arbitrary *origin of integration* a. It is usual to suppose a finite. It has, however, been pointed out by Weyl‡ that for some purposes, and particularly in the theory of trigonometrical series, when $f(t)$ is periodic, it is more appropriate to take $a = -\infty$. The integral (1.1) will then usually be non-absolutely convergent at the lower limit.

We state here a number of theorems concerning fractional integrals which appear to be new and to have a number of interesting applications. We postpone a more detailed account of our researches, since we are at present by no means certain that we have found the most appropriate methods of proof.

2. We say that f belongs to the class L^p, where $p > 1$, if f and $|f|^p$ are integrable, and that f satisfies a Lipschitz condition of order r if

$$f(x+h) - f(x) = O(|h|^r),$$

uniformly throughout the interval (A, B). We write

$$p' = \frac{1}{p}, \quad q' = \frac{1}{q}.$$

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* See G. H. Hardy and M. Riesz, "The general theory of Dirichlet's series", *Cambridge Math. Tracts* (18), 22, and the writings there referred to.

† In the sense of Lebesgue.

‡ H. Weyl, "Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung", *Vierteljahrsschrift d. naturf. Ges. in Zürich*, 62 (1917), 296–302.

Our starting point is a theorem concerning series of positive terms, proved by Prof. G. Pólya and ourselves in a paper presented recently to the Society.

THEOREM 1.—*Suppose that a_m and b_n are positive,*

$$p > 1, \quad q > 1, \quad p'+q' > 1,$$

and

$$\lambda = 2-p'-q';$$

and that Σa_m^p and Σb_n^q are convergent. Then

$$\Sigma\Sigma \frac{a_m b_n}{|m-n|^\lambda} \quad (m \neq n) \tag{2.1}$$

is convergent.

This theorem may be compared with others already known. If we replace $|m-n|$ by $m+n$ in (2.1), we obtain a simple generalization of a well known theorem of Hilbert.* This theorem, however, lies less deep than Theorem 1, and is true when

$$p'+q' = 1, \quad \lambda = 1,$$

when Theorem 1 becomes false.†

From Theorem 1 we deduce

THEOREM 2.—*If f and g belong to L^p and L^q respectively, and p and q satisfy the conditions of Theorem 1, then*

$$\iint \frac{f(x)\,g(y)}{|x-y|^\lambda}\,dx\,dy$$

is convergent.

THEOREM 3.—*If f belongs to L^p and $0 < \alpha < p'$, then f_α belongs to L^ϖ, where*

$$\varpi = \frac{p}{1-p\alpha}.$$

THEOREM 4.—*If f belongs to L^p and $p' < \alpha < p'+1$, then f_α satisfies a Lipschitz condition of order $\alpha-p'$.*

THEOREM 5.—*If f belongs to L^p and $0 < \alpha < 1$, then*

$$\int_A^B |f_\alpha(x)-f_\alpha(x-h)|^p\,dx = O(|h|^{p\alpha}).$$

* See G. H. Hardy, "Note on a theorem of Hilbert concerning series of positive terms", *Proc. London Math. Soc.* (2), 23 (1925), xlv–xlvi (*Records* for April 24, 1924); and memoirs there referred to.

† The series
$$\Sigma\Sigma \frac{a_m a_n}{m-n}$$
is then convergent (as has been shown by Marcel Riesz), but not necessarily absolutely.

It is understood here that $f(x) = 0$ outside (A, B).

THEOREM 6.—*If*

$$\int_A^B |g(x)-g(x-h)|^p\,dx = O(|h|^{p\alpha}),$$

then g possesses derivatives, belonging to L^p, of all orders less than α.

The last two theorems were suggested by the theorems of Weyl* that (1) the α-th integral of a continuous function satisfies a Lipschitz condition of order α, and (2) that a function which satisfies such a condition has continuous derivatives of all orders less than α.

3. THEOREM 7.—*If f belongs to L^p, and*

$$1 < p \leqslant 2, \quad p \leqslant q \leqslant \frac{p}{p-1},$$

then

$$\Sigma n^{-\kappa}(|a_n|^q + |b_n|^q),$$

where

$$\kappa = \frac{p+q-pq}{p},$$

and a_n and b_n are the Fourier constants of f in (A, B), is convergent.

When $q = p/(p-1)$, Theorem 5 reduces to a theorem of Hausdorff;† and the general theorem is deducible from Hausdorff's theorem and the special result for $q = p$. It may be observed that, if we assert only that

$$\Sigma n^{-\kappa-\epsilon}(|a_n|^q + |b_n|^q)$$

is convergent for every positive ϵ, then the condition $q \geqslant p$ becomes unnecessary. This less precise theorem is, however, not new, for Young‡ has proved the convergence of

$$\Sigma n^{-\lambda}(|a_n| + |b_n|) \quad (\lambda > p'),$$

and the result follows from this and Hausdorff's theorem.

There is an analogue of Theorem 7 for *moment constants*, viz.

THEOREM 8.—*If f belongs to L^p, $p \leqslant q$, and*

$$a_n = \int_0^1 x^n f(x)\,dx,$$

* H. Weyl, *l.c.*

† F. Hausdorff, "Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen", *Math. Zeitschrift*, 16 (1923), 163–167.

‡ W. H. Young, "On the multiplication of successions of Fourier constants", *Proc. Royal Soc.* (A), 87 (1912), 331-339.

then $$\Sigma n^{-\kappa} a_n^q$$
is convergent.

This is a much easier theorem, which does not involve the difficulties characteristic of such theorems as 2, 3, or 7.

Theorems 7 and 8 may be generalized in various directions, to series involving products of Fourier or moment constants of several functions, to multiple series of any order, and so forth.

4. We conclude by quoting a theorem of a somewhat different character.

THEOREM 9.—*Suppose that $f(z) = \Sigma a_n z^n$ is an analytic function of $z = re^{i\theta}$, regular for $r < 1$, and that*

$$\int_0^{2\pi} |f(re^{i\theta})|\, d\theta$$

is bounded for $r < 1$. Then $$\Sigma \frac{|a_n|}{n}$$
is convergent.

An equivalent theorem, stated in terms of functions of a real variable, is

THEOREM 10.—*Suppose that the two conjugate series*

$$\Sigma(a_n \cos n\theta + b_n \sin n\theta), \qquad \Sigma(b_n \cos n\theta - a_n \sin n\theta),$$

are both Fourier series. Then

$$\Sigma \frac{|a_n| + |b_n|}{n}$$

is convergent.

Finally, Theorem 9 may be generalized in a different direction.

THEOREM 11.—*If $0 < p \leqslant 1$ and*

$$\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta$$

is bounded, then each of the series

$$\Sigma n^{-1/p} |a_n|, \quad \Sigma n^{p-2} |a_n|^p$$

is convergent.

The last result may be regarded as an extension, to values of p not greater than 1, of the extreme case of Theorem 7 in which $q = p$.

CORRECTION

First line following the statement of Theorem 7. For 'Theorem 5' read 'Theorem 7'.

COMMENT

The results described here are proved in full in 1928, 5 and 1932, 4.

Some properties of fractional integrals. I.

Von

G. H. Hardy in Oxford und J. E. Littlewood in Cambridge.

1. Introduction.

1.1. In this memoir we present the first systematic treatment of certain theorems some of which we enunciated in a short note published in 1924 [1]). Our theme is the properties of the "Riemann-Liouville" integrals and derivatives of arbitrary order of functions of certain standard classes, in particular the "Lebesgue classes L^p", the "Lipschitz classes Lip. k", and the more general classes of functions which satisfy "integrated Lipschitz conditions". We shall give the formal definitions of these classes in a moment.

Our arguments in this memoir are entirely "real" and direct. Our results have many interesting applications to the theory of analytic functions and the theory of Fourier series; but we make no use of these theories here, our only weapons being elementary inequalities and the ordinary methods of the theory of functions of a real variable. It seems clear indeed that the "right" proofs of all the theorems which we prove here are of this "elementary" character. All our variables and functions are accordingly real.

1.2. We say (following F. Riesz [2])) that $f(x)$ belongs to the class L^p, where $p \geqq 1$, in a finite interval (a, b), if $f(x)$ and $|f(x)|^p$ are integrable in (a, b) in the sense of Lebesgue. The class L^1 or L is the class of integrable functions.

We say that $f(x)$ belongs to the class Lip. k, where $0 \leqq k \leqq 1$, in (a, b) if

$$f(x) - f(x-h) = O(h^k)$$

[1]) Hardy and Littlewood, **4**.

[2]) F. Riesz, **11**.

1928, 5 (with J. E. Littlewood) *Mathematische Zeitschrift*, 27, 565–606.

uniformly for $a \leqq x - h < x \leqq b$; and to the class Lip.* k if

$$f(x) - f(x-h) = o(|h|^k)$$

when $h \to 0$, uniformly in x. Thus the class Lip.* 0 is the class of continuous functions.

We say that $f(x)$ belongs to the class Lip. (k, p), where $p \geqq 1$, $0 \leqq k \leqq 1$, in (a, b), if $f(x)$ belongs to L in an interval including (a, b) in its interior, and

$$\int_a^b |f(x) - f(x-h)|^p \, dx = O(|h|^{kp}); \tag{1.21}$$

and to the class Lip.* (k, p) if it satisfies the analogous equation with o.

If $f(x)$ belongs to Lip. (k, p) it necessarity belongs to L^p. For if we select a (small) positive h and write

$$\Phi_h = \Phi_h(x) = \frac{1}{h}\int_0^h f(x-u)\,du$$

we have

$$f - \Phi_h = \frac{1}{h}\int_0^h (f(x) - f(x-u))\,du.$$

Hence, by Hölder's inequality,

$$|f - \Phi_h|^p \leqq \frac{1}{h}\int_0^h |f(x) - f(x-u)|^p\,du,$$

$$\int_a^b |f - \Phi_h|^p\,dx \leqq \frac{1}{h}\int_0^h du \int_a^b |f(x) - f(x-u)|^p\,dx < \frac{A}{h}\int_0^h u^{kp}\,du < A,$$

where the A's are constants; and the result follows, since Φ_h is continous.

The "α-th integral" $f_\alpha(x)$ of $f(x)$, where α is any positive number and $f(x)$ any function of L, was defined by Liouville and Riemann[3]) by the equation

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt. \tag{1.22}$$

This agrees with the ordinary definition when α is a positive integer[4]).

[3]) Liouville, 8; Riemann, 10; naturally for less general f.

[4]) See for example Jordan, 7.

The derivatives $f^{\beta}(x)$ or $D^{\beta}f(x)$ may then be defined for $0<\beta<1$ by

$$f^{\beta}(x)=\frac{d}{dx}f_{1-\beta}(x),$$

and for larger values of β by further differentiations.

We may call these integrals and derivatives "right-handed" integrals and derivatives "with origin a". It is obvious that we may equally define, for example, a "left-handed" integral "with origin b". All the theorems which we prove in one case will apply, with the obvious changes, to the other. But it is plain that the differences between integrals, whether right or left-handed, with different origins, though not of importance for our purposes here, are not entirely trivial.

In applications, we are usually concerned with *periodic* functions; and it was observed by Weyl[5]) that the definitions just given, attaching as they do a particular rôle to the origin a, are not altogether appropriate in this case. Weyl pointed out that, if we suppose (as we may do without real loss of generality[6]) that the mean value of $f(x)$ over a period is zero, then (1.22) converges (at $t=x$) or diverges with

$$(1.23)\qquad f_{\alpha}^{*}(x)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-t)^{\alpha-1}f(t)\,dt,$$

and used this as the definition of the α-th integral. We use both definitions here. Up to § 5.5 inclusive we distinguish the two and state our theorems for both. By this time it has become plain that the distinction is not important for our purposes, and we follow Weyl's definition as the easier to work with and the more appropriate for the ends which we have in view.

1.3. If $f(x)$ is integrable, $f_{\alpha}(x)$ exists, for any $\alpha>0$, for almost all x, and is integrable. Our first object is to determine the Lebesgue class to which $f_{\alpha}(x)$ necessarily belongs when $f(x)$ belongs to L^{p}. Our principal result (Theorem 4) shews that if $p>1$ and $0<\alpha<\frac{1}{p}$ then $f_{\alpha}(x)$ belongs to $L^{\frac{p}{1-p\alpha}}$. This theorem is deduced from a theorem (Theorem 3) concerning the convergence of double integrals of the type

$$\iint\frac{f(x)\,g(y)}{|x-y|^{\lambda}}\,dx\,dy$$

which is very interesting in itself.

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[5]) Weyl, 18.

[6]) Subtracting a constant from $f(x)$.

We prove these theorems in § 3, § 2 being devoted to the corresponding theorems for series, from which the integral theorems are deduced by passages to the limit. The theorems of § 2 are due to Pólya and ourselves jointly, and a good deal of § 2 is substantially a reproduction of part of a joint memoir published in the *Proceedings of the London Mathematical Society* [7]). We have inserted this section partly because (using a suggestion of M. Riesz) we are able to simplify the proof of Theorem 1 to a certain extent, but more because these theorems are the real origin of the most interesting part of the memoir, and we have thought it worth while to make the memoir a whole in itself at the cost of a little repetition. In § 4 we give a series of generalisations of the results of § 3. These involve no new difficulty of idea, but it seems worth while to develop Theorems 3 and 4 to their fullest extent.

In § 5 we pass to functions of the Lipschitz classes. Theorem 4 ceases to be significant when $\alpha > \frac{1}{p}$, and the result which replaces it (Theorem 12) is that $f_\alpha(x)$ belongs to the Lipschitz class Lip.* $\left(\alpha - \frac{1}{p}\right)$. In Theorem 14 we suppose that $f(x)$ itself belongs to a Lipschitz class of order k, and prove (what is to be expected) that, so long as $k + \alpha < 1$, $f_\alpha(x)$ belongs to the class of order $k + \alpha$. Theorem 19 is a very interesting theorem of Weyl [8]), to which we add a complement in Theorem 20. The remaining theorems of this section are subsidiary. The general lesson of the latter theorems of the section is that to belong to a Lipschitz class of order k is nearly, though not quite, the same thing as to have continuous derivatives of all orders less than k.

Finally, in § 6 we consider the corresponding questions for the generalised Lipschitz classes Lip. (k, p), the general lesson being that to belong to such a class is nearly, though not quite, the same thing as to possess derivatives, belonging to L^p, of all orders less than k.

2. Theorems concerning series.

2.1. Theorem 1. *Suppose that*

$$S = \sum_{m,n=-h}^{h} c_{m-n}\, a_m\, b_n\,,$$

where

$$c_\nu = c_{-\nu}, \qquad c_0 \geqq c_1 \geqq c_2 \geqq \ldots,$$

and the a's and b's are positive (in the wider sense) and given in every

[7]) Hardy, Littlewood, Pólya, 6.

[8]) Weyl, 8.

respect except arrangement. Then, among the arrangements for which S assumes its maximum value, there is one in which

(2.11) $$a_m - a_{m'} \geqq 0\,, \quad b_n - b_{n'} \geqq 0$$

whenever

(2.12) $$|m| < |m'|\,, \quad |n| < |n'|\,,$$

and all of

(2.13) $$a_{-m} - a_m\,, \quad b_{-n} - b_n \qquad (m, n = 1, 2, 3, \ldots)$$

have the same sign.

It is convenient to regard the summations as extended over all integral values, all but a finite number of the a's and b's being zero.

Given any integer p, we can associate the a's and b's either in pairs

(2.14) $$(a_{p-i}, a_{p+i}), \qquad (b_{p-j}, b_{p+j}) \qquad (i, j = 1, 2, 3, \ldots)$$

or in pairs

(2.15) $$(a_{p-i}, a_{p+i+1}), \qquad (b_{p-j}, b_{p+j+1}) \qquad (i, j = 0, 1, 2, \ldots).$$

In the first case the elements a_p and b_p are unpaired, and the pairs include, for appropriate p, i, j, any pair of elements the difference of whose rank is even. In the second case all pairs with odd difference of rank occur for appropriate p, i, j. Our argument applies to either case, but is slightly simpler in expression in the second (since no element is unpaired). We therefore write it out in the first case; and we may suppose, for reasons of symmetry, that $p \geqq 0$, so that

$$|p - i| \leqq |p + i|\,, \quad |p - j| \leqq |p + j|\,.$$

We denote by I or J a value of i or j for which $a_{p-I} < a_{p+I}$ or $b_{p-J} < b_{p+J}$. We shall prove that there is a maximum arrangement in which, whatever p, there are no I's or J's; and it is plainly sufficient to show that, whatever p, the value of S is not diminished by the substitution Ω_p defined by the permutation of every pair

$$(a_{p-I}, a_{p+I}), \quad (b_{p-J}, b_{p+J}).$$

We divide up S into the following partial sums:

(1) S_1: $m = p$; $\quad n = p$.

(2) S_2: $m = p$; $\quad n = p - j,\ p + j\ (j \neq J)$.

(3) S_3: $m = p - i,\ p + i\ (i \neq I)$; $\quad n = p$.

(4) S_4: $m = p$; $\quad n = p - J,\ p + J$.

(5) S_5: $m = p - I,\ p + I$; $\quad n = p$.

(6) $S_6:\ m = p - i,\ p + i\ (i \neq I);\quad n = p - j,\ p + j\ (j \neq J).$

(7) $S_7:\ m = p - i,\ p + i\ (i \neq I);\quad n = p - J,\ p + J.$

(8) $S_8:\ m = p - I,\ p + I;\quad n = p - j,\ p + j\ (j \neq J).$

(9) $S_9:\ m = p - I,\ p + I;\quad n = p - J,\ p + J.$

It is evident that S_1, S_2, S_3 and S_6 are not affected by Ω_p. S_4 is

$$a_p \sum_J (c_J b_{p-J} + c_{-J} b_{p+J}),$$

which is unaltered because $c_J = c_{-J}$. Hence S_4 is unaltered; and similarly S_5 is unaltered. S_9 is

$$\sum_{I,J} (c_{-I+J} a_{p-I} b_{p-J} + c_{-I-J} a_{p-I} b_{p+J} + c_{I+J} a_{p+I} b_{p-J} + c_{I-J} a_{p+I} b_{p+J}),$$

and is also unaltered. It is therefore sufficient to prove that S_7 and S_8 are not diminished, and plainly we need only prove this for S_7.

In S_7, the contribution of the pair (a_{p-i}, a_{p+i}) is

$$a_{p-i} \sum_J (c_{-i+J} b_{p-J} + c_{-i-J} b_{p+J}) + a_{p+i} \sum_J (c_{i+J} b_{p-J} + c_{i-J} b_{p+J}).$$

The increment in this produced by Ω_p is

$$-(a_{p-i} - a_{p+i}) \sum_J (c_{i-J} - c_{i+J})(b_{p-J} - b_{p+J}) \geqq 0,$$

since the three brackets are respectively positive, positive and negative.

It follows that there is a maximum arrangement which satisfies the conditions of the theorem, in so far as they concern pairs the difference of whose rank is even. The same argument, based upon the pairing (2.15), leads to the same conclusion for pairs the difference of whose rank is odd[9]); and this proves the theorem. We have in fact proved that there is a maximum arrangement in which all the differences (2.13) are positive or zero; but this adds nothing to what is stated in the theorem, since S is plainly unaltered if m and n are interchanged with $-m$ and $-n$.

2.2. Theorem 2. *If*

(2.21) $$a_m \geqq 0,\quad b_n \geqq 0,\quad r > 1,\quad s > 1,\quad \frac{1}{r} + \frac{1}{s} > 1$$

and the series

(2.22) $$A = \sum a_m^r, \quad B = \sum b_n^s$$

are convergent, then

(2.23) $$T = \sum{}' \frac{a_m b_n}{|m-n|^\lambda} \leqq K A^{\frac{1}{r}} B^{\frac{1}{s}},$$

---

[9]) In this case the sums $S_1, S_2, \ldots, S_5$ do not occur.

where

$$\lambda = 2 - \frac{1}{r} - \frac{1}{s}, \tag{2.24}$$

$K = K(r, s)$ *is a function of* r *and* s *only, and the summation extends over all unequal integral values of* m *and* n.

It is obviously sufficient to prove the inequality for

$$T_h = \sum_{m,n=-h}^{h}{}' \frac{a_m b_n}{|m-n|^{\lambda}}.$$

This form is not as it stands of the type S, but it does not exceed the form S in which

$$c_0 = 1, \quad c_\nu = |\nu|^{-\lambda} \qquad (\nu \neq 0).$$

We may suppose that, in S, the a's and b's are arranged in the most unfavourable manner, defined by Theorem 1. In this case S does not exceed the sum of four sums of the type

$$S^* = \sum{}' c_{m-n} a_m b_n.$$

where m and n now run from 0 to h and a_m and b_n are decreasing functions of m and n. Further

$$S^* = S_0 + S_1 + S_2,$$

where

$$S_0 = \sum a_m b_m, \qquad S_1 = \sum_{m>n} \frac{a_m b_n}{|m-n|^{\lambda}},$$

and S_2 is a similar sum over $n > m$. It is therefore sufficient to prove that S_0 and S_1 satisfy inequalities of the type (2.23). It will be convenient to suppose now, as plainly we may, that m and n run from 1 to h only.

We use three known inequalities[10]:

(i) *If* $r > 1$, $s > 1$, $\frac{1}{r} + \frac{1}{s} \geqq 1$ *and* u_ν *and* v_ν *are positive, then*

$$\sum u_\nu v_\nu \leqq \left(\sum u_\nu^r\right)^{\frac{1}{r}} \left(\sum v_\nu^s\right)^{\frac{1}{s}}.$$

(ii) *If* u_ν *and* v_ν *are positive and monotonic, and vary in opposite directions, then*

$$h \sum_{\nu=1}^{h} u_\nu v_\nu \leqq \sum_{\nu=1}^{h} u_\nu \sum_{\nu=1}^{h} v_\nu.$$

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[10]) For detailed references, see Hardy, Littlewood, Pólya, 6, 274. The first inequality, with $\frac{1}{r} + \frac{1}{s} = 1$ (Hölder's inequality), and its analogue for integrals, are of course used repeatedly throughout the memoir.

(iii) *If* $r > 1$, u_ν *is positive, and*

$$U_\nu = u_1 + u_2 + \ldots + u_\nu,$$

then

$$\sum_{\nu=1}^{h} \left(\frac{U_\nu}{\nu}\right)^r \leqq \left(\frac{r}{r-1}\right)^r \sum_{\nu=1}^{h} u_\nu^r.$$

The sum S_0 is at once disposed of by (i), and we need only consider S_1. Now

$$(2.25) \qquad S_1 = \sum_{m=2}^{h} a_m \sum_{n=1}^{m-1} \frac{b_n}{(m-n)^\lambda}.$$

But $(m-n)^{-\lambda}$ increases with n and b_n decreases. Hence, writing

$$b_1 + b_2 + \ldots + b_n = B_n,$$

we have, by (ii),

$$\sum_{n=1}^{m-1} \frac{b_n}{(m-n)^\lambda} \leqq \frac{B_{m-1}}{m-1} \sum_{n=1}^{m-1} \frac{1}{(m-n)^\lambda} \leqq \frac{B_{m-1}}{(1-\lambda)(m-1)^\lambda}.$$

Substituting in (2.25), and applying (i), with

$$r' = \frac{r}{r-1} > s$$

in place of s, we obtain

$$(2.26) \qquad S_1 \leqq \frac{1}{1-\lambda} \sum_{m=2}^{h} a_m \frac{B_{m-1}}{(m-1)^\lambda}$$

$$= \frac{1}{1-\lambda} \sum_{m=1}^{h-1} a_{m+1} \frac{B_m}{m^\lambda} \leqq \frac{A^{\frac{1}{r}}}{1-\lambda} \left(\sum_{m=1}^{h} \frac{B_m^{r'}}{m^{\lambda r'}}\right)^{\frac{1}{r'}}.$$

Now $B_m^s \leqq m^{s-1} B$, by (i), with s and s' for r and s.[11] Hence

$$m^{-\lambda r'} B_m^{r'} \leqq m^{-\lambda r'} B_m^s (m^{s-1} B)^{\frac{r'-s}{s}} = B^{\frac{r'-s}{s}} \left(\frac{B_m}{m}\right)^s,$$

since

$$-\lambda r' + \frac{(r'-s)(s-1)}{s} = -s.$$

Hence (2.26) gives

$$S_1 \leqq \frac{A^{\frac{1}{r}}}{1-\lambda} B^{\frac{r'-s}{r's}} \left(\sum_{m=1}^{h} \left(\frac{B_m}{m}\right)^s\right)^{\frac{1}{r'}} \leqq \frac{A^{\frac{1}{r}}}{1-\lambda} B^{\frac{1}{s}-\frac{1}{r'}} \left(\frac{s}{s-1}\right)^{\frac{s}{r'}} B^{\frac{1}{r'}} = K A^{\frac{1}{r}} B^{\frac{1}{s}},$$

by (iii). This proves the theorem.

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[11]) We use the notation $c' = \dfrac{c}{c-1}$ generally, when $c > 1$.

3. The fundamental theorems for integrals.

3.1. We shall say that $f(x)$ belongs to L^p $(p \geqq 1)$ in (a, ∞), if it belongs to L^p in (a, b) for any finite $b > a$, and

$$\int_a^\infty |f(x)|^p \, dx$$

is convergent. For a finite interval, the class L^p includes the class L^q if $p > q$, but this is not true for an infinite interval.

Theorem 3. *If $f(x)$ and $g(y)$ belong to L^r and L^s respectively in $(0, \infty)$,*

$$f(x) \geqq 0, \quad g(y) \geqq 0, \quad r > 1, \quad s > 1, \quad \frac{1}{r} + \frac{1}{s} > 1, \tag{3.11}$$

and

$$F = \int_0^\infty f^r(x)\, dx, \qquad G = \int_0^\infty g^s(y)\, dy, \tag{3.12}$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{|x-y|^\lambda}\, dx\, dy \leqq K F^{\frac{1}{r}} G^{\frac{1}{s}}, \tag{3.13}$$

λ and K being defined as in Theorem 1[12]).

(i) We suppose first that $f(x)$ and $g(y)$ are zero for $x > \xi$ or $y > \xi$ and continuous in $(0, \xi)$.[13]) We denote the square $0 \leqq x \leqq \xi$, $0 \leqq y \leqq \xi$ by D, and the part of it in which $|x - y| \geqq \varepsilon > 0$ by $D(\varepsilon)$. It is plainly sufficient to prove that

$$J(\varepsilon) = \iint\limits_{D(\varepsilon)} \frac{f(x)\, g(y)}{|x-y|^\lambda}\, dx\, dy \leqq K F^{\frac{1}{r}} G^{\frac{1}{s}} \tag{3.14}$$

for every ε.

We divide D into squares $D_{m,n}$ by the lines

$$x = x_m = \frac{m\xi}{\nu}, \qquad y = y_n = \frac{n\xi}{\nu} \qquad (0 \leqq m \leqq \nu,\ 0 \leqq n \leqq \nu),$$

calling $D_{m,n}$ the square corresponding to $m-1$, m, $n-1$, n and $D_{m,n}(\varepsilon)$ any $D_{m,n}$ which has a point in common with $D(\varepsilon)$; so that

$$J(\varepsilon) = \lim_{\nu \to \infty} \sum \iint\limits_{D_{m,n}(\varepsilon)}.$$

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[12]) Hardy and Littlewood **4**, (Theorem 2).

[13]) One-sidedly at the ends.

It is plain that we may suppose ν so large that

$$|m-n| \geqq 2, \qquad |x-y| \geqq |m-n| \frac{\xi}{2\nu}$$

at every point of every $D_{m,n}(\varepsilon)$. Also, in $D_{m,n}(\varepsilon)$,

$$f(x)g(y) \leqq f(x_m)g(y_n) + \eta_\nu,$$

where η_ν tends uniformly to 0 when $\nu \to \infty$. It follows that

$$J(\varepsilon) \leqq \overline{\lim} \left(\frac{2\nu}{\xi}\right)^\lambda \sum \iint\limits_{D_{m,n}(\varepsilon)} \frac{f(x_m)g(y_n)+\eta_\nu}{|m-n|^\lambda} dx\,dy.$$

But

$$\overline{\lim} \left(\frac{2\nu}{\xi}\right)^\lambda \sum \iint \frac{\eta_\nu}{|m-n|^\lambda} dx\,dy = 2^\lambda \xi^{2-\lambda} \overline{\lim} \left(\eta_\nu \nu^{\lambda-2} \sum{}' \frac{1}{|m-n|^\lambda}\right)$$
$$\leqq K\xi^{2-\lambda}\, \overline{\lim}\, \eta_\nu = 0,$$

by Theorem 2. Hence, again using Theorem 2, we obtain

$$J(\varepsilon) \leqq 2^\lambda \overline{\lim} \left(\frac{\xi}{\nu}\right)^{2-\lambda} \sum{}' \frac{f(x_m)g(y_n)}{|m-n|^\lambda}$$
$$\leqq K\, \overline{\lim} \left(\frac{\xi}{\nu}\right)^{2-\lambda} \left(\sum_1^\nu f^r(x_m)\right)^{\frac{1}{r}} \left(\sum_1^\nu g^s(y_n)\right)^{\frac{1}{s}}$$
$$= K\, \overline{\lim} \left(\frac{\xi}{\nu}\sum_1^\nu f^r(x_m)\right)^{\frac{1}{r}} \left(\frac{\xi}{\nu}\sum_1^\nu g^s(y_n)\right)^{\frac{1}{s}}$$
$$= K F^{\frac{1}{r}} G^{\frac{1}{s}}.$$

This proves (3.14), and therefore (3.13) in the particular case considered.

3.2. (ii) We suppose next that $f(x)$ and $g(y)$ are any functions of L^r and L^s in $(0, \xi)$, and zero outside.

We can determine positive and continuous functions $f^*(x)$ and $g^*(y)$ such that

$$\int f^{*r} dx \leqq 2^r M, \qquad \int g^{*s} dy \leqq 2^s M,$$
$$\int f^* dx \leqq 2M^{\frac{1}{r}} \xi^{\frac{r-1}{r}}, \quad \int g^* dy \leqq 2M^{\frac{1}{s}} \xi^{\frac{s-1}{s}},$$

where $M = \mathrm{Max}(F, G)$, so that

$$\int f\,dx \leqq M^{\frac{1}{r}} \xi^{\frac{r-1}{r}}, \quad \int g\,dy \leqq M^{\frac{1}{s}} \xi^{\frac{s-1}{s}};$$

and also

$$\int |f-f^*|\,dx, \quad \int |g-g^*|\,dy, \quad \int |f^r-f^{*r}|\,dx, \quad \int |g^s-g^{*s}|\,dy$$

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are all less than a given positive η. We have then

$$\iint_{D(\varepsilon)} \frac{fg}{|x-y|^{\lambda}}\,dx\,dy \leqq \iint_{D} \frac{f^* g^*}{|x-y|^{\lambda}}\,dx\,dy + \iint_{D(\varepsilon)} \frac{|fg-f^*g^*|}{|x-y|^{\lambda}}\,dx\,dy$$

$$\leqq K\Big(\int f^{*r}\,dx\Big)^{\frac{1}{r}} \Big(\int g^{*s}\,dy\Big)^{\frac{1}{s}} + \varepsilon^{-\lambda} \iint_{D(\varepsilon)} |fg - f^*g^*|\,dx\,dy$$

$$\leqq K\Big(\int f^{r}\,dx+\eta\Big)^{\frac{1}{r}} \Big(\int g^{s}\,dy+\eta\Big)^{\frac{1}{s}} + \varepsilon^{-\lambda} \iint_{D} (g|f-f^*|+f^*|g-g^*|)\,dx\,dy$$

$$\leqq K\Big(\int f^{r}\,dx+\eta\Big)^{\frac{1}{r}} \Big(\int g^{s}\,dy+\eta\Big)^{\frac{1}{s}} + 2\eta\,\varepsilon^{-\lambda}\Big(M^{\frac{1}{r}}\xi^{\frac{r-1}{r}} + M^{\frac{1}{s}}\xi^{\frac{s-1}{s}}\Big).$$

Making $\eta \to 0$, we obtain (3.14) and so (3.13).

(iii) The final result of the theorem now follows in the obvious manner by making $\xi \to \infty$.

3.3. Theorem 4. *Suppose that $f(x)$ belongs to L^p, where $p>1$, in (a, b), where $-\infty < a < b \leqq \infty$; that*

$$0 < \alpha < \frac{1}{p}, \qquad q = \frac{p}{1-p\alpha}; \tag{3.31}$$

and that $f_\alpha(x)$ is the α-th integral of $f(x)$, with origin a. Then $f_\alpha(x)$ belongs to L^q in (a, b); and

$$\int_a^b |f_\alpha(x)|^q\,dx \leqq K\Big(\int_a^b |f(x)|^p\,dx\Big)^{\frac{q}{p}}, \tag{3.32}$$

where $K=K(p,\alpha)$ is a function of p and α only.

It is evident, first, that we may suppose without real loss of generality that $f \geqq 0$.

Next, we may suppose that $b=\infty$. For suppose that the theorem has been proved in this case, that f satisfies the conditions of the theorem, and that f^* is equal to f in (a, b) and to 0 outside this interval. Then f^* belongs to L^p in (a, ∞), and

$$\int_a^b f_\alpha^q\,dx = \int_a^b f_\alpha^{*q}\,dx \leqq \int_a^\infty f_\alpha^{*q}\,dx \leqq K\Big(\int_a^\infty f^{*p}\,dx\Big)^{\frac{q}{p}} = K\Big(\int_a^b f^{p}\,dx\Big)^{\frac{q}{p}}.$$

Finally, we may obviously suppose that $a=0$.

In order that

$$\int_0^\infty f_\alpha^q\,dx \leqq K\Big(\int_0^\infty f^p\,dx\Big)^{\frac{q}{p}},$$

it is necessary and sufficient that

$$\int_0^\infty f_\alpha(y)\, g(y)\, dy \leqq K\left(\int_0^\infty f^p(x)\, dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^{q'}(y)\, dy\right)^{\frac{1}{q'}}$$

for every $g(y)$ of $L^{q'}$.[14]) But the left hand side here is

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty g(y)\, dy \int_0^y f(x)(y-x)^{\alpha-1} dx \leqq \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{|x-y|^\lambda}\, dx\, dy$$

$$\leqq K\left(\int_0^\infty f^p(x)\, dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^{q'}(y)\, dy\right)^{\frac{1}{q'}},$$

by Theorem 3, with

$$r = p > 1, \qquad s = q' > 1, \qquad \lambda = 2 - \frac{1}{p} - \frac{1}{q'} = 1 - \frac{1}{p} + \frac{1}{q} = 1 - \alpha,$$

$$\frac{1}{r} + \frac{1}{s} = 1 + \alpha > 1.$$

3.4. Theorem 5. *The result of Theorem 4 is also true when* $a = 0$, $b = 2\pi$, $f(x)$ *has period* 2π *and mean value* 0, *and the origin of integration is* $-\infty$.

We write

$$(3.41) \qquad f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x f(t)(x-t)^{\alpha-1} dt = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{-2\pi} + \frac{1}{\Gamma(\alpha)} \int_{-2\pi}^x$$

$$= \varphi(x) + \psi(x).$$

By Theorem 4, $\psi(x)$ belongs to L^q and

$$(3.42) \qquad \int_0^{2\pi} |\psi|^q dx \leqq \int_{-2\pi}^{2\pi} |\psi|^q dx \leqq K\left(\int_{-2\pi}^{2\pi} |f|^p dx\right)^{\frac{q}{p}} \leqq K\left(\int_0^{2\pi} |f|^p dx\right)^{\frac{q}{p}}.$$

On the other hand $\varphi(x)$ is continous in $(0, 2\pi)$, the integral which defines it being uniformly convergent; and, by the second mean value theorem, there is a $\xi < -2\pi$ such that

$$(3.43) \qquad |\varphi(x)| = \left| \frac{(x+2\pi)^{\alpha-1}}{\Gamma(\alpha)} \int_\xi^{-2\pi} f(t)\, dt \right|$$

$$\leqq \frac{K}{2\pi} \int_0^{2\pi} |f(t)|\, dt \leqq K\left(\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^p dt\right)^{\frac{1}{p}}.$$

The conclusion follows from (3.41), (3.42) and (3.43).

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[14]) F. Riesz, **11**. The K here is not the same K (actually it is the q-th root of the earlier K).

It should be noticed that the conclusion of Theorem 4 is not necessarily true when the origin of integration is $c < a$ (the reason being naturally that f does not necessarily belong to L^p in (c, a)). For example, if

$$f(x) = (a-x)^{-\beta} \quad (x < a), \qquad f(x) = 0 \quad (x \geqq a),$$

and $c < a$, $0 < \alpha < \beta < 1$, then $f_\alpha(x)$ differs from

$$\frac{1}{\Gamma(\alpha)} \int_{-\infty}^{a} (a-t)^{-\beta} (x-t)^{\alpha-1} dt$$

by a continuous function. This is a multiple of $(x-a)^{\alpha-\beta}$, and does not belong to L^q in (a, b) if $q(\beta-\alpha) \geqq 1$, although f is zero in (a, b).

3. 5. Before going further we show by examples that the inequalities which occur as conditions in Theorems 3 and 4 cannot be widened.

(i) Theorem 3 is not true when $p > 1$, $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. In fact the integral (3.13) is then divergent if $f = g = 1$ in the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ and $f = g = 0$ outside. What takes the place of Theorem 3 in this case is M. Riesz's theorem that

$$\iint \frac{f(x)\, g(y)}{x-y}\, dx\, dy$$

is convergent as a "principal value", *i. e.* as the limit of an integral over $D(\varepsilon)$.[15]

(ii) Theorem 3 is not true when $p = 1$, $q > 1$. For then it would imply that

$$\int_0^{\infty} f(x)\, dx \int_0^{x} \frac{g(y)}{(x-y)^{1-\frac{1}{q}}}\, dy$$

converges whenever f belongs to L and g to L^q. This would imply that

$$\int_0^{x} \frac{g(y)}{(x-y)^{1-\frac{1}{q}}}\, dy$$

is, for every g of L^q, a bounded function of x; and this is false, since

$$\int_0^{x} \left(\frac{1}{(x-y)^{1-\frac{1}{q}}} \right)^{\frac{q}{q-1}} dy = \int_0^{x} \frac{dy}{x-y}$$

is divergent.

On the other hand the theorem is true (and trivial) when $p = 1$, $q = 1$.

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15) See M. Riesz, **12**, **13**; Titchmarsh, **16**.

(iii) Theorem 4 is not true when $p = 1$, for any α between 0 and 1. Take

$$f(x) = x^{-1}\left(\log\frac{1}{x}\right)^{-\beta}$$

where $\beta > 1$, for $0 < x \leqq \frac{1}{2}$, and $f(x) = 0$ otherwise. Then f belongs to L. For small positive x, we have

$$f_\alpha(x) > K\int_0^x \frac{(x-t)^{\alpha-1}}{t\left(\log\frac{1}{t}\right)^\beta}\,dt > K x^{\alpha-1}\left(\log\frac{1}{x}\right)^{1-\beta}.$$

Hence f_α does not belong to $L^{\frac{1}{1-\alpha}}$ unless

$$\frac{1-\beta}{1-\alpha} < -1, \qquad \beta > 2-\alpha,$$

which is more than $\beta > 1$.

It is true that, when $p = 1$, f_α *belongs to*

$$L^{\frac{p}{1-p\alpha}-\varepsilon}$$

for every positive ε. This theorem, which lies much less deep than Theorem 4, is in fact a simple corollary of an inequality of Young[16]).

(iv) A theorem which suggests itself naturally, as a limiting case of Theorem 4 in which $\alpha = \frac{1}{p}$, $q = \infty$, is this: *if* $p > 1$ *and* $\alpha = \frac{1}{p}$, *then* $f_\alpha(x)$ *is bounded.* This proposition is false. If it were true, it would follow that

$$\int_0^a f_\alpha(x)\,g(x)\,dx = \frac{1}{\Gamma(\alpha)}\int_0^a g(x)\,dx\int_0^x f(t)\,(x-t)^{\alpha-1}\,dt$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^a f(t)\,dt\int_t^a g(x)\,(x-t)^{\alpha-1}\,dx$$

exists whenever f belongs to L^p and g to L; and so that

$$\int_t^a g(x)\,(x-t)^{\alpha-1}\,dx$$

belongs to $L^{p'}$, *i. e.* to $L^{\frac{1}{1-\alpha}}$, whenever g belongs to L. This assertion is plainly equivalent[17]) to the assertion of Theorem 4 in the case $p = 1$, when we have seen that it is false.

[16]) For a proof see Hardy, 2.

[17]) Allowing for the reversal of the "direction of integration".

4. Generalisations of the theorems of § 3.

4.1. Theorem 6. *If $f(x)$ and $g(y)$ satisfy the conditions of Theorem 3, and*

$$r>1, \quad s>1, \quad \frac{1}{r}+\frac{1}{s}\geqq 1, \tag{4.11}$$

$$h<1-\frac{1}{r}, \quad k<1-\frac{1}{s}, \quad h+k\geqq 0, \tag{4.12}$$

$$h+k>0 \quad \textit{if} \quad \frac{1}{r}+\frac{1}{s}=1, \tag{4.13}$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x^h y^k\,|x-y|^{\lambda-h-k}}\,dx\,dy<K F^{\frac{1}{r}} G^{\frac{1}{s}}, \tag{4.14}$$

$K=K(r,s,h,k)$ being now a function of r, s, h, k only.

(i) Suppose first that $\frac{1}{r}+\frac{1}{s}>1$, so that $0<\lambda<1$. Then Theorem 5 is a corollary of Theorem 3. For we may divide the quarter-plane of integration into an angle A in which $\frac{1}{2}x\leqq y\leqq 2x$ and two residual angles A'. In A', $\frac{x}{|x-y|}$ and $\frac{y}{|x-y|}$ lie between positive bounds, and in A

$$x^h y^k\geqq K\,|x-y|^{h+k}.$$

Thus the integral (4.14) does not exceed a constant multiple of the integral (3.13).

(ii) We may therefore suppose that $\frac{1}{r}+\frac{1}{s}=1$, $\lambda=1$. If

$$\Omega(x,y)=x^{-h}y^{-k}\,|x-y|^{h+k-1}$$

then the integrals

$$\int_0^\infty \Omega(u,1)\,u^{-\frac{1}{r}}\,du=\int_0^\infty u^{-h-\frac{1}{r}}\,|u-1|^{h+k-1}\,du,$$

$$\int_0^\infty \Omega(1,v)\,v^{-\frac{1}{s}}\,dv=\int_0^\infty v^{-k-\frac{1}{s}}\,|1-v|^{h+k-1}\,dv$$

converge to the common value

$$J=\frac{\pi\,\Gamma(h+k)}{\Gamma\left(1+h-\frac{1}{r}\right)\Gamma\left(1+k-\frac{1}{s}\right)}\left(\frac{1}{\sin\left(h+\frac{1}{r}\right)\pi}+\frac{1}{\sin\left(k+\frac{1}{s}\right)\pi}\right).$$ [18]

[18]) This expression must be replaced by its limiting value if $h+\frac{1}{r}$ or $k+\frac{1}{s}$ is zero or a negative integer.

And

$$\int_0^\infty\!\!\int_0^\infty f(x)\,g(y)\,\Omega(x,y)\,dx\,dy = \int_0^\infty\!\!\int_0^\infty \left(f(x)\left(\frac{x}{y}\right)^{\frac{1}{rs}}\Omega^{\frac{1}{r}}\cdot g(y)\left(\frac{y}{x}\right)^{\frac{1}{rs}}\Omega^{\frac{1}{s}}\right)dx\,dy$$

$$\leqq \left[\int_0^\infty f^r\,dx\int_0^\infty \left(\frac{x}{y}\right)^{\frac{1}{s}}\Omega\,dy\right]^{\frac{1}{r}}\left[\int_0^\infty g^s\,dy\int_0^\infty \left(\frac{y}{x}\right)^{\frac{1}{r}}\Omega\,dx\right]^{\frac{1}{s}}$$

$$= J\left(\int_0^\infty f^r\,dx\right)^{\frac{1}{r}}\left(\int_0^\infty g^s\,dy\right)^{\frac{1}{s}};$$

which completes the proof of the theorem.

4.2. When $\frac{1}{r}+\frac{1}{s}=1$, the constant J is the "best possible" constant. We do not delay to prove this, but we show that the inequalities (4.12) and (4.13) for h and k are best possible inequalities.

In the first place, it is essential that $h<1-\frac{1}{r}$ and $k<1-\frac{1}{s}$. If $h=1-\frac{1}{r}$, for example, there are functions of L^r for which

$$\int_0 x^{-h}f(x)\,dx$$

diverges, and then the double integral is certainly divergent.

Next, it is essential that $\lambda-h-k<1$ $\left(\text{and so } h+k>0 \text{ if } \frac{1}{r}+\frac{1}{s}=1\right)$; for otherwise the integral may diverge (along the line $x=y$) even for continuous f and g.

It is less obvious that $h+k$ must not be negative when $\frac{1}{r}+\frac{1}{s}>1$.[19]) Suppose however that

$$f(x)=(\log x)^{-\frac{1}{r}-\delta}\qquad (\delta>0,\ 2^m\leqq x\leqq 2^m+1,\ m=1,2,\dots)$$

and $f(x)=0$ otherwise, and that $g(y)$ is defined in a similar manner with $-\frac{1}{s}-\delta$; so that $f(x)\,g(y)=0$ outside the strip $|x-y|\leqq 2$. Then

$$\int_0^\infty f^r\,dx<K\sum_1^\infty m^{-1-r\delta},$$

so that F is convergent. Similarly G is convergent. The double integral, on the other hand, is plainly greater than

$$K\sum_1^\infty (2^m)^{-h-k}\,m^{-\frac{1}{r}-\frac{1}{s}-2\delta},$$

which is divergent if $h+k<0$.

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[19]) It might be supposed that $h+k>\lambda-1$ would be sufficient.

4.3. Theorem 7. *Suppose that*

$$p>1, \quad l>\frac{1}{p}-1, \tag{4.31}$$

$$0 \leqq m-l \leqq \alpha < \operatorname{Min}\left(m-l+\frac{1}{p},\ -l+\frac{1}{p}\right), \tag{4.32}$$

$$m-l>0 \quad \textit{if} \quad \alpha=m-l, \tag{4.33}$$

$$q=\frac{p}{1-p(\alpha+l-m)}, \tag{4.34}$$

that $x^{-l} f(x)$ *belongs to* L^p *in* $(0, \infty)$, *and that* $f_\alpha(x)$ *is the* α*-th integral of* $f(x)$ *with origin* 0. *Then* $x^{-m} f_\alpha(x)$ *belongs to* L^q, *and*

$$\int_0^\infty (x^{-m} f_\alpha(x))^q\, dx \leqq K\left(\int_0^\infty (x^{-l} f(x))^p\, dx\right)^{\frac{q}{p}}, \tag{4.35}$$

where $K=K(p, \alpha, l, m)$.

It follows from Theorem 6 that, if f belongs to L^r and g to L^s, then

$$\int_0^\infty f(x)\, \varphi(x)\, dx,$$

where

$$\varphi(x)=x^{-h} \int_0^x g(y)\, y^{-k}(x-y)^{h+k-\lambda}\, dy,$$

is convergent (and satisfies the appropriate inequality). If now we replace $y^{-k} g$ by G, and argue as in the proof of Theorem 4, we obtain

$$\int_0^\infty (x^{-h} G_{h+k-\lambda+1})^{r'}\, dx \leqq K\left(\int_0^\infty (x^k G)^s\, dx\right)^{\frac{r'}{s}}.$$

We now write

$$G=f, \quad h=m, \quad k=-l, \quad s=p, \quad \alpha=h+k-\lambda+1,$$

so that

$$r'=\frac{p}{1-p(\alpha+l-m)}=q,$$

and we obtain (4.35).

It remains to verify that the conditions of Theorem 7 are equivalent to those of Theorem 6. Of the former

$$p>1, \quad l>\frac{1}{p}-1, \quad 0 \leqq m-l, \quad m-l \leqq \alpha$$

are equivalent to

$$s>1, \quad k<1-\frac{1}{s}, \quad h+k \geqq 0, \quad \frac{1}{r}+\frac{1}{s} \geqq 1$$

(the glosses on the last pairs of inequalities also corresponding). Finally $\alpha < -l + \frac{1}{p}$ corresponds to $h < 1 - \frac{1}{r}$, and $\alpha < m - l + \frac{1}{p}$ corresponds to $r > 1$, since it is $1 - \frac{1}{r'} < 1$ or $\frac{1}{r} < 1$ and the preceding inequalities show that r is positive.

4.4. Theorem 4 is the special case $l = m = 0$ of Theorem 7. There are other interesting special cases, but two remarks are called for before we state them.

The inequalities of Theorem 7 are, essentially, best possible inequalities like those of Theorem 6. The "gloss" (4.33) on the inequalities, however, now excludes a case, viz. $\alpha = 0$, $l = m$, $p = q$, in which the result of the theorem is true but trivial (the two integrals in the inequality being identical). The gloss corresponds to the fact that the definition of f_α fails when $\alpha = 0$, though

$$\lim_{\alpha \to 0} f_\alpha(x) = \lim_{\alpha \to 0} \frac{1}{\Gamma(\alpha)} \int_0^x f(t)(x-t)^{\alpha-1}\, dt = f(x)$$

for almost all x. In our statement of the particular cases we therefore suppress the gloss.

Next we observe that some of our specialisations introduce the combination

$$\Gamma(\alpha) x^{-\alpha} f_\alpha(x) = \frac{1}{x} \int_0^x \left(1 - \frac{t}{x}\right)^{\alpha-1} f(t)\, dt.$$

This function is plainly, for fixed $f(t)$ and x, a decreasing function of α, and we are thus sometimes able to extend the scope of a theorem, by the removal of an upper bound for α, in spite of the fact that the argument by which it was proved depends upon what were, in their original form, "best possible" inequalities.

4.5. Taking $l = 0$ in Theorem 7, we obtain

Theorem 8. *If*

$$p > 1, \quad 0 \leqq m \leqq \alpha < \frac{1}{p}, \quad q = \frac{p}{1 - p(\alpha - m)},$$

then

$$\int_0^\infty (x^{-m} f_\alpha)^q\, dx \leqq K \left(\int_0^\infty f^p\, dx \right)^{\frac{q}{p}}.$$

Restating this in a slightly different notation, and using the last remark of § 4.4, we obtain

Theorem 9. *If*

$$p > 1, \qquad 0 \leqq \alpha < \frac{1}{p}, \qquad p \leqq q \leqq \frac{p}{1-p\alpha},$$

then

$$\int_0^\infty x^{-\frac{p-q+pq\alpha}{p}} f_\alpha^q \, dx \leqq K \left(\int_0^\infty f^p \, dx \right)^{\frac{q}{p}}.$$

The result is still true for $\alpha \geqq \frac{1}{p}$, *when the second inequality for* q *may be omitted.*

If we put $m = l + \alpha$ in Theorem 7 and (after the last remark of § 4.4) drop the upper bound for α, we obtain

Theorem 10. *If*

$$p > 1, \qquad l > \frac{1}{p} - 1, \qquad \alpha \geqq 0,$$

then

$$\int_0^\infty (x^{-l-\alpha} f_\alpha)^p \, dx \leqq K \int_0^\infty (x^{-l} f)^p \, dx.$$

If we put $l = 0$ in Theorem 10, or $m = \alpha$ in Theorem 8, we obtain

Theorem 11. *If* $p > 1$, $\alpha \geqq 0$, *then*

$$\int_0^\infty (x^{-\alpha} f_\alpha)^p \, dx \leqq K \int_0^\infty f^p \, dx.$$

4.6. We shall not give a formal proof that all the inequalities which occur as conditions in Theorems 7—11 are best possible inequalities. The following examples will probably be sufficient to convince the reader that this is so.

(i) The result of Theorem 9 is (sometimes) false if $q < p$ or

$$q > \frac{p}{1-p\alpha}.$$

(a) Suppose that $\varrho > 0$, that

$$f(x) = x^{-\frac{1}{p}} \left(\log \frac{1}{x} \right)^{-\varrho}$$

if $0 < x \leqq \frac{1}{2}$, and that $f(x) = 0$ otherwise. Then

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{-\frac{1}{p}} \left(\log \frac{1}{t} \right)^{-\varrho} dt > K x^{\alpha - \frac{1}{p}} \left(\log \frac{1}{x} \right)^{-\varrho}$$

for small x. The right hand side of (4.35) is convergent if $p\varrho > 1$, the left hand side only if $q\varrho > 1$. Hence the result cannot be true unless $q \geqq p$.

(b) Suppose that $f(x) = \frac{1}{\log x}$ if $2^m \leqq x \leqq 2^m + 1$, $m = 1, 2, \ldots$, and that $f(x) = 0$ otherwise. Then the right hand side of (4.35) is convergent, since $\sum m^{-p}$ is convergent. If now $2^m + \frac{1}{2} \leqq x \leqq 2^m + 1$, we have

$$f_\alpha(x) > K \int_{2^m}^{2^m + \frac{1}{2}} f(t)\,dt > \frac{K}{m}.$$

Thus the convergence of the left hand side implies the convergence of

$$\sum m^{-q} 2^{-m\frac{p-q+pq\alpha}{p}},$$

and this series is divergent if $q > \frac{p}{1-p\alpha}$, so that the result is false in this case also.

(ii) The result of Theorem 10 is (sometimes) false if

$$l \leqq \frac{1}{p} - 1.$$

Suppose, for example, that $\alpha = 1$. Then f_1 is (for any positive f) an increasing function of x, and the integral on the left hand side of (4.35) is divergent if $l \leqq \frac{1}{p} - 1$.

(iii) The result of Theorem 11 is (sometimes) false if $p = 1$. This may be shown by the example of § 3.5 (iii), in which

$$f(x) = x^{-1}\left(\log \frac{1}{x}\right)^{-\beta} \qquad \left(0 < x \leqq \frac{1}{2}\right),$$

where $\beta > 1$. Here

$$f_\alpha(x) > K x^{\alpha-1}\left(\log \frac{1}{x}\right)^{1-\beta}$$

for small x. The right hand side of (4.35) is convergent if $\beta > 1$, the left hand side only if $\beta > 2$.

5. Lipschitz conditions.

5.1. Theorem 12. *If*

(5.11) $$p > 1, \quad \frac{1}{p} < \alpha < \frac{1}{p} + 1$$

or

(5.111) $$p = 1, \quad 1 \leqq \alpha < 2,$$

f belongs to L^p in (a, b), where $a < b < \infty$, and f_α is the α-th integral

of f, *with origin* a, *then* f_α *belongs to* Lip.* $\left(\alpha-\frac{1}{p}\right)$ *in* (a, b), *i. e.*

$$f_\alpha(x)-f_\alpha(x-h)=o\left(h^{\alpha-\frac{1}{p}}\right)$$

when $a \leqq x-h<x \leqq b$ *and* $h \rightarrow 0$, *uniformly in* x.

We consider first the case $p>1$. We may take $a=0$; and we suppose first that $x \geqq 2h$. We have

$$\begin{aligned}
(5.12)\qquad \Gamma(\alpha)\Delta &= \Gamma(\alpha)\Delta_h f_\alpha=\Gamma(\alpha)(f_\alpha(x)-f_\alpha(x-h)) \\
&= \int_0^x f(t)(x-t)^{\alpha-1}\,dt-\int_0^{x-h} f(t)(x-h-t)^{\alpha-1}\,dt \\
&= \int_0^x f(x-u)\,u^{\alpha-1}\,du-\int_h^x f(x-u)(u-h)^{\alpha-1}\,du \\
&= \int_0^h f(x-u)\,u^{\alpha-1}\,du+\int_h^{2h} f(x-u)\left(u^{\alpha-1}-(u-h)^{\alpha-1}\right)du \\
&\quad+\int_{2h}^{\delta} f(x-u)\left(u^{\alpha-1}-(u-h)^{\alpha-1}\right)du+\int_{\delta}^{x} f(x-u)\left(u^{\alpha-1}-(u-h)^{\alpha-1}\right)du \\
&= I_1+I_2+I_3+I_4 .
\end{aligned}$$

Here δ is any positive number between $2h$ and x.

Now

$$(5.13)\qquad |I_1| \leqq \left(\int_0^h |f(x-u)|^p\,du\right)^{\frac{1}{p}}\left(\int_0^h u^{\frac{(\alpha-1)p}{p-1}}\,du\right)^{\frac{p-1}{p}} = o(1)\,O\left(h^{\alpha-\frac{1}{p}}\right)=o\left(h^{\alpha-\frac{1}{p}}\right),$$

since

$$\frac{(\alpha-1)p}{p-1}>-1, \qquad \left(\frac{(\alpha-1)p}{p-1}+1\right)\frac{p-1}{p}=\alpha-\frac{1}{p};$$

and the same argument may be applied to

$$\int_h^{2h} f(x-u)\,u^{\alpha-1}\,du, \qquad \int_h^{2h} f(x-u)(u-h)^{\alpha-1}\,du,$$

so that

$$(5.14)\qquad I_2=o\left(h^{\alpha-\frac{1}{p}}\right).$$

Next, in I_3 we have

$$|u^{\alpha-1}-(u-h)^{\alpha-1}|\leqq h\,|1-\alpha|\,(u-h)^{\alpha-2}$$

and so

$$(5.15)\qquad |I_3|\leqq h\,|1-\alpha|\int_{2h}^{\delta}|f(x-u)|\,(u-h)^{\alpha-2}\,du$$

$$\leqq h\,|1-\alpha|\left(\int_{2h}^{\delta}|f(x-u)|^p\,du\right)^{\frac{1}{p}}\left(\int_{2h}^{\delta}(u-h)^{\frac{(\alpha-2)p}{p-1}}\,du\right)^{\frac{p-1}{p}}$$

$$\leqq|1-\alpha|\left(\frac{p-1}{p+1-\alpha p}\right)^{\frac{p-1}{p}}h^{\alpha-\frac{1}{p}}\left(\int_{2h}^{\delta}|f(x-u)|^p\,du\right)^{\frac{1}{p}}\leqq\varepsilon_\delta\,h^{\alpha-\frac{1}{p}},$$

where ε_δ is independent of x and h and tends to zero with δ.

Finally

$$(5.16)\qquad |I_4|\leqq h\,|1-\alpha|\int_{\delta}^{x}|f(x-u)|\,(u-h)^{\alpha-2}\,du=O(h)=o\left(h^{\alpha-\frac{1}{p}}\right)$$

uniformly in x, for any fixed δ. It is plain that the conclusion follows from (5.12)—(5.16), by choice first of δ and then of h.

When $h\leqq x\leqq 2h$, the proof is simpler. In this case the three integrals I_2, I_3, I_4 are replaced by

$$I_2^*=\int_{h}^{x}f(x-u)\,(u^{\alpha-1}-(u-h)^{\alpha-1})\,du,$$

which divides into two parts to each of which our first argument applies.

If $p=1$, we have $\alpha-1\geqq 0$, and so

$$|I_1|\leqq h^{\alpha-1}\int_{0}^{h}|f(x-u)|\,du=o(h^{\alpha-1});$$

$I_2=o(h^{\alpha-1})$, similarly;

$$|I_3|\leqq h(\alpha-1)\cdot h^{\alpha-2}\int_{2h}^{\delta}|f(x-u)|\,du\leqq\varepsilon_\delta\,h^{\alpha-1}$$

(since $\alpha<2$); and

$$I_4=O(h)=o(h^{\alpha-1})$$

as before. In this case the theorem is true (and trivial) for $\alpha=1$.

5.2. The result of Theorem 12 is not true, in the cases $p > 1$, $\alpha = \frac{1}{p}$ or $\alpha = \frac{1}{p} + 1$, even if it is stated in the less strong form that f_α belongs to Lip. $\left(\alpha - \frac{1}{p}\right)$. In the first place, when $\alpha = \frac{1}{p}$, we have seen already (§ 3.5) that f_α is not necessarily bounded. If f_α belonged to Lip. 1, when $\alpha = \frac{1}{p} + 1$, f_α would have almost everywhere a derivative bounded in the set of points in which it exists, and $f_{\alpha-1}$ would be equal, except at a null set, to a bounded function. Our discussion of the case $\alpha = \frac{1}{p}$ shows that this is not necessarily true.

Theorem 13. *The result of Theorem* 12 *is still true in the case considered in Theorem* 5.

For (in the notation of Theorem 5) $\psi(x)$ belongs to Lip.* $\left(\alpha - \frac{1}{p}\right)$ in $(-2\pi, 2\pi)$, and *a fortiori* in $(0, 2\pi)$, while $\varphi(x)$ has a continuous derivative.

Theorem 14. *Suppose that*

$$k \geqq 0, \quad \alpha > 0, \quad k + \alpha < 1,$$

and that $f(x)$ *belongs to* Lip. k *in* (a, b), *it being understood that this hypothesis includes*

$$f(x) = O((x-a)^k)$$

for small $x - a$. *Then* $f_\alpha(x)$, *defined with origin* a, *belongs to* Lip. $(k+\alpha)$ *in* (a, b), *and*

$$f_\alpha(x) = O((x-a)^{k+\alpha})$$

for small $x - a$.

The gloss on the hypothesis means simply that $f(a) = 0$. It is clear that such a gloss is necessary. Thus $f = 1$ belongs to Lip. k in (a, b), while f_α is a multiple of $(x-a)^\alpha$.

We take $a = 0$. Then $f(0) = 0$ and

$$f(x) - f(x-h) = O(h^k)$$

uniformly for $0 \leqq x - h < x \leqq b$; and we have to show that

$$f_\alpha(x) - f_\alpha(x-h) = O(h^{k+\alpha})$$

uniformly for $0 \leqq x - h < x \leqq b$, and that $f_\alpha(h) = O(h^{k+\alpha})$.

The last result is immediate; for the first, we have

$$\Gamma(\alpha)\Delta = \int_0^x f(x-u)\,u^{\alpha-1}\,du \;-\int_h^x f(x-u)(u-h)^{\alpha-1}\,du$$

$$= \frac{f(x)}{\alpha}(x^\alpha - (x-h)^\alpha) - \int_0^x (f(x)-f(x-u))\,u^{\alpha-1}\,du$$

$$+\int_h^x (f(x)-f(x-u))(u-h)^{\alpha-1}\,du$$

$$= \frac{f(x)}{\alpha}(x^\alpha - (x-h)^\alpha) - \int_0^h (f(x)-f(x-u))\,u^{\alpha-1}\,du$$

$$-\int_h^x (f(x)-f(x-u))(u^{\alpha-1}-(u-h)^{\alpha-1})\,du$$

$$= J_1 + J_2 + J_3,$$

say. In the first place

$$J_1 = O(x^k \cdot h \cdot x^{\alpha-1}) = O(h x^{k+\alpha-1}) = O(h^{k+\alpha})$$

if $x > 2h$, and $J_1 = O(x^{k+\alpha}) = O(h^{k+\alpha})$ if $h \leqq x \leqq 2h$. Secondly

$$J_2 = O\Bigl(\int_0^h u^{k+\alpha-1}\,du\Bigr) = O(h^{k+\alpha}).$$

Finally

$$J_3 = O\Bigl(\int_h^x u^k((u-h)^{\alpha-1} - u^{\alpha-1})\,du\Bigr)$$

$$= O\Bigl(h^{k+\alpha}\int_1^{\frac{x}{h}} w^k((w-1)^{\alpha-1} - w^{\alpha-1})\,dw\Bigr) = O(h^{k+\alpha}),$$

since the integral is convergent up to ∞.

5.4. Theorem 15. *If $f(x)$ belongs to* Lip.* k, *it being understood that this includes $f(x) = o((x-a)^k)$ for small $x-a$, then $f_\alpha(x)$ belongs to* Lip.* $(k+\alpha)$, *with the corresponding gloss.*

In this case we obtain

$$J_1 = o(h^{k+\alpha}), \qquad J_2 = o(h^{k+\alpha}),$$

by the same argument as before. We write

$$J_3 = -\int_h^\delta - \int_\delta^x = J_3' + J_3''.$$

Here $|J_3'| < \varepsilon_\delta h^{k+\alpha}$, where ε_δ is independent of x and h and tends to zero with δ, and J_3'' is $O(h) = o(h^{k+\alpha})$. Hence

$$J_3 = o(h^{k+\alpha}).$$

It is plain that we have proved incidentally

Theorem 16. *If $f(x)$ is integrable and satisfies a Lipschitz condition of order k at a point, then $f_\alpha(x)$ satisfies one of order $k+\alpha$ at the same point, the conditions being of the same (O or o) type.*

Theorem 17. *If $f(x)$ satisfies the conditions of Theorem* 14 *or Theorem* 15, *except for the gloss on the Lipschitz condition, then $f_\alpha(x)$ belongs to* Lip. $(k+\alpha)$, *or* Lip.* $(k+\alpha)$, *in any interval (a', b') interior to (a, b).*

For

$$\Gamma(\alpha) f_\alpha(x) = \int_a^x (f(t) - f(a))(x-t)^{\alpha-1}\, dt + \frac{f(a)}{\alpha}(x-a)^\alpha .$$

Theorem 18. *If $f(x)$ is periodic, has mean value* 0, *and belongs to* Lip. k *or* Lip.* k, *then $f_\alpha(x)$ belongs to* Lip. $(k+\alpha)$ *or* Lip.* $(k+\alpha)$.

We have

$$\Gamma(\alpha) f_\alpha(x) = \int_{-\infty}^{x} f(t)(x-t)^{\alpha-1}\, dt = \int_{-\infty}^{-c} + \int_{-c}^{x},$$

where $c > 0$. The first term has a continuous derivative, while the second, by Theorem 17, belongs to Lip. $(k+\alpha)$ or Lip.* $(k+\alpha)$ in $(0, 2\pi)$.

5.5. It is to be observed that none of these theorems are true when $k > 0$, $k + \alpha = 1$. Thus Weierstrass's function

$$f(x) = \sum a^{-nk} \cos a^n x \qquad (a > 1,\ 0 < k < 1)$$

satisfies a Lipschitz condition of order k (of O type) uniformly in any interval[20]). But $f_{1-k}(x)$, which is a linear combination of

$$\sum a^{-n} \cos a^n x, \qquad \sum a^{-n} \sin a^n x,$$

does not satisfy one of order 1, since it is nowhere differentiable.

When $k = 0$, the hypothesis asserts simply that $f(x)$ is bounded (or continuous). This case of the theorem is due to Weyl[21]); and in this case the result is true, and trivial, in O form, when $\alpha = 1$.

[20]) Hardy, **3**.
[21]) Weyl, **18**.

In what follows, in order to avoid complications concerning the origin of integration, glosses on Lipschitz conditions, and so forth, we confine ourselves to the case in which the origin is $-\infty$. We suppose then henceforth that $f(x)$ has period 2π, that

$$\int_0^{2\pi} f(x)\,dx = 0,$$

and that the origin of integration is $-\infty$, so that $f_\alpha(x)$ is periodic, and has also mean value 0.

We define $f^\beta(x)$, the β-th derivative of $f(x)$, for $0<\beta<1$, by

$$f^\beta(x) = \frac{d}{dx} f_{1-\beta}(x).$$

5.6. Theorem 19[22]. *If $0<\beta<k\leqq 1$ and $f(x)$ belongs to* Lip. k, *then $f^\beta(x)$ exists and is continuous; and $f(x)$ is the β-th integral of $f^\beta(x)$.*

We write

$$f_{1-\beta,\varepsilon}(x) = \frac{1}{\Gamma(1-\beta)} \int_{-\infty}^{x-\varepsilon} f(t)\,(x-t)^{-\beta}\,dt \tag{5.61}$$

for $\varepsilon \geqq 0$, so that $f_{1-\beta,0} = f_{1-\beta}$. Then, if $\varepsilon>0$,

$$\begin{aligned} \Gamma(1-\beta)\, f'_{1-\beta,\varepsilon}(x) &= \varepsilon^{-\beta} f(x-\varepsilon) - \beta \int_{-\infty}^{x-\varepsilon} f(t)\,(x-t)^{-\beta-1}\,dt \\ &= \beta \int_{-\infty}^{x-\varepsilon} (f(x)-f(t))\,(x-t)^{-\beta-1}\,dt - \varepsilon^{-\beta}(f(x)-f(x-\varepsilon)). \end{aligned} \tag{5.62}$$

But

$$f(x)-f(t) = O(|x-t|^k)$$

uniformly in any intervals of values of x and t, and $\beta+1-k<1<\beta+1$. It follows that, when $\varepsilon\to 0$, the right hand side of (5.62) tends uniformly to

$$g(x) = \beta \int_{-\infty}^{x} (f(x)-f(t))\,(x-t)^{-\beta-1}\,dt.$$

Also

$$\begin{aligned} f_{1-\beta}(x_1) - f_{1-\beta}(x_2) &= \lim_{\varepsilon\to 0} (f_{1-\beta,\varepsilon}(x_1) - f_{1-\beta,\varepsilon}(x_2)) \\ &= \lim_{\varepsilon\to 0} \int_{x_2}^{x_1} f'_{1-\beta,\varepsilon}(x)\,dx = \frac{1}{\Gamma(1-\beta)} \int_{x_2}^{x_1} g(x)\,dx, \end{aligned}$$

[22] Weyl, **18**. We have added the last clause.

so that $\Gamma(1-\beta)\, f_{1-\beta}(x)$ is the integral of $g(x)$, and

$$g(x)=\Gamma(1-\beta)\, f'_{1-\beta}(x)=\Gamma(1-\beta)\, f^{\beta}(x).$$

Finally, since the $(1-\beta)$-th integral of the β-th integral of an integrable function is the integral of that function, $\Gamma(1-\beta)\, f(x)$ and $g_\beta(x)$ (which are indeed continuous) have the same $(1-\beta)$-th integral, viz. $\Gamma(1-\beta)\, f_{1-\beta}(x)$, *i. e.*

$$\int_{-\infty}^{x} (\Gamma(1-\beta)\, f(t)-g_\beta(t))\,(x-t)^{-\beta}\, dt=0.$$

It follows that $\Gamma(1-\beta)\, f(t)$ and $g_\beta(t)$, being continuous, are identical.

5.7. Theorem 20. *If the conditions of Theorem* 19 *are satisfied, then $f^{\beta}(x)$ belongs to* Lip. $(k-\beta)$. *If also $f(x)$ belongs to* Lip.* k, *then $f^{\beta}(x)$ belongs to* Lip.* $(k-\beta)$.

We suppose first that hypothesis and conclusion are of the weaker type. We have, by Theorem 19,

$$g(x)=\Gamma(1-\beta)\, f^{\beta}(x)=\beta\int_{-\infty}^{x} (f(x)-f(t))\,(x-t)^{-\beta-1}\, dt,$$

(5.71)
$$\begin{aligned}
\frac{1}{\beta}\Delta g(x) &= \frac{1}{\beta}(g(x)-g(x-h)) \\
&= \int_{-\infty}^{x} (f(x)-f(t))(x-t)^{-\beta-1}\,dt-\int_{-\infty}^{x-h} (f(x-h)-f(t))\,(x-h-t)^{-\beta-1}\,dt \\
&= \int_{0}^{\infty} (f(x)-f(x-u))\,u^{-\beta-1}\,du-\int_{h}^{\infty} (f(x-h)-f(x-u))\,(u-h)^{-\beta-1}\,du \\
&= -\int_{h}^{\infty} (f(x-h)-f(x-u))\,((u-h)^{-\beta-1}-u^{-\beta-1})\,du \\
&\quad +\int_{h}^{\infty} (f(x)-f(x-h))\,u^{-\beta-1}\,du+\int_{0}^{h} (f(x)-f(x-u))\,u^{-\beta-1}\,du \\
&= J_1+J_2+J_3,
\end{aligned}$$

say. Here

$$\begin{aligned}
J_1 &= O\left(\int_{h}^{\infty} (u-h)^{k}\,((u-h)^{-\beta-1}-u^{-\beta-1})\,du\right) \\
&= O\left(h^{k-\beta}\int_{1}^{\infty} (w-1)^{k}\,((w-1)^{-\beta-1}-w^{-\beta-1})\,dw\right)=O(h^{k-\beta}),
\end{aligned}$$

since $\beta+1-k<1$ and $\beta+2-k>1$. Also

$$J_2=(f(x)-f(x-h))\frac{h^{-\beta}}{\beta}=O(h^{k-\beta});$$

and

$$J_3=O\left(\int_0^h u^{k-\beta-1}\,du\right)=O(h^{k-\beta});$$

which proves the theorem in O form.

When hypothesis and conclusion have the sharper form, the treatment of J_2 and J_3 is unaltered. As regards J_1, we write

$$J_1=-\int_h^\infty=-\int_h^{Ch}-\int_{Ch}^\infty=J_1'+J_1''.$$

Plainly J_1'' is less than a constant multiple of

$$\int_{Ch}^\infty (u-h)^k\left((u-h)^{-\beta-1}-u^{-\beta-1}\right)du$$

$$=h^{k-\beta}\int_C^\infty (w-1)^k\left((w-1)^{-\beta-1}-w^{-\beta-1}\right)dw<\varepsilon h^{k-\beta}$$

by choice of C; while, when C is fixed, $J_1'=o(h^{k-\beta})$. This proves the theorem in o form.

5.8. Theorems 19 and 20 are false for $k=\beta$. In fact Weierstrass's function $\sum a^{-nk}\cos a^n x$ has not at any point a derivative of order k.

6. Integrated Lipschitz conditions.

6.1. In what follows we make repeated use, explicit or tacit, of Minkowski's inequality

$$\left(\sum_i\left(\sum_j a_{i,j}\right)^p\right)^{\frac{1}{p}}\leqq\sum_j\left(\sum_i a_{i,j}^p\right)^{\frac{1}{p}},$$

where $p\geqq 1$, $a_{i,j}\geqq 0$, and the summations are finite or infinite; of the corresponding integral inequality

$$\left(\int dx\left(\int f(x,y)\,dy\right)^p\right)^{\frac{1}{p}}\leqq\int dy\left(\int f^p(x,y)\,dx\right)^{\frac{1}{p}};$$

or the corresponding mixed inequalities with $\sum$ and $\int$. We also use freely the ideas of "strong" and "weak" convergence, due to F. Riesz[22a]).

[22a]) F. Riesz, **11**.

Theorem 21. *If $p \geqq 1$, $0 < \alpha < 1$, and $f(x)$ belongs to L^p, then*

$$\int_a^b |f_\alpha(x) - f_\alpha(x-h)|^p\, dx = o(h^{p\alpha})$$

for any finite interval (a, b).

We have

$$g(x) = \Gamma(\alpha) f_\alpha(x) = \int_{-\infty}^{x} f(t)(x-t)^{\alpha-1}\, dt \tag{6.11}$$

$$= \int_{x-Ch}^{x} + \int_{x-Dh}^{x-Ch} + \int_{-\infty}^{x-Dh} = g_1 + g_2 + g_3,$$

where C and D are constants at our disposal. It is plainly sufficient to prove (i) that

$$\int_a^b |\Delta g_2|^p\, dx = o(h^{p\alpha})$$

for any fixed C and D, and (ii) that

$$\int_a^b |\Delta g_1|^p\, dx \leqq \varepsilon h^{p\alpha}, \qquad \int_a^b |\Delta g_3|^p \leqq \varepsilon h^{p\alpha} \tag{6.12}$$

for $h \leqq 1$, sufficiently small values of C, and sufficiently large values of D.

(i) In the first place, we have

$$\Delta g_2 = \int_{x-Dh}^{x-Ch} f(t)(x-t)^{\alpha-1}\, dt - \int_{x-Dh-h}^{x-Ch-h} f(t)(x-h-t)^{\alpha-1}\, dt$$

$$= \int_{x-Dh}^{x-Ch} (x-t)^{\alpha-1}(f(t) - f(t-h))\, dt = O\left(h^{\alpha-1}\int_{x-Dh}^{x-Ch} |f(t) - f(t-h)|\, dt\right),$$

$$|\Delta g_2|^p = O\left(h^{\alpha p-1}\int_{x-Dh}^{x-Ch} |f(t) - f(t-h)|^p\, dt\right),$$

$$\int_a^b |\Delta g_2|^p\, dx = O\left(h^{\alpha p-1}\int_a^b dx\int_{x-Dh}^{x-Ch} |f(t) - f(t-h)|^p\, dt\right).$$

The double integral does not exceed

$$\int_{a-Dh}^{b-Ch} |f(t) - f(t-h)|^p\, dt \int_{t+Ch}^{t+Dh} dx = O\left(h\int_{a-Dh}^{b-Ch} |f(t) - f(t-h)|^p\, dt\right) = o(h),$$

since

$$\int_{\alpha}^{\beta} |f(t) - f(t-h)|^p \, dt$$

tends to zero with h, for any f of L^p. Hence

$$\int_{a}^{b} |\Delta g_2|^p \, dx = o(h^{\alpha p})$$

for every fixed C and D.

(ii) In order to prove the first of the inequalities (6.12), it is plainly enough to prove that

$$(6.13) \qquad \int_{a}^{b} |g_1(x)|^p \, dx \leqq \varepsilon h^{p\alpha}, \qquad \int_{a}^{b} |g_1(x-h)|^p \, dx \leqq \varepsilon h^{p\alpha}$$

if C is sufficiently small. Now

$$g_1(x) = \int_{x-Ch}^{x} f(t)(x-t)^{\alpha-1} \, dt = \int_{0}^{Ch} f(x-u) u^{\alpha-1} \, du,$$

$$\left(\int_{a}^{b} |g_1(x)|^p \, dx\right)^{\frac{1}{p}} \leqq \left(\int_{a}^{b} dx \left(\int_{0}^{Ch} |f(x-u)| \, u^{\alpha-1} \, du\right)^p\right)^{\frac{1}{p}}$$

$$\leqq \int_{0}^{Ch} u^{\alpha-1} \, du \left(\int_{a}^{b} |f(x-u)|^p \, dx\right)^{\frac{1}{p}} \leqq \frac{(Ch)^\alpha}{\alpha} \left(\int_{a-Ch}^{b} |f(t)|^p \, dt\right)^{\frac{1}{p}} \leqq \varepsilon^{\frac{1}{p}} h^\alpha,$$

if C is sufficiently small. Similarly

$$g_1(x-h) = \int_{x-Ch-h}^{x-h} f(t)(x-h-t)^{\alpha-1} \, dt = \int_{0}^{Ch} f(x-h-u) u^{\alpha-1} \, du,$$

and the second of the inequalities (6.13) may be proved in the same manner.

(iii) Finally we have

$$g_3(x) = \int_{-\infty}^{x-Dh} f(t)(x-t)^{\alpha-1} \, dt = \int_{Dh}^{\infty} f(x-u) u^{\alpha-1} \, du,$$

$$g_3(x-h) = \int_{-\infty}^{x-Dh-h} f(t)(x-h-t)^{\alpha-1} \, dt = \int_{Dh+h}^{\infty} f(x-u)(u-h)^{\alpha-1} \, du,$$

$$\varDelta g_3 = g_3(x) - g_3(x-h) = -\int_{Dh+h}^{\infty} f(x-u)\left((u-h)^{\alpha-1} - u^{\alpha-1}\right)du$$
$$+\int_{Dh}^{Dh+h} f(x-u)\,u^{\alpha-1}\,du = \psi + \chi,$$

say. It is enough to prove that

$$\left(\int_a^b |\psi|^p\,dx\right)^{\frac{1}{p}} \leqq \varepsilon^{\frac{1}{p}}\,h^{\alpha}, \qquad \left(\int_a^b |\chi|^p\,dx\right)^{\frac{1}{p}} \leqq \varepsilon^{\frac{1}{p}}\,h^{\alpha}$$

if D is sufficiently large.

Now

$$\left(\int_a^b |\psi|^p\,dx\right)^{\frac{1}{p}} \leqq \left(\int_a^b dx\left(\int_{Dh+h}^{\infty} |f(x-u)|\left((u-h)^{\alpha-1} - u^{\alpha-1}\right)du\right)^p\right)^{\frac{1}{p}}$$
$$\leqq \int_{Dh+h}^{\infty}\left(\left((u-h)^{\alpha-1} - u^{\alpha-1}\right)\left(\int_a^b |f(x-u)|^p\,dx\right)^{\frac{1}{p}}\right)du,$$

which is less than a constant multiple of

$$\int_{Dh+h}^{\infty}\left((u-h)^{\alpha-1} - u^{\alpha-1}\right)du = h^{\alpha}\int_{D+1}^{\infty}\left((w-1)^{\alpha-1} - w^{\alpha-1}\right)dw < \varepsilon^{\frac{1}{p}}\,h^{\alpha}$$

if D is sufficiently large.

On the other hand

$$\int_a^b |\chi|^p\,dx \leqq \int_a^b dx\left(\int_{Dh}^{Dh+h} |f(x-u)|\,u^{\alpha-1}\,du\right)^p$$

$$\leqq (Dh)^{p\alpha-p}\int_a^b dx\left(\int_{Dh}^{Dh+h} |f(x-u)|\,du\right)^p \leqq (Dh)^{p\alpha-p}\,h^{p-1}\int_a^b dx\int_{Dh}^{Dh+h} |f(x-u)|^p\,du$$

$$= D^{(\alpha-1)p}\,h^{p\alpha-1}\int_a^b dx\int_{x-Dh-h}^{x-Dh} |f(t)|^p\,dt \leqq D^{(\alpha-1)p}\,h^{p\alpha-1}\int_{a-Dh-h}^{b-Dh} |f(t)|^p\,dt\int_{t+Dh}^{t+Dh+h} dx$$

$$= D^{(\alpha-1)p}\,h^{p\alpha}\int_{a-Dh-h}^{b-Dh} |f(t)|^p\,dt \leqq \varepsilon\,h^{p\alpha},$$

if D is sufficiently large.

This completes the proof of Theorem 21. When $\alpha = 1$, the theorem assumes a different form.

6. 2. Theorem 22. *If $p \geqq 1$ and $f(x)$ belongs to L^p, then*

$$\int_a^b \left| \frac{f_1(x) - f_1(x-h)}{h} \right|^p dx \to \int_a^b |f(x)|^p \, dx.$$

By Minkowski's inequality, the difference of the p-th roots of the two integrals does not exceed the p-th root of

$$\int_a^b dx \left| \frac{1}{h} \int_{-h}^{0} (f(x+t) - f(x)) \, dt \right|^p$$

$$\leqq \int_a^b dx \left(\frac{1}{h} \int_{-h}^{0} |f(x+t) - f(x)|^p \, dt \right) = \frac{1}{h} \int_0^h F(t) \, dt,$$

where

$$F(t) = \int_a^b |f(x+t) - f(x)|^p \, dx.$$

Since $F(t)$ tends to zero with t, the theorem follows.

It is plain that we may also write the conclusion of Theorem 21 in either of the forms

$$\int_a^b |f_\alpha(x+h) - f_\alpha(x)|^p \, dx = o(h^{p\alpha}),$$

$$\int_a^b |f_\alpha(x+h) - f_\alpha(x-h)|^p \, dx = o(h^{p\alpha}),$$

and that similar modifications may be made in that of Theorem 22.

6. 3. Theorem 23. *If $p \geqq 1$, $0 < k \leqq 1$, $f(x)$ belongs to L^p, and*

$$\int_a^b |\Delta f|^p \, dx = O(h^{pk}),$$

where Δf is one of

$$f(x) - f(x-h), \quad f(x+h) - f(x), \quad f(x+h) - f(x-h),$$

holds over every finite interval (a, b), then $f^\beta(x)$ exists for almost all x, and belongs to L^p, for every $\beta < k$ and over every finite interval. Also $f(x)$ is the β-th integral of $f^\beta(x)$.

It is indifferent which of the three conditions we select. Suppose for example that we select the third (as apparently the weakest). Then

$$\int_a^b |f(x)-f(x-2h)|^p\,dx = \int_{a-h}^{b-h} |f(t+h)-f(t-h)|^p\,dx$$

$$\leqq \int_{a-1}^{b} |f(t+h)-f(t-h)|^p\,dx = O(h^{pk})$$

if $0<h<1$. We take the first condition and we suppose first that $p>1$.

We write, as in § 5.6,

$$f_{1-\beta,\varepsilon}(x) = \frac{1}{\Gamma(1-\beta)} \int_{-\infty}^{x-\varepsilon} f(t)(x-t)^{-\beta}\,dt \qquad (\varepsilon \geqq 0), \tag{6.31}$$

and, for $\varepsilon>0$,

$$\begin{aligned} g_\varepsilon = g_\varepsilon(x) &= \Gamma(1-\beta) f'_{1-\beta,\varepsilon}(x) \\ &= \beta \int_{-\infty}^{x-\varepsilon} (f(x)-f(t))(x-t)^{-\beta-1}\,dt - \varepsilon^{-\beta}(f(x)-f(x-\varepsilon)). \end{aligned} \tag{6.32}$$

It is then sufficient to prove that

$$\int_a^b |g_\varepsilon - g_{\varepsilon'}|^p\,dx \to 0 \tag{6.33}$$

over any interval (a, b), when ε and ε' tend to zero. For, if this is so, g_ε tends strongly to a g of L^p, and the integral of g is the limit of that of g_ε, i. e. $\Gamma(1-\beta) f_{1-\beta}(x)$. Hence $f_{1-\beta}(x)$ is the integral of a g belonging to L^p. It follows that $f^\beta(x)$ exists for almost all x and belongs to L^p. Finally, the $(1-\beta)$-th integral of the β-th integral of any integrable function is the integral of that function, so that $f(x)$ and $g_\beta(x)$ have the same $(1-\beta)$-th integral, and are therefore equivalent.

Writing (6.32) in the form

$$g_\varepsilon = \beta g_{\varepsilon,1} + g_{\varepsilon,2},$$

it is enough to show that

$$\int_a^b |g_{\varepsilon,i} - g_{\varepsilon',i}|^p\,dx \to 0 \qquad (i=1,2). \tag{6.34}$$

Now if $\varepsilon<\varepsilon'$ we have

$$\int_a^b |g_{\varepsilon,1} - g_{\varepsilon',1}|^p \, dx \leqq \int_a^b dx \left(\int_{x-\varepsilon'}^{x-\varepsilon} |f(x) - f(t)| (x-t)^{-\beta-1} dt \right)^p$$

$$= \int_a^b dx \left(\int_\varepsilon^{\varepsilon'} |f(x) - f(x-u)| \, u^{-\beta-1} du \right)^p$$

$$= \int_a^b dx \left(\int_\varepsilon^{\varepsilon'} |f(x) - f(x-u)| \, u^{-\beta-\frac{1}{p}-\delta} \cdot u^{-\frac{1}{p'}+\delta} du \right)^p$$

$$\leqq \left(\int_\varepsilon^{\varepsilon'} u^{-1+p'\delta} du \right)^{p-1} \int_a^b dx \int_\varepsilon^{\varepsilon'} |f(x) - f(x-u)|^p \, u^{-p\beta-1-p\delta} du$$

for any $\delta > 0$. This is

$$o\left(\int_a^b dx \int_\varepsilon^{\varepsilon'} |f(x) - f(x-u)|^p \, u^{-p\beta-1-p\delta} du \right)$$

$$= o\left(\int_\varepsilon^{\varepsilon'} u^{-p\beta-1-p\delta} du \int_a^b |f(x) - f(x-u)|^p dx \right)$$

$$= o\left(\int_\varepsilon^{\varepsilon'} u^{p(k-\beta-\delta)-1} du \right) = o(1),$$

if $0 < \delta < k - \beta$. This proves (6.34) for $i = 1$.

Also

$$\int_a^b |g_{\varepsilon,2}|^p \, dx = O\left(\varepsilon^{-p\beta} \int_a^b |f(x) - f(x-\varepsilon)|^p dx \right) = O(\varepsilon^{p(k-\beta)}) = o(1);$$

and *a fortiori*

$$\int_a^b |g_{\varepsilon,2} - g_{\varepsilon',2}|^p \, dx = o(1).$$

This completes the proof of the theorem when $p > 1$.

Next suppose $p = 1$. In this case

$$\int_a^b |g_{\varepsilon,1} - g_{\varepsilon',1}| \, dx \leqq \int_a^b dx \int_\varepsilon^{\varepsilon'} |f(x) - f(x-u)| \, u^{-\beta-1} du$$

$$= \int_\varepsilon^{\varepsilon'} u^{-\beta-1} du \int_a^b |f(x) - f(x-u)| \, du = o\left(\int_\varepsilon^{\varepsilon'} u^{-\beta-1+k} du \right) = o(1);$$

while the proof of the similar equation for $g_{\varepsilon,2}$ is unaltered. Hence (6.33) is still true. From this it follows that there is a g of L such that

$$\int g_\varepsilon(x)\,dx \to \int g(x)\,dx,$$

and the proof is then completed as before.

We have proved incidentally that $g_{\varepsilon,2}(x) \to 0$ in the sense of mean convergence with index p, and that

$$\Gamma(1-\beta) f^{\beta}(x) = \beta \int_{-\infty}^{x} (f(x)-f(t))(x-t)^{-\beta-1}\,dt \tag{6.35}$$

if the integral be interpreted as the limit of $\int_{-\infty}^{x-\varepsilon}$ in the same sense.

6.4. If $\beta = k < 1$ then (whether $p > 1$ or $p = 1$) the result is false, Weierstrass's function providing a "Gegenbeispiel". But if $\beta = k = 1$ we have

Theorem 24. *Suppose that $f(x)$ has the period 2π, that $p \geqq 1$, and that*

$$\int_{-\pi}^{\pi} \left| \frac{f(x)-f(x-h)}{h} \right|^{p} dx \leqq C. \tag{6.41}$$

Then (i) *if $p > 1$, f is equivalent to an integral g, and*

$$\int_{-\pi}^{\pi} |g'(x)|^{p}\,dx \leqq C; \tag{6.42}$$

(ii) *if $p = 1$, f is equivalent to a function g of bounded variation, and*

$$\int_{-\pi}^{\pi} |dg| \leqq C. \tag{6.43}$$

We state our result for a complete period of $f(x)$ in order to obtain precise inequalities.

We use the following lemma.

Lemma. *Given an integrable $f(x)$, we can find a sequence of integrals $\varphi_n(x)$ such that*

(i) $\varphi_n(x) \to f(x)$ *for almost all x*,

(ii) $$\int_{-\pi}^{\pi} |\varphi_n(x) - \varphi_n(x-h)|^{p}\,dx \leqq \int_{-\pi}^{\pi} |f(x)-f(x-h)|^{p}\,dx.$$

In fact, if we take

$$\varphi_n(x) = n\left(f_1\left(x+\frac{1}{n}\right) - f_1(x)\right),$$

and write h for $\frac{1}{n}$ and Δf for $f(x)-f(x-h)$, then (i) is true, and

$$\int_{-\pi}^{\pi} |\Delta\varphi_n|^p\,dx = \int_{-\pi}^{\pi} \left| n\left(\Delta f_1\left(x+\frac{1}{n}\right) - \Delta f_1(x)\right)\right|^p dx$$

$$= \int_{-\pi}^{\pi} dx \left|\frac{1}{h}\int_0^h \Delta f(x+t)\,dt\right|^p \leqq \int_{-\pi}^{\pi} dx \left(\frac{1}{h}\int_0^h |\Delta f(x+t)|\,dt\right)^p$$

$$\leqq \int_{-\pi}^{\pi} dx \frac{1}{h}\int_0^h |\Delta f(x+t)|^p\,dt = \frac{1}{h}\int_0^h dt \int_{-\pi}^{\pi} |\Delta f(x+t)|^p\,dt$$

$$= \int_{-\pi}^{\pi} |\Delta f(x)|^p\,dx.$$

6.5. It follows from (6.41) and the lemma that

$$\int_{-\pi}^{\pi} \left|\frac{\Delta\varphi_n}{h}\right|^p dx \leqq C.$$

Fixing n, and making $h \to 0$, we obtain

$$(6.44)\qquad \int_{-\pi}^{\pi} |\varphi_n'|^p\,dx = \int_{-\pi}^{\pi} \lim \left|\frac{\Delta\varphi_n}{h}\right|^p dx \leqq \underline{\lim} \int_{-\pi}^{\pi} \left|\frac{\Delta\varphi_n}{h}\right|^p dx \leqq C.$$ [23])

This is true for $p \geqq 1$. If $p > 1$, there is a subsequence φ_{n_i}' which converges weakly to an h for which

$$\int_{-\pi}^{\pi} |h|^p\,dx \leqq C,$$

and

$$\int_{x_1}^{x_2} h(x)\,dx = \lim_{i\to\infty} \int_{x_1}^{x_2} \varphi_{n_i}'(x)\,dx = \lim_{i\to\infty} (\varphi_{n_i}(x_2) - \varphi_{n_i}(x_1)) = f(x_2) - f(x_1)$$

for almost all (x_1, x_2); which proves the theorem.

If $p = 1$, suppose that e is any finite set of non-overlapping intervals (ξ, η). Then, by (6.44),

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[23]) See Pollard, **9**; Schlesinger and Plessner, **14**.

$$\sum_e |\varphi_n(\eta) - \varphi_n(\xi)| \leqq \int_e |\varphi_n'(x)|\, dx \leqq C.$$

Making $n \to \infty$,

$$\sum_e |f(\eta) - f(\xi)| \leqq C,$$

for all sets e whose extremities do not fall in a certain set of measure zero, which again proves the theorem.

6.5. Theorem 25. *If*

$$p \geqq 1, \quad k \geqq 0, \quad \alpha > 0, \quad k + \alpha < 1, \tag{6.51}$$

$f(x)$ belongs to L^p, and

$$\int_a^b |f(x) - f(x-h)|^p\, dx = O(h^{pk}) \tag{6.52}$$

for every finite (a, b), then

$$\int_a^b |f_\alpha(x) - f_\alpha(x-h)|^p\, dx = O(h^{p(k+\alpha)}). \tag{6.53}$$

If the O in the hypothesis is replaced by o, the same change may be made in the conclusion.

We prove the theorem only in O form, leaving the o form to the reader.

We have

$$\begin{aligned} \Gamma(\alpha)\Delta &= \Gamma(\alpha)(f_\alpha(x) - f_\alpha(x-h)) \\ &= \int_{-\infty}^{x} f(t)(x-t)^{\alpha-1}\, dt - \int_{-\infty}^{x-h} f(t)(x-h-t)^{\alpha-1}\, dt \\ &= \int_0^\infty f(x-u)\, u^{\alpha-1}\, du - \int_h^\infty f(x-u)\,(u-h)^{\alpha-1}\, du \\ &= \int_h^\infty (f(x) - f(x-u))((u-h)^{\alpha-1} - u^{\alpha-1})\, du \\ &\quad - \int_0^h (f(x) - f(x-u))\, u^{\alpha-1}\, du = J_1 + J_2, \end{aligned} \tag{6.54}$$

say. It is enough to prove that

$$\int_a^b |J_1|^p\, dx = O(h^{p(k+\alpha)}), \qquad \int_a^b |J_2|^p\, dx = O(h^{p(k+\alpha)}). \tag{6.56}$$

Now

$$\left(\int_a^b |J_1|^p\,dx\right)^{\frac{1}{p}} \leqq \left(\int_a^b dx\left(\int_h^\infty |f(x)-f(x-u)|((u-h)^{\alpha-1}-u^{\alpha-1})\,du\right)^p\right)^{\frac{1}{p}}$$

$$\leqq \int_h^\infty ((u-h)^{\alpha-1}-u^{\alpha-1})\,du\left(\int_a^b |f(x)-f(x-u)|^p\,du\right)^{\frac{1}{p}}$$

$$= O\left(\int_h^\infty u^k((u-h)^{\alpha-1}-u^{\alpha-1})\,du\right)$$

$$= O\left(h^{k+\alpha}\int_1^\infty w^k((w-1)^{\alpha-1}-w^{\alpha-1})\,dw\right) = O(h^{k+\alpha}),$$

since $\alpha > 0$, $k+\alpha < 1$; which proves the first of (6.56). And

$$\left(\int_a^b |J_2|^p\,dx\right)^{\frac{1}{p}} \leqq \left(\int_a^b dx\left(\int_0^h |f(x)-f(x-u)|\,u^{\alpha-1}\,du\right)^p\right)^{\frac{1}{p}}$$

$$\leqq \int_0^h u^{\alpha-1}\,du\left(\int_a^b |f(x)-f(x-u)|^p\,dx\right)^{\frac{1}{p}} = O\left(\int_0^h u^{k+\alpha-1}\,du\right) = O(h^{k+\alpha}),$$

which completes the proof of the theorem.

The theorem is not true when $k+\alpha = 1$ (whether $p > 1$ or $p = 1$). A Gegenbeispiel is provided by $\sum a^{-kn}\cos a^n x$, which satisfies the hypothesis whatever p. If the conclusion were true, it would follow that a certain linear combination of the functions

$$\sum a^{-n}\cos a^n x, \qquad \sum a^{-n}\sin a^n x$$

has a derivative belonging to L^p, which is false.

6.6. Theorem 26. *If $p > 1$, $0 < k \leqq 1$, $f(x)$ belongs to L^p, and*

$$\int_a^b |f(x)-f(x-h)|^p\,dx = O(h^{pk}) \tag{6.61}$$

over every finite interval (a, b), then $f^\beta(x)$ exists for almost all x, and belongs to L^p, for every $\beta < k$, and

$$\int_a^b |f^\beta(x)-f^\beta(x-h)|^p\,dx = O(h^{p(k-\beta)}). \tag{6.62}$$

If the hypothesis is satisfied in o form, so is the conclusion.

We prove the theorem in the O form only. We have seen already that $f^{\beta}(x)$ exists and belongs to L^{p} (Theorem 21), and we have only to prove (6.62). Also, by (6.35),

$$\Gamma(1-\beta) f^{\beta}(x) = \beta \int_{-\infty}^{x} (f(x) - f(t)(x-t)^{-\beta-1}\, dt,$$

in the sense that the left hand side is the limit to which $\beta \int_{-\infty}^{x-\varepsilon}$ converges strongly with index p.

Writing now, as in Theorem 19, $g(x) = \Gamma(1-\beta) f^{\beta}(x)$, we have

$$(6.63)\quad \frac{1}{\beta} \Delta g(x) = -\int_{h}^{\infty} (f(x-h) - f(x-u))\left((u-h)^{-\beta-1} - u^{-\beta-1}\right) du$$
$$+ \int_{h}^{\infty} (f(x) - f(x-h))\, u^{-\beta-1}\, du + \int_{0}^{h} (f(x) - f(x-u))\, u^{-\beta-1}\, du$$
$$= J_1 + J_2 + J_3.$$

This equation is formally identical with (5.71) in the proof of Theorem 20, but now J_1 is a 'strong limit' of $\int_{h+\varepsilon}^{\infty}$, and J_3 is a strong limit of $\int_{\varepsilon}^{h}$.

We suppose $\varepsilon < h$, and write

$$J_1(\varepsilon) = -\int_{h+\varepsilon}^{\infty} (f(x-h) - f(x-u))\left((u-h)^{-\beta-1} - u^{-\beta-1}\right) du,$$

$$J_3(\varepsilon) = \int_{\varepsilon}^{h} (f(x) - f(x-u))\, u^{-\beta-1}\, du.$$

We then have

$$\int_{a}^{b} |J_1|^{p}\, dx = \lim_{\varepsilon \to 0} \int_{a}^{b} |J_1(\varepsilon)|^{p}\, dx, \qquad \int_{a}^{b} |J_3|^{p}\, dx = \lim_{\varepsilon \to 0} \int_{a}^{b} |J_3(\varepsilon)|^{p}\, dx;$$

and it is sufficient to prove that

$$(6.64)\quad \int_{a}^{b} |J_1(\varepsilon)|^{p}\, dx = O(h^{p(k-\beta)}), \qquad \int_{a}^{b} |J_3(\varepsilon)|^{p}\, dx = O(h^{p(k-\beta)}),$$

$$(6.65)\quad \int_{a}^{b} |J_2|^{p}\, dx = O(h^{p(k-\beta)}),$$

the first two equations uniformly in ε, for (say) $0 < \varepsilon \leq h$.

(i) We have

$$\left(\int_a^b |J_1(\varepsilon)|^p\,dx\right)^{\frac{1}{p}}$$

$$\leqq \left(\int_a^b dx \left(\int_{h+\varepsilon}^{\infty} |f(x-h)-f(x-u)|\,((u-h)^{-\beta-1}-u^{-\beta-1})\,du\right)^p\right)^{\frac{1}{p}}$$

$$\leqq \int_{h+\varepsilon}^{\infty} ((u-h)^{-\beta-1}-u^{-\beta-1})\,du \left(\int_a^b |f(x-h)-f(x-u)|^p\,dx\right)^{\frac{1}{p}}$$

$$= O\left(\int_{h+\varepsilon}^{\infty} ((u-h)^{-\beta-1}-u^{-\beta-1})(u-h)^k\,du\right) = O\left(\int_h\right)$$

$$= O\left(h^{k-\beta}\int_1^{\infty} (w-1)^k\,((w-1)^{-\beta-1}-w^{-\beta-1})\,dw\right) = O(h^{k-\beta}),$$

since $k-\beta > 0$, $k-\beta-1 < 0$; and uniformly in ε. This proves the first of (6.64).

(ii) Next we have

$$\left(\int_a^b |J_3(\varepsilon)|^p\,dx\right)^{\frac{1}{p}} \leqq \left(\int_a^b dx \left(\int_\varepsilon^h |f(x)-f(x-u)|\,u^{-\beta-1}\,du\right)^p\right)^{\frac{1}{p}}$$

$$\leqq \int_\varepsilon^h u^{-\beta-1}\,du \left(\int_a^b |f(x)-f(x-u)|^p\,dx\right)^{\frac{1}{p}}$$

$$= O\left(\int_\varepsilon^h u^{k-\beta-1}\,du\right) = O\left(\int_0^h u^{k-\beta-1}\,du\right) = O(h^{k-\beta}),$$

uniformly in ε.

(iii) Finally

$$\beta\,|J_2| = h^{-\beta}\,|f(x)-f(x-h)|,$$

$$\int_a^b |J_2|^p\,dx = O\left(h^{-p\beta}\int_a^b |f(x)-f(x-h)|^p\,dx\right) = O(h^{p(k-\beta)}).$$

This completes the proof of Theorem 26 when $p > 1$. The proof when $p = 1$ is again the same in principle but simpler.

6.7. There are many further interesting questions concerning Lipschitz conditions and "integrated Lipschitz conditions" of the type (1.21). In particular we may ask whether a function which satisfies an integrated Lipschitz condition Lip. (p, k), necessarily satisfies other conditions

Lip. (p', k') with different indices, or a Lipschitz condition of the ordinary type. Some results in this direction have been proved by Titchmarsh[24]), and further results may be deduced as corollaries from the theorems which we have proved. A complete discussion, however, is very much facilitated by the use of complex function theory, and we postpone it to a later memoir. Our main result is that (1) *if* $p \geqq 1$ *and* $0 < k < \frac{1}{p}$, *then a function which belongs to* Lip. (k, p) *belongs also to* Lip. $\left(k - \frac{1}{p} + \frac{1}{p'}, p'\right)$, *where p' is any number for which*

$$p < p' < \frac{p}{1 - kp};$$

(2) *if* $k > \frac{1}{p}$ *then it belongs to the class stated for* $p' > p$, *and is equivalent to a function of the class* Lip. $\left(k - \frac{1}{p}\right)$.[25]) The theorems proved in this memoir are not quite sufficient to establish this, and we appeal to a theorem in complex function theory, viz. that *a complex function which belongs to* Lip. (k, p) *on the unit circle, and whose Fourier series is of power series type*[26]), *is the boundary function of an analytic function* $f(z)$ *for which*

$$\int |f'(re^{i\theta})|^p \, d\theta = O\left(\frac{1}{(1-r)^{(1-k)p}}\right).$$

The classes Lip. (k, p) are also of interest in the convergence theory of Fourier series. If $k > \frac{1}{p}$ then $f(x)$ belongs (as just stated) to the class Lip. $\left(k - \frac{1}{p}\right)$, and its Fourier series is uniformly convergent[27]). The case $k = \frac{1}{p}$ is particularly interesting. We have in fact the very curious theorem that *if* $p \geqq 1$ *and*

$$\int_{-\pi}^{\pi} |f(\theta + h) - f(\theta - h)|^p \, d\theta = O(h),$$

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[24]) Titchmarsh, **17**.

[25]) The limiting case $p' = \infty$.

[26]) See Hardy and Littlewood, **4**.

[27]) If $p \leqq 2$, *absolutely*; in fact $\Sigma |c_n|^{\varkappa}$, where c_n is the (complex) Fourier constant of $f(x)$, is convergent if

$$\varkappa > \frac{p}{pk + p - 1}.$$

This was first proved for $p = 2$ by Szász (**15**), and later generally by Titchmarsh (**17**). Szász, in his turn, was generalising a theorem of S. Bernstein (**1**). Titchmarsh proves the corresponding theorem for Fourier transforms.

The series is also summable $(C, -k + \delta)$ for every positive δ. That this is so for a function of Lip. k is proved by Zygmund, **19**.

and in particular if $f(\theta)$ is the $\frac{1}{p}$-th integral of a function of L^p, then the Fourier series of $f(\theta)$ is convergent, and indeed summable $\left(C, -\frac{1}{p}+\delta\right)$ for every positive δ, wherever it is summable by any Cesàro mean. For the Fourier series of such a function, the convergence problem (in the classical sense) admits of a complete solution[28]).

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[28]) We give proofs of all the theorems stated in § 6. 7 in a note 'A convergence criterion for Fourier series' which will be published in this journal later.

(Eingegangen am 21. Mai 1927.)

CORRECTIONS

p. 566, *line* 10. Read necessarily.

p. 567, (1.23). The integral here is not necessarily convergent at $-\infty$ if $\alpha \geqslant 1$.

pp. 568–70, *proof of Theorem* 1. See the comments below.

p. 573, *last line of statement of Theorem* 3. For Theorem 1 read Theorem 2.

p. 577, § 3.5 (i), (ii). For p, q read r, s throughout.

pp. 579–83, *proofs of Theorems* 6–10. See the comments below.

p. 590, *last line*. For $f_{1-\beta,\epsilon}(x)$ read $f'_{1-\beta,\epsilon}(x)$.

p. 592, *line* 9. For J''_1 read $|J''_1|$.

p. 599, *Theorem* 24. The proof of this theorem is incomplete, and a correction is given in 1932, 4, p. 439.

p. 601, *last line*, and *p.* 602, *line* 6. For (6.56) read (6.55).

p. 602, *first line of statement of Theorem* 26. For $p > 1$ read $p \geqslant 1$.

p. 603, *line* 2. For Theorem 21 read Theorem 23.

p. 603, *line* 4. Insert bracket after $f(t)$.

COMMENTS

pp. 568–76. Theorems 1–5 form the core of this paper, and have led to much further work on inequalities and fractional integrals.

Theorem 1 and its proof are substantially reproduced from 1926, 6. It was later observed by R. Rado that this proof is inconclusive, and a corrected version is given by Hardy, Littlewood, and Pólya, *Inequalities*, pp. 265–76, together with generalizations of Theorem 1 by themselves and Gabriel. Another generalization of Theorem 1, for integrals, was proved by F. Riesz (see Hardy, Littlewood, and Pólya, *Inequalities*, pp. 288–91, and C. A. Rogers, *J. London Math. Soc.* 32 (1957), 102–8), and Theorem 3 can be deduced directly from this, without passing via sums (see *Inequalities*, loc. cit., and F. F. Bonsall, *Quart. J. of Math.* (2), 2 (1951), 135–50).

An alternative proof of Theorem 3 using interpolation arguments has been given by E. M. Stein and G. Weiss, *J. Math. Mech.* 8 (1959), 263–84, and an 'elementary' proof, using only Hölder's and Minkowski's inequalities, has been given by V. A. Solonnikov, *Vestnik Leningrad Univ.*, 17 (1962), 150–3. An alternative proof of the case of Theorem 4 in which $p = 2$ and q is an even integer is given in 1927, 11.

The results of Theorems 3–5 have been extended in various ways to functions defined on n-dimensional Euclidean space (see, for example, S. Soboleff, *Dokladi Ak. Nauk. U.S.S.R.* 20 (1938), 5, G. D. Thorin, *Comm. Math. Lund*, 1948, 1–57, N. du Plessis, *Trans. Amer. Math. Soc.* 80 (1955), 124–34, and 84 (1957), 262–72, R. O'Neil, *Duke Math. J.* 30 (1963), 129–42).

pp. 579–83. The proof of the case (i) of Theorem 6 given here is invalid (the statement concerning A' on p. 579, line 13, is incorrect). A correct proof of this case has been given by E. M. Stein and G. Weiss, *J. Math. Mech.* 7 (1958), 503–14, together with extensions of the theorem to n dimensions. The proof of Theorem 7 via Theorem 6 is therefore valid, as are also the deductions of Theorems 8, 9, and 11. On the other hand, the deduction of Theorem 10 from Theorem 7 on p. 583 is incorrect as it stands, and the condition $l < 1/p$ has to be added to the hypotheses of Theorem 10 to make the deduction valid (if $l \geqslant 1/p$, there is no α satisfying (4.32), so that in this case Theorem 7 is vacuous).

The following theorem, which contains Theorems 6–11, and also the cases $r < \infty$ of Theorems 5 and 7 of 1936, 2, has been proved by T. M. Flett, *Proc. Glasgow Math. Assoc.* 4 (1958), 7–15. The theorem can also be deduced from the integral analogue of the inequality for sums given by Hardy and Littlewood in 1935, 4.

(A) *Let* $f(t) \geqslant 0$ *for* $t > 0$, *let*

$$f_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-u)^{\alpha-1} f(u)\, du \quad (\alpha > 0), \qquad f_0(t) = f(t),$$

and let p, q, α, γ *satisfy one of the following sets of conditions:*

(i) $1 < p \leqslant q < \infty$, $\alpha \geqslant 1/p - 1/q$, $\gamma > -1$,
(ii) $1 = p < q < \infty$, $\alpha > 1/p - 1/q$, $\gamma > -1$,
(iii) $p = q = 1$, $\alpha \geqslant 0$, $\gamma > -1$.

Then
$$\left\{\int\limits_0^\infty t^{-1-q\gamma-q\alpha} f_\alpha^q(t)\, dt\right\}^{1/q} \leqslant A(p,q,\alpha,\gamma) \left\{\int\limits_0^\infty t^{-1-p\gamma} f^p(t)\, dt\right\}^{1/p}. \tag{1}$$

The following remarks explain the relation of (A) to the theorems of this paper and of 1936, 2. If in (1) we put $t = 1/s$, $u = 1/v$, $g(s) = s^{-1-\alpha} f(1/s)$, $\gamma' = -1-\gamma-\alpha$ (so that $\alpha < -\gamma'$), we obtain

$$\left\{\int\limits_0^\infty s^{-1-q\gamma'-q\alpha}\, ds \left\{\int\limits_s^\infty (v-s)^{\alpha-1} g(v)\, dv\right\}^q\right\}^{1/q} \leqslant A(p,q,\alpha,\gamma) \left\{\int\limits_0^\infty s^{-1-p\gamma'} g^p(s)\, ds\right\}^{1/p}. \tag{2}$$

On combining (1) and (2), we obtain:

(B) *If* p, q, α, γ *satisfy one of* (i), (ii), (iii), *and in addition* $\alpha < -\gamma$, *then*

$$\left\{\int\limits_0^\infty t^{-1-q\gamma-q\alpha}\, dt \left\{\int\limits_0^\infty |t-u|^{\alpha-1} f(u)\, du\right\}^q\right\}^{1/q} \leqslant A(p,q,\alpha,\gamma) \left\{\int\limits_0^\infty t^{-1-p\gamma} f^p(t)\, dt\right\}^{1/p}.$$

Conversely, (B) implies (A), so that (A) and (B) are equivalent. To prove this, we note that (B) trivially implies the inequality (1) under the additional hypothesis that $\alpha < -\gamma$. Using the last remark in § 4.4, viz. that $\Gamma(\alpha) t^{-\alpha} f_\alpha(t)$ is a decreasing function of α for $\alpha > 0$, we deduce that (B) implies:

(C) *If* p, q, α, γ *satisfy one of* (i), (ii), (iii), *and in addition* $\gamma < -(1/p - 1/q)$, *then* (1) *holds.*

If now we replace $f(t)$ in (1) by $t^{-\mu} f(t)$, where $\mu > 0$, and observe that

$$t^{-\mu} \int\limits_0^t (t-u)^{\alpha-1} f(u)\, du \leqslant \int\limits_0^t (t-u)^{\alpha-1} u^{-\mu} f(u)\, du, \tag{3}$$

we see that if (1) holds for some $\gamma = \gamma_0 > -1$, then it holds for all $\gamma \geqslant \gamma_0$. The upper bound for γ in (C) may therefore be deleted, and we thus obtain (A).

The chain of argument in the present paper is as follows. Theorem 6 is easily seen to be equivalent to B (i), and, as remarked above, this trivially implies A (i) under the additional hypothesis that $\alpha < -\gamma$. This latter result (with the additional hypothesis) is Theorem 7, which in turn gives C (i), exactly as above. Theorems 8 and 9 are the cases $\gamma = -1/p$ of Theorem 7 and C (i), respectively, Theorem 10 (as corrected) is the case $p = q$ of C (i), and Theorem 11 is the case $p = q$, $\gamma = -1/p$ of C (i). The passage from (C) to (A) via (3), which shows that Theorem 7 implies the cases $r < \infty$ of Theorems 5 and 7 of 1936, 2, does not seem to have been noticed.

p. 581. Two theorems which are limiting cases of Theorems 7–9 have been proved by A. Zygmund, *Trans. Amer. Math. Soc.* 36 (1934), 586–617, and T. M. Flett, loc. cit.

pp. 584–606. Alternative proofs of some of the theorems proved here are given in 1932, 4.

p. 599. An alternative proof of Theorem 24, valid also for functions defined over the interval $(-\infty, \infty)$, has been given by P. L. Butzer, *Math. Annalen*, 142 (1961), 259–69.

Some properties of conjugate functions.

By *G. H. Hardy* and *J. E. Littlewood* in Cambridge.

1. Introduction.

1.1. This paper is a sequel to our paper 5. There we were concerned primarily with the mean values

$$(1.1.1)\qquad M_\lambda(f) = M_\lambda(r, f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\, f(re^{i\theta})\,|^\lambda\, d\theta\right)^{\frac{1}{\lambda}}$$

of an analytic function

$$(1.1.2)\qquad f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$$

regular for $r < 1$, λ being any positive number. We had also to consider the corresponding mean values, such as

$$(1.1.3)\qquad M_\lambda(u) = M_\lambda(r, u) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\, u(r, \theta)\,|^\lambda\, d\theta\right)^{\frac{1}{\lambda}},$$

of the harmonic components of f, but in these we supposed always that $\lambda > 1$, reserving for the present paper the more difficult questions which arise when $\lambda \leqq 1$.

We begin by stating shortly what is known. Suppose first that $\lambda > 1$. If $M_\lambda(r, u)$ is bounded then $M_\lambda(r, v)$ and $M_\lambda(r, f)$ are also bounded; in other words, u, v, and f belong together to the 'complex Lebesgue class' L_λ. This is the well known theorem of M. Riesz, stated in terms of the means (1.1.1) and (1.1.3), with $r < 1$. The theorem may also be stated in terms of the 'boundary functions' $U(\theta)$, $V(\theta)$, and $F(\theta)$, by saying that U, V, and F belong together to the ordinary Lebesgue class L^λ. There is in short, for our purposes, no material difference between harmonic and analytic functions when $\lambda > 1$[0]).

All this ceases to be true when $\lambda = 1$. Riesz's theorem, and many other theorems concerning harmonic functions, such as those which, in previous papers, we have called 'Max' and '$p \to q$', are then false. The place of Riesz's theorem is taken by two recent theorems of Kolmogoroff and Zygmund.

Theorem of Kolmogoroff[1]). *If $U(\theta)$ belongs to L, then any conjugate $V(\theta)$ of $U(\theta)$ belongs to L^λ for every $\lambda < 1$, and $U + iV$ is the boundary function of an analytic function of the complex class L^λ.*

Theorem of Zygmund[2]). *If $U(\theta)$ is measurable and $U(\theta)\log^+|U(\theta)|$ belongs to L, then any conjugate $V(\theta)$ of $U(\theta)$ belongs to L, and $U + iV$ is the boundary function of an analytic function of the complex class L.*

[0]) There is a certain ambiguity in the phrase 'boundary function'. We may say that $u(r, \theta)$ has the boundary fnnction $U(\theta)$ if either (a) $u \to U$ when $r \to 1$, for almost all θ, or (b) $\int |\, u - U\,|^\lambda d\theta \to 0$ (*i. e.* if U is a 'strong limit' of u).

It is indifferent here which definition we adopt. The theorems of Kolmogoroff and Zygmund are true, and our two statements of Riesz's theorem are strictly equivalent, with either definition.

[1]) Kolmogoroff (7); see also Littlewood (9) and Hardy (1).

[2]) Zygmund (17); see also Titchmarsh (16), Littlewood (13), Tamarkine (14), and Zygmund (17, 18).

1931, 2 (with J. E. Littlewood) *Journal für Mathematik*, 167, 405–23.

Finally suppose that $\lambda < 1$. It is to be expected that Riesz's theorem will be false in this case also, and this is easily proved. But in this case the theorem is false in a much more comprehensive sense. Kolmogoroff's theorem might suggest that, if u belongs to L^λ, then v, while not necessarily belonging to L^λ, must belong to L^μ for $0 < \mu < \lambda$; but this also is false, and in fact v need not belong to L^μ for any positive μ. This (which is naturally a good deal more difficult to prove) was first shown by Littlewood[3]), by an example drawn from the theory of the modular functions. We deal with the matter more fully in § 4.

It may be worth while to state explicitly a 'reason' which underlies these distinctions, and which is to be found in the theory of 'subharmonic' functions. The function $|f|^\lambda$ is subharmonic, and so $M_\lambda(f)$ is an increasing function of r, for all positive λ. The functions $|u|^\lambda$, $|v|^\lambda$ are subharmonic when $\lambda > 1$, but not when $\lambda < 1$; and in this case $M_\lambda(u)$ and $M_\lambda(v)$ are not necessarily monotonic.

1.2. The content of the paper is as follows. §§ 2—3 are 'positive'. In § 2 we show that the symmetry of Riesz's theorem is restored if we consider 'orders of infinity'; if

$$M_\lambda(r, u) = O((1 - r)^{-a}) \quad (a > 0), \tag{1.2.1}$$

then v has the same property. This theorem (Theorem 4) is surprisingly difficult to prove.

In § 3 we return to the case $a = 0$. We suppose $M_\lambda(r, u)$ bounded, and ask what is the most that is true of $M_\lambda(r, v)$. The result (Theorem 7) is that

$$M_\lambda(r, v) = O\left(\log \frac{1}{1 - r}\right); \tag{1.2.2}$$

a result easy to prove when $\lambda \geqq 1$ but difficult when $\lambda < 1$. We have introduced a variation of method into the proof of Theorem 7, in spite of a slight sacrifice of space. Theorems 4 and 7 both depend upon Theorem 2, and each also depends on a secondary theorem of some intrinsic interest, Theorem 3 in the first case and Theorem 6 in the second. We have already published two proofs of Theorem 3 in our paper 5, and we do not repeat either of them here. Theorem 6 may be proved by an adaptation of the first of these proofs, and this would be the most natural course to follow. We have used a quite different method (which will also prove Theorem 3, and is in fact the method by which we first obtained that theorem) because it seems to us both interesting in itself and a possible line of attack on other problems.

Finally, § 4 is 'negative'. We consider a number of special functions, with the primary object of elucidating our assertions at the end of § 1.1.

2. Proof of Theorem 4.

2.1. *Two functional inequalities.* We begin with some elementary lemmas, which we prove in a form sharper than is necessary for our applications of them. They have all the same character; a sequence or function satisfies an inequality, and the lemma asserts that either (a) it is bounded, or (b) it tends to infinity, at any rate for appropriately selected values of the variable, with great rapidity.

Lemma 1. *If*

$$u_n \leqq \alpha + \beta\, u_{n+1},$$

where $0 < \beta < 1$, *and* $u_0 > \dfrac{\alpha}{1 - \beta}$, *then there is a positive K such that*

$$u_n > K\, \beta^{-n}.$$

---

[3]) Littlewood (8, 516—517). The proof is not given in detail. An incidental result of some interest in itself is that a harmonic u for which $M_\lambda(u)$ is bounded need not necessarily possess a boundary function U. See also § 4. 9 of this paper.

It is plain that $u_n \geqq U_n$, where U_n is defined by $U_0 = u_0$ and $U_n = \alpha + \beta\, U_{n+1}$. Since

$$U_n = \frac{\alpha}{1-\beta} + \beta^{-n}\left(u_0 - \frac{\alpha}{1-\beta}\right),$$

the conclusion follows. It is to be observed that K depends on u_0 (and so on the sequence considered) as well as on α and β; the conclusion is equivalent to

$$\varliminf \beta^n u_n > 0.$$

Lemma 2. *If*

$$\alpha > 0,\ \beta > 0,\ 0 < \gamma < 1,\ u_0 > \xi,$$

where ξ *is the positive root of*

$$\xi = \alpha + \beta\, \xi^\gamma,$$

and

$$u_n \leqq \alpha + \beta u_{n+1}^\gamma,$$

then there is a positive K *for which*

(2.1.1) $$e^{-K\gamma^{-n}} u_n \to \infty.$$

Here $u_n \geqq U_n$, where U_n is defined by $U_0 = u_0$ and $U_n = \alpha + \beta\, U_{n+1}^\gamma$. We prove first that $U_n \to \infty$. If $U_{n-1} > \xi$ then $U_{n-1} > \alpha + \beta\, U_{n-1}^\gamma$ and so $U_n > U_{n-1}$. Hence U_n increases to a limit l, where $\xi < l \leqq \infty$. But if l is finite then $l = \alpha + \beta\, l^\gamma$, which contradicts $l > \xi$. Hence $U_n \to \infty$.

Next, we have

$$\log U_{n+1} = \frac{1}{\gamma}\log(U_n - \alpha) - \frac{1}{\gamma}\log\beta > \frac{1}{\gamma}\log U_n - \frac{2}{\gamma}\,|\log\beta\,|$$

for large n, say for $n \geqq n_0$. We may also suppose n_0 large enough to make

$$\log U_{n_0} > \frac{2\,|\log\beta\,|}{1-\gamma}.$$

If then we write $V_n = \log U_{n_0+n}$, we have

$$V_n < 2\,|\log\beta\,| + \gamma\, V_{n+1},\quad V_0 > \frac{2\,|\log\beta\,|}{1-\gamma}.$$

Hence, by Lemma 1, there is a positive K for which $V_n > K\gamma^{-n}$ or

$$U_{n_0+n} > e^{K\gamma^{-n}};$$

and so

$$u_p \geqq U_p < e^{K\gamma^{n_0-p}} \qquad (p \geqq n_0).$$

This is equivalent to (2.1.1), with a change of K. Here also K depends on the sequence as well as on α, β, and γ.

Lemma 3. *If*

$$f(x) \leqq \alpha + \beta f(cx) \qquad (x > x_0 > 0),$$

where

$$0 < \beta < 1,\ c > 1,$$

then either

$$f(x) \leqq \frac{\alpha}{1-\beta} \qquad (x > x_0),$$

or

$$\varlimsup_{x\to\infty} x^{-\mu} f(x) > 0,$$

where

$$\mu = \frac{\log(1/\beta)}{\log c}.$$

In particular, in the second case

$$\varlimsup_{x\to\infty} \frac{f(x)}{\log x} = \infty.$$

For if there is an $X > x_0$ for which $f(X) > \dfrac{\alpha}{1-\beta}$, and we take $u_n = f(c^n X)$, then u_n satisfies the conditions of Lemma 1. Hence, for certain x,

$$f(x) = f(c^n X) > K\beta^{-n} = Kc^{n\mu} = K\left(\frac{x}{X}\right)^{\mu}.$$

Lemma 4. *If*

$$f(x) \leqq \alpha + \beta(f(cx))^{\gamma} \qquad (x > x_0 > 0),$$

where

$$\alpha > 0,\ \beta > 0,\ 0 < \gamma < 1,\ c > 1,$$

then either

$$f(x) \leqq \xi \qquad (x > x_0),$$

where ξ *is defined as in Lemma* 2, *or*

$$\varlimsup_{x\to\infty} e^{-Kx^{\nu}} f(x) = \infty, \tag{2.1.2}$$

where

$$\nu = \frac{\log(1/\gamma)}{\log c},$$

for some positive K. *In particular, in the second case,*

$$\varlimsup_{x\to\infty} x^{-k} f(x) = \infty$$

for all k.

For if there is an $X > x_0$ for which $f(x) > \xi$, and we take $u_n = f(c^n X)$, then u_n satisfies the conditions of Lemma 2. Hence there is a K such that

$$e^{-K\gamma^{-n}} f(c^n X) \to \infty.$$

If we write

$$x = c^n X,\quad \gamma^{-n} = c^{n\nu} = \left(\frac{x}{X}\right)^{\nu},$$

this is equivalent to (2.1.2). It is to be observed that the K of this lemma depends upon the function f as well as upon the numbers α, β, γ, c.

2.2. In what follows we suppose always that

$$f(z) = \Sigma c_n z^n = u + iv$$

is regular for $r = |z| < 1$. $B = B(\lambda, a)$ is a function of λ and a only, A an absolute constant. A and B vary from one occurrence to another. C, on the other hand, is a number which preserves its identity throughout a theorem and its proof.

Our hypotheses bear on u, and we suppose always that that conjugate v is chosen which vanishes at the origin.

Lemma 5. *If* $\lambda > 0$, $a \geqq 0$, *and*

$$M_{\lambda}(u) \leqq C(1-r)^{-a},$$

then $f(z)$ *is of finite order, i. e.*

$$|f(z)| \leqq BC(1-r)^{-B} \tag{2.2.1}$$

(*and* $|c_n| \leqq BC(n+1)^{B}$).

It is familiar that the two last assertions are equivalent.

We may plainly suppose that $C = 1$ and $\lambda \leqq 1$[1]). There is a θ_n such that

$$|c_n|\, r^n = \left| \frac{1}{\pi} \int_{-\pi}^{\pi} u(r, \theta) \cos(n\theta - \theta_n)\, d\theta \right| \leqq \frac{1}{\pi} \int_{-\pi}^{\pi} |u|\, d\theta$$

$$\leqq M^{1-\lambda}(r) \frac{1}{\pi} \int_{-\pi}^{\pi} |u|^{\lambda} d\theta \leqq 2(1 - r)^{-\lambda a} M^{1-\lambda}(r),$$

where $M(r)$ is the maximum modulus of $f(z)$ on $|z| = r$. The result follows at once if $\lambda = 1$.

If $\lambda < 1$ then

$$M(\varrho) \leqq \Sigma |c_n| r^n \left(\frac{\varrho}{r}\right)^n \leqq \frac{2r}{r - \varrho} (1 - r)^{-\lambda a} M^{1-\lambda}(r). \tag{2.2.2}$$

We now write

$$r = e^{-1/t}, \quad \varrho = e^{-1/\tau},$$

where $t > \tau > 0$, $t > \frac{1}{2}$; and

$$\log M(\varrho) = g(\tau), \quad 1 - \lambda = b.$$

Then (2.2.2) becomes

$$g(\tau) - bg(t) = \log \frac{2\, e^{-1/t}}{e^{-1/t} - e^{-1/\tau}} + \lambda a \log \frac{1}{1 - e^{-1/t}}. \tag{2.2.3}$$

It is easily verified that

$$\log \frac{1}{1 - e^{-1/t}} < A + \log^+ t$$

and

$$\log \frac{2 e^{-1/t}}{e^{-1/t} - e^{-1/\tau}} < A + \log^+ \frac{t\tau}{t - \tau} < A + \log^+ t + \log^+ \tau + \log^+ \frac{1}{t - \tau},$$

$\log^+ x$ meaning as usual Max $(\log x, 0)$. Thus (2.2.3) gives

$$g(\tau) - bg(t) < B + B \log^+ t + B \log^+ \tau + B \log^+ \frac{1}{t - \tau}.$$

Herein we substitute successively the pairs $\left(\tau, \tau + \frac{1}{2}\right)$, $\left(\tau + \frac{1}{2}, \tau + \frac{1}{2} + \frac{1}{2^2}\right), \ldots$ for τ and t. We thus obtain

$$g(\tau) - bg\left(\tau + \frac{1}{2}\right) \leqq B \log^+ \left(\tau + \frac{1}{2}\right) + B,$$

$$bg\left(\tau + \frac{1}{2}\right) - b^2 g\left(\tau + \frac{1}{2} + \frac{1}{2^2}\right) \leqq B\, b \log^+ \left(\tau + \frac{1}{2} + \frac{1}{2^2}\right) + 2\, b\, B,$$

. . . .

and so

$$g(\tau) \leqq b^n g\left(\tau + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\right) + B \log(\tau + 1) \sum_0^n b^m + B \sum_0^n (m + 1)\, b^m.$$

When $n \to \infty$, the first term on the right tends to zero and the two series to finite sums, so that we obtain

$$\log M(\varrho) = g(\tau) < B + B \log(1 + \tau) < B + B \log \frac{1}{1 - \varrho},$$

which is equivalent to (2.2.1).

[1]) The result is in any case well-known for $\lambda > 1$.

2.3. *Theorem* 1. *If* $0 < \lambda \leqq 1$, $a \geqq 0$,

and

(2.3.1) $$M_\lambda(u) \leqq C(1-r)^{-a},$$

then

(2.3.2) $$|c_n| \leqq BC(n+1)^{a+\frac{1}{\lambda}-1},$$

(2.3.3) $$|f(z)| \leqq BC(1-r)^{-a-\frac{1}{\lambda}},$$

(2.3.4) $$|f'(z)| \leqq BC(1-r)^{-a-1-\frac{1}{\lambda}}.$$

It is not strictly necessary that we should prove Theorem 1 here, since the results can be made corollaries of the later Theorem 4, and in the proof of Theorem 4 we require only the cruder result of Lemma 5. It is however of some interest to prove Theorem 1 as simply as possible; the proof is in any case rather more difficult than might have been expected.

It is sufficient to prove (2.3.3), and we deduce this also from (2.2.2). Taking now

$$\varrho = e^{-1/\tau},\ r = e^{-1/2\tau},\ h(\tau) = \log\left((1-\varrho)^{a+\frac{1}{\lambda}} M(\varrho)\right),$$

we obtain

$$h(\tau) - b\,h(2\tau) \leqq \log\frac{2r}{r-\varrho} + \lambda a \log\frac{1}{1-r} - \left(a+\frac{1}{\lambda}\right)\log\frac{1}{1-\varrho}$$
$$+ (1-\lambda)\left(a+\frac{1}{\lambda}\right)\log\frac{1}{1-r} = \log\frac{2r(1-r)}{1-\varrho} + \left(a+\frac{1}{\lambda}\right)\log\frac{1-\varrho}{1-r} < B.$$

Hence, by Lemma 3, either $h(\tau)$ is bounded by a B, when the conclusion follows, or

$$\overline{\lim}\,\frac{h(\tau)}{\log\tau} = \infty;$$

and the second possibility is ruled out by Lemma 5.

2.4. *Lemma* 6. *If* $\psi(z)$ *is indefinitely differentiable then*

$$\left(z\frac{d}{dz}\right)^n \psi(z) = \sum_{m=1}^{n} a_{m,n}\, z^m\, \psi^{(m)}(z),$$

where

$$0 \leqq a_{m,n} \leqq \frac{2^n(n-1)!}{(m-1)!}$$

The proof is immediate by induction.

2.5. *Theorem* 2. *If the conditions of Theorem* 1 *are satisfied, then*

$$M_\lambda(f') \leqq BC(1-r)^{-a-1}.$$

It is in the proof of this theorem that we meet the principal difficulty of our work.

We denote by P, P_1, P_2 the points r, $r^{\frac{1}{2}}$, $r^{\frac{1}{4}}$ on the radius θ, and by γ the circle whose centre is P and which passes through P_1; its radius is

$$\sigma = \sigma(r) = r^{\frac{1}{2}} - r.$$

It is easily verified that $0 \leqq \sigma \leqq \frac{1}{4}$ and that the angle subtended at P_2 by γ is less than $2\arcsin\frac{2}{3}$.

We write

(2.5.1) $$T(r) = (1-r)^{-\lambda a},$$

(2.5.2) $$\mu = \mu(r, \theta) = \operatorname*{Max}_{\gamma} |f'(z)|,$$

(2.5.3) $$J(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |z f'(z)|^{\lambda} d\theta .$$

By 'Max' [1])

(2.5.4) $$\int_{-\pi}^{\pi} (r\mu)^{\lambda} d\theta \leqq BJ(r^{\frac{1}{4}}) .$$

If $n > 1$,

$$|f^{(n)}(z)| = \left|\left(\frac{d}{dz}\right)^{n-1} f'(z)\right| \leqq \frac{(n-1)!}{\sigma^{n-1}} \mu,$$

for z in γ. Hence we derive first

(2.5.5) $$\left|\frac{\partial^n u}{\partial r^n}\right| \leqq \frac{(n-1)!}{\sigma^{n-1}} \mu \leqq \frac{2^n n!}{\sigma^{n-1}} \mu .$$

Also

$$\left|\frac{\partial^n u}{\partial \theta^n}\right| = \left|\Re\left(iz\frac{d}{dz}\right)^n f(z)\right| \leqq \sum_1^n a_{m,n} r^m |f^{(m)}(z)| \leqq r\mu \sum_1^n a_{m,n} \frac{(m-1)!}{\sigma^{m-1}},$$

where $a_{m,n}$ is the coefficient of Lemma 6. Hence

(2.5.6.) $$\left|\frac{\partial^n u}{\partial \theta^n}\right| \leqq r\mu\, 2^n (n-1)! \sum_1^n \frac{1}{\sigma^{m-1}} \leqq r\mu\, 2^n n!\, \sigma^{-n+1},$$

an inequality like (2.5.5) except for the additional factor r.

2.6. Suppose that $h = \eta\sigma$; η will ultimately be supposed to be appropriately small, and is in any case less than $\frac{1}{4}$.

We have

$$|u(r+h, \theta) - u(r, \theta) - hu_r(r, \theta)| \leqq \sum_2^{\infty} \frac{h^n}{n!} \left|\frac{\partial^n u}{\partial r^n}\right|$$
$$\leqq \mu \sum_2^{\infty} (2h)^n \sigma^{-n+1} = \mu\sigma \sum_2^{\infty} (2\eta)^m \leqq A\mu\sigma\eta^2,$$

and so

(2.6.1) $$\left|\sigma \frac{\partial u}{\partial r}\right| \leqq \frac{|u(h+r, \theta)| + |u(r, \theta)|}{\eta} + A\mu\sigma\eta .$$

Similarly

$$|u(r, \theta+h) - u(r, \theta) - hu_\theta(r, \theta)| \leqq \sum_2^{\infty} \frac{h^n}{n!} \left|\frac{\partial^n u}{\partial \theta^n}\right| \leqq r\mu \sum_2^{\infty} (2h)^n \sigma^{-n+1} \leqq Ar\mu\sigma\eta^2,$$

(2.6.2.) $$\left|\sigma \frac{\partial u}{\partial \theta}\right| \leqq \frac{|u(r, \theta+h)| + |u(r, \theta)|}{\eta} + Ar\mu\sigma\eta .$$

Hence, since

$$|zf'(z)| \leqq r\left|\frac{\partial u}{\partial r}\right| + \left|\frac{\partial u}{\partial \theta}\right|,$$

and $r < 1$, we have

(2.6.3) $$\sigma |zf'(z)| \leqq \eta^{-1}(|u(r+h, \theta)| + |u(r, \theta+h)| + |u(r, \theta)|) + A r \mu \sigma \eta .$$

---

[1]) Theorem 27 of 4 (repeated as Theorem 33 of 5). γ lies within a 'kite-shaped region' S associated with P_2.

Since

$$(\Sigma a)^\lambda \leqq \Sigma a^\lambda$$

if $a \geqq 0$ and $0 < \lambda \leqq 1$, we may raise every term in (2.6.3) to the power λ. Doing this, and integrating with respect to θ, we obtain

$$\sigma^\lambda J(r) \leqq \eta^{-\lambda}(M_\lambda^\lambda(r+h, u) + 2M_\lambda^\lambda(r, u)) + B(\eta\sigma)^\lambda \int_{-\pi}^{\pi} (r\mu)^\lambda \, d\theta .$$

Using (2.5.1) and (2.5.4), and observing that $1 - r - h > A(1 - r)$, and so $T(r + h) < BT(r)$, we obtain

(2.6.4) $$\sigma^\lambda J(r) < B_1 \eta^{-\lambda} T(r) + B_2 \eta^\lambda \sigma^\lambda J(r^{\frac{1}{4}}) .$$

2.7. We now take [1])

$$\eta^{2\lambda} = \frac{B_1 T(r)}{4^{2\lambda} B_1 T(r) + B_2 \sigma^\lambda J(r^{\frac{1}{4}})} .$$

This gives

$$\begin{aligned}\sigma^\lambda J(r) &< B\sqrt{T(r)}\sqrt{T(r) + \sigma^\lambda J(r^{\frac{1}{4}})} + B\sqrt{T(r)}\sqrt{\sigma^\lambda J(r^{\frac{1}{4}})} \\ &< B\,T(r) + B\sqrt{T(r)\cdot \sigma^\lambda J(r^{\frac{1}{4}})} \, ;\end{aligned}$$

and so, since

$$\sigma^\lambda(r) \leqq B\sigma^\lambda(r^{\frac{1}{4}}), \qquad T(r) > B\,T(r^{\frac{1}{4}}) ,$$

(2.7.1) $$\frac{\sigma^\lambda(r)\, J(r)}{T(r)} < B + B\sqrt{\frac{\sigma^\lambda(r^{\frac{1}{4}})\, J(r^{\frac{1}{4}})}{T(r^{\frac{1}{4}})}} .$$

Writing finally

$$r = e^{-\frac{1}{x}}, \quad r^{\frac{1}{4}} = e^{-\frac{1}{4x}}, \quad \frac{\sigma^\lambda(r)\, J(r)}{T(r)} = \psi(x) ,$$

(2.7.1) becomes

$$\psi(x) < B + B\sqrt{\psi(4x)} .$$

It follows from Lemma 4 that either $\psi(x)$ is less than a B or else

$$\overline{\lim}\, x^{-k} \psi(x) = \infty$$

for all k. The latter alternative would mean that

$$\overline{\lim}\, (1 - r)^k \frac{\sigma^\lambda(r)}{T(r)} J(r) = \infty$$

for all k, and Theorem 1 (or Lemma 5) shows that this is impossible. Hence $\psi(x)$ is less than a B, *i. e.*

$$r^\lambda \int_{-\pi}^{\pi} |f'|^\lambda \, d\theta < B(1 - r)^{-a\lambda - \lambda} .$$

Finally we may omit the factor r^λ, since we may obviously do so when $r > \frac{1}{2}$ and since the integral increases with r. This completes the proof of the theorem.

[1]) If we omitted the first term in the denominator, we should make the two terms on the right of (2.6.4) equal. We have however to secure that $\eta < \frac{1}{4}$. The B_1, B_2 here are the same as in (2.6.4).

2.8. *Theorem* 3. *If* $\lambda > 0, a > 0$,

$$M_\lambda(f') \leqq C(1-r)^{-a-1},$$

and $f(0) = 0$, *then*

$$M_\lambda(f) \leqq BC(1-r)^{-a}$$

This is proved in 5 (Theorem 4 b).

2.9. Suppose now that (2.3.1) is true, with $a > 0$. Then

$$M_\lambda(f') \leqq BC(1-r)^{-a-1}$$

by Theorem 2, and so

$$M_\lambda(f - u(0)) \leqq BC(1-r)^{-a}$$

by Theorem 3. Also, taking $r = 0$ in (2.3.1), $|u(0)| < BC$. Hence we obtain

Theorem 4. *If* $0 < \lambda \leqq 1$, $a > 1$, *and*

$$M_\lambda(u) \leqq C(1-r)^{-a},$$

then

$$M_\lambda(v) \leqq BC(1-r)^{-a}, \quad M_\lambda(f) \leqq BC(1-r)^{-a}.$$

The theorem is of course true for all $\lambda > 0$, but is a corollary of Riesz's theorem when $\lambda > 1$.

We observe before passing on that we may prove, by very similar reasoning,

Theorem 5. *If* $0 < \lambda \leqq 1$ *and*

$$\iint |u|^\lambda r\,dr\,d\theta \leqq C,$$

the integral being extended over the unit circle, then

$$\iint |v|^\lambda r\,dr\,d\theta \leqq BC.$$

There is a simple proof when $\lambda = 1$. In this case, if we replace u and v by their derivatives with respect to r (or θ), which are also conjugate, we obtain the theorem that *if u is of bounded variation in the circle* (in the sense of Tonelli) *then so also is v.* Theorem 5 is again true for all positive λ, but is a corollary of Riesz's theorem when $\lambda > 1$.

3. Proof of Theorem 7.

3.1. In this section we vary our method, for the reasons explained in § 1.2.

Theorem 6. *If* $\lambda > 0$,

$$M_\lambda(f') \leqq \frac{C}{1-r},$$

and $f(0) = 0$, *then*

(3.1.1) $$M_\lambda(f) \leqq BC + BC\left(\log\frac{1}{1-r}\right)^\gamma \text{ [1]},$$

where

(3.1.2) $$\gamma = \frac{1}{\lambda}\ (\lambda < 1), \quad \gamma = 1\ (\lambda > 1).$$

The case $\lambda \geqq 1$ is easy. Suppose that

$$f^*(r,\theta) = \int_0^r |f'(\varrho e^{i\theta})|\,d\varrho$$

(so that $|f| \leqq f^*$). Then

---

[1]) We might replace the term BC by BCr, were there any advantage in doing so.

$$M_\lambda(f) \leqq M_\lambda(f^*) = \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \Big(\int_0^r |f'(\varrho e^{i\theta})|\, d\varrho\Big)^\lambda\Big)^{\frac{1}{\lambda}}$$

$$\leqq \int_0^r d\varrho \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'|^\lambda\, d\theta\Big)^{\frac{1}{\lambda}} \leqq C\int_0^r \frac{d\varrho}{1-\varrho} = C\log\frac{1}{1-r}.$$

In this case we may replace the two B of (3.1.1) by 0 and 1 respectively.

We may therefore suppose $\lambda < 1$.

3.2. *Lemma* 7. *Suppose that A and B are given and that $F(x, \theta)$ is continuous and $\frac{\partial F(x, \theta)}{\partial x}$ bounded in every rectangle R or*

$$A < \alpha \leqq x \leqq \beta < B, \qquad -\pi \leqq \theta \leqq \pi;$$

and let $H(x)$ be, for each x, the set of θ in which $F > 0$. Then

$$\int_{H(x_2)} F(x_2, \theta)\, d\theta - \int_{H(x_1)} F(x_1, \theta)\, d\theta = \int_{x_1}^{x_2} dx \int_{H(x)} \frac{\partial F}{\partial x}\, d\theta.$$

Let

$$G = \operatorname{Max}(F, 0).$$

A moment's consideration shows that G has everywhere a forward x-derivative G_x which is either F_x or 0 (and is accordingly bounded), and that $G_x \geqq 0$ in the plane set S in which $G = 0$ [1]).

Let S_1 be the set (included in S) in which $G = 0, G_x > 0$ [2]). Then S_1 is of (plane) measure zero. For (1) if P or (ξ, φ) is a point of S_1, there is a linear interval $\xi < x < \xi + \delta$, $\theta = \varphi$ in which $G > 0$. But (2) if the (plane) measure of S_1 is positive, there is a line $\theta = \varphi$ on which its (linear) measure is positive, and a (ξ, φ) on this line at which it has unit (linear) density; and this contradicts (1).

Now, since G_x is bounded, we have

$$\int_{H(x_2)} F(x_2, \theta) d\theta - \int_{H(x_1)} F(x_1,\theta)\, d\theta = \int_{-\pi}^{\pi} (G(x_2, \theta) - G(x_1, \theta))\, d\theta = \iint_R G_x\, dx\, d\theta.$$

Since S_1 has measure zero we may replace R here by $R - S_1$; and since $G_x = 0$ in $S - S_1$, we may replace $R - S_1$ by $R - S$. But $G_x = F_x$ in $R - S$, so that

$$\int_{H(x_2)} F(x_2, \theta)\, d\theta - \int_{H(x_1)} F(x_1, \theta)\, d\theta = \iint_{R-S} F_x dx\, d\theta = \int_{x_1}^{x_2} dx \int_{H(x)} F_x d\theta.$$

3.3. We now write

$$f^*(r, \theta) = \int_0^r \operatorname*{Max}_{t \leqq \varrho} |f'(te^{i\theta})|\, d\varrho;$$

this function plainly majorises the function called f^* in § 3.1. The derivative

$$\frac{\partial f^*}{\partial r} = \operatorname*{Max}_{t \leqq r} |f'(te^{i\theta})|$$

is continuous, and

---

[1]) If P is a point of the rectangle, and we distinguish the five cases (1) $F > 0$ at P, (2) $F < 0$, (3) $F = 0$, $F_x > 0$, (4) $F = 0$, $F_x < 0$, (5) $F = 0$, $F_x = 0$, then $G_x = F_x$ in cases (1), (3), (5), $G_x = 0$ in cases (2), (4), (5). The last four cases correspond to S.

[2]) Case (3) only.

$$\frac{\partial^2 f^*}{\partial r^2} = \frac{\partial}{\partial r} \operatorname*{Max}_{t \leq r} |f'(te^{i\theta})|$$

is bounded and non-negative [3]).

We also write

$$F(r, \theta) = f^{*\lambda} - (1-r)^\lambda \left(\frac{\partial f^*}{\partial r}\right)^\lambda,$$

and denote by $H = H(r)$ the set of θ in which $F > 0$, by CH the complementary set. Since $f^* \leqq r\dfrac{\partial f^*}{\partial r}$, F is negative for all θ when r is sufficiently small, and $H(a)$ is nul for sufficiently small a.

Plainly

$$(3.3.1) \qquad \int_{CH} f^{*\lambda}\, d\theta \leqq (1-r)^\lambda \int_{CH} \left(\frac{\partial f^*}{\partial r}\right)^\lambda d\theta \leqq (1-r)^\lambda \int_{-\pi}^{\pi} \left(\frac{\partial f^*}{\partial r}\right)^\lambda d\theta.$$

Next, if $0 < a < r < 1$, we have, by Lemma 7,

$$(3.3.2) \qquad \int_{H(r)} F(r, \theta)\, d\theta - \int_{H(a)} F(a, \theta)\, d\theta = \int_a^r d\varrho \int_{H(\varrho)} \frac{\partial F}{\partial \varrho}\, d\theta.$$

Also

$$\frac{\partial F}{\partial \varrho} = \lambda f^{*\lambda-1} \frac{\partial f^*}{\partial \varrho} + \lambda(1-\varrho)^{\lambda-1} \left(\frac{\partial f^*}{\partial \varrho}\right)^\lambda - \lambda(1-\varrho)^\lambda \left(\frac{\partial f^*}{\partial \varrho}\right)^{\lambda-1} \frac{\partial^2 f^*}{\partial \varrho^2}.$$

The last term is negative, and

$$f^{*\lambda-1} < (1-\varrho)^{\lambda-1} \left(\frac{\partial f^*}{\partial \varrho}\right)^{\lambda-1},$$

for θ of $H(\varrho)$, since $\lambda - 1 < 0$. Hence, for θ of $H(\varrho)$,

$$\frac{\partial F}{\partial \varrho} < 2\,\lambda(1-\varrho)^{\lambda-1} \left(\frac{\partial f^*}{\partial \varrho}\right)^\lambda.$$

And if we suppose a small enough to make $H(a)$ nul, (3.2.2) gives

$$\int_{H(r)} F(r, \theta)\, dr \leqq 2\lambda \int_0^r (1-\varrho)^{\lambda-1}\, d\varrho \int_{-\pi}^{\pi} \left(\frac{\partial f^*}{\partial \varrho}\right)^\lambda d\theta,$$

or

$$(3.3.3) \qquad \int_{H(r)} f^{*\lambda}\, d\theta \leqq (1-\varrho)^\lambda \int_{-\pi}^{\pi} \left(\frac{\partial f^*}{\partial r}\right)^\lambda d\theta + 2\lambda \int_0^r (1-\varrho)^{\lambda-1} d\varrho \int_{-\pi}^{\pi} \left(\frac{\partial f^*}{\partial \varrho}\right)^\lambda d\theta.$$

Finally, combining (3.3.1) and (3.3.3), and observing that, by 'Max',

$$\int_{-\pi}^{\pi} \left(\frac{\partial f^*}{\partial r}\right)^\lambda d\theta = \int_{-\pi}^{\pi} (\operatorname*{Max}_{t \leq r} |f'(te^{i\theta})|)^\lambda\, d\theta \leqq BC(1-r)^{-\lambda},$$

we obtain

$$\int_{-\pi}^{\pi} f^{*\lambda}\, d\theta \leqq BC + BC \int_0^r \frac{dr}{1-\varrho} = BC + BC \log \frac{1}{1-r}.$$

This plainly includes (3.1.1).

[3]) It is to secure this that we use the new and not the old f^*. The second derivative is either $\dfrac{\partial}{\partial r} |f'(re^{i\theta})|$ or 0.

3.4. *Theorem* 7. *If* $0 < \lambda \leqq 1$ *and*

$$M_\lambda(u) \leqq C,$$

then

$$M_\lambda(v) \leqq BC + BC\left(\log\frac{1}{1-r}\right)^{\frac{1}{\lambda}}.$$

This follows from Theorems 2 and 6 in the same way that Theorem 4 followed from Theorems 2 and 3.

4. Negative results.

4.1. In this section we consider a number of special functions. Our object is to show, so far as we can, that the theorems of §§ 2—3 are the best of their kind, and that the possibilities which they leave open may actually occur. There are several points at which our results are not as complete as we should desire.

An elementary example.

4.2. Suppose that k is zero or a positive integer, and that

$$f(z) = e^{\frac{1}{2}k\pi i}(1-z)^{-k-1}.$$

We prove that $M_\lambda(u)$ is bounded when

$$\lambda = \frac{1}{k+1}. \tag{4.2.1}$$

Since

$$\int |f|^\lambda\, d\theta = \int |1-z|^{-1}\, d\theta \sim \log\frac{1}{1-r},$$

it will follow that the logarithmic factor in Theorem 7 cannot be improved, at any rate when k is of the form (4.2.1). We have no example to prove that the same is true generally, though this is hardly doubtful.

We write

$$z = e^{-Z},\ Z = X + iY = Re^{i\Theta} = \log\frac{1}{z},$$

so that $r = e^{-X}$, $\theta = -Y$. What we have to prove is plainly equivalent to

$$\int |\Re(e^{\frac{1}{2}k\pi i} Z^{-k-1})|^\lambda\, dY = O(1), \tag{4.2.2}$$

where the integration is along a line $X = \delta$, $-a < Y < a$, and $\delta \to 0$.

From

$$R\cos\Theta = \delta,\ R\sin\Theta = Y,$$

we deduce

$$dY = \frac{RdR}{\sqrt{R^2-\delta^2}},$$

and the integral (4.2.2) splits up into two, each of the form

$$\int_\delta^{\sqrt{a^2+\delta^2}} |\cos(\tfrac{1}{2}k\pi - (k+1)\Theta)|^\lambda \frac{dR}{\sqrt{R^2-\delta^2}}, \tag{4.2.3}$$

but Θ lying between 0 and $\frac{1}{2}\pi$ in the first integral and between $-\frac{1}{2}\pi$ and 0 in the second. The significant parts of the integrals are those for which R/δ is large, *i. e.* Θ near $\frac{1}{2}\pi$ or $-\frac{1}{2}\pi$.

In the first integral we put $\Theta = \frac{1}{2}\pi - \psi$, when

$$\cos\left(\tfrac{1}{2}k\pi - (k+1)\Theta\right) = \sin(k+1)\psi = O(\psi) = O\left(\frac{\delta}{R}\right).$$

It follows that (4.2.3) is less than a constant multiple of

$$\int_\delta^\infty \left(\frac{\delta}{R}\right)^\lambda \frac{dR}{\sqrt{R^2-\delta^2}},$$

and is therefore bounded.

In the second integral we put $\Theta = -\frac{1}{2}\pi + \psi$, when

$$\cos\left(\tfrac{1}{2}k\pi - (k+1)\Theta\right) = (-1)^k \sin(k+1)\psi;$$

it is here that the fact that k is integral is relevant. The proof then goes as before.

The function $\sum \dfrac{z^{a_n}}{1-z^{2a_n}}$.

4.3. In this paragraph we consider the function

$$f(z) = \sum_0^\infty \frac{z^{a_n}}{1-z^{2a_n}}, \tag{4.3.1}$$

where a_n is a positive integer and tends rapidly to infinity with n. Actually we shall be concerned only with the cases

$$\text{(A)}\quad a_n = 10^n, \quad \text{(B)}\quad a_n = 10^{10^n};$$

but it will be obvious that our arguments apply more generally. Our object is to give as simple an example as possible of a function $f = u + iv$ such that $M_\lambda(u)$ is bounded for every $\lambda < 1$, while $M_\lambda(v)$ is not bounded for any positive λ.

4.4. Suppose first, rather more generally, that $a_n = a^n$, where a is an integer greater than 1, that η_n is real and $|\eta_n| < C$, and that

$$f(z) = \sum \eta_n \frac{z^{a_n}}{1-z^{2a_n}}. \tag{4.4.1}$$

We shall prove that $M_\lambda(u)$ is bounded for $0 < \lambda < 1$. In proving this, we may plainly suppose $\lambda > \dfrac{1}{2}$.

We have

$$u(r,\theta) = \sum \eta_n \frac{R(1-R^2)\cos\Theta}{1-2R^2\cos 2\Theta + R^4}, \tag{4.4.2}$$

where

$$R = r^{a_n},\ \Theta = a_n\theta;$$

and so

$$\int |u|^\lambda d\theta \leqq C \sum R^\lambda (1-R^2)^\lambda \int_{-\pi}^{\pi} \frac{d\theta}{(1-2R^2\cos 2\Theta + R^4)^\lambda}.$$

Here we may replace Θ by θ. Further, we may ignore the terms of the series for which $R < \dfrac{1}{2}$, since these contribute

$$\sum_{r^{a^n} < \frac{1}{2}} r^{\lambda a^n} O(1) = O\left(2^{-\lambda} + 2^{-a\lambda} + 2^{-a^2\lambda} + \cdots\right) = O(1).$$

Finally we may replace $1 - 2R^2 \cos 2\Theta + R^4$ by $(1-R)^2 + \theta^2$ in the terms which remain. Making these simplifications, and observing that

$$\int \frac{d\theta}{((1-R)^2 + \theta^2)^\lambda} = O((1-R)^{1-2\lambda})$$

(since $\lambda > \frac{1}{2}$), we find that

$$\int |u|^\lambda \, d\theta = O(\Sigma R^\lambda (1-R)^\lambda (1-R)^{1-2\lambda}) = O(\Sigma R^\lambda (1-R)^{1-\lambda}).$$

But

$$(1-R)^{1-\lambda} = (1-r^{a^n})^{1-\lambda} < a^{(1-\lambda)n} (1-r)^{1-\lambda},$$

and so

$$\int |u|^\lambda \, d\theta = O((1-r)^{1-\lambda} \Sigma a^{(1-\lambda)n} r^{\lambda a^n}) = O((1-r)^{1-\lambda} (1-r)^{-1+\lambda}) = O(1).$$

In particular, the proof applies to case (A). *A fortiori* it gives the corresponding result in case (B), since the (a_n) of case (B) is a selection from the (a_n) of case (A).

4.5. It is more difficult to prove that $M_\lambda(v)$ is not bounded. We have not succeeded in finding a proof in case (A) (though we have no doubt of the conclusion), and our proof in case (B) is indirect.

If $M_\lambda(v)$, and so $M_\lambda(f)$, is bounded, $f(re^{i\theta})$ has (by the theorem of Fatou and F. and M. Riesz) a radial limit for almost all θ. We shall prove that, when $a_n = 10^{10^n}$, this is untrue.

We take

$$r = e^{-t}, \quad \frac{1}{t} = 10^{10^{N+\frac{1}{2}}},$$

where N is an integer, and write

$$f = \sum_0^N + \sum_{N+1}^\infty = f_1 + f_2.$$

Then

$$r^{10^{10^{N+1}}} = e^{-10^{10^{N+1} - 10^{N+\frac{1}{2}}}}$$

is extremely small when N is large, and f_2 tends rapidly to zero, uniformly in θ.

Next

$$\frac{z^{a_n}}{1 - z^{2a_n}} - \frac{e^{a_n i\theta}}{1 - e^{2a_n i\theta}} = -\frac{(1-r^{a_n})\, e^{a_n i\theta} + r^{a_n}(1-r^{a_n})\, e^{3a_n i\theta}}{(1 - r^{2a_n} e^{2a_n i\theta})(1 - e^{2a_n i\theta})} = O((1-r^{a_n}) \operatorname{cosec}^2 a_n \theta),$$

uniformly in θ; and so

$$(4.5.1) \qquad f_1 - \sum_0^N \frac{e^{a_n i\theta}}{1 - e^{2a_n i\theta}} = O(N(1 - r^{a_N}) \operatorname*{Max}_{n \leq N} \operatorname{cosec}^2 a_n \theta).$$

Since

$$1 - r^{a_N} = 1 - e^{-10^{10^N - 10^{N+\frac{1}{2}}}} = O(10^{-10^{N+\frac{1}{2}} + 10^N}) = O(10^{-K \cdot 10^N}),$$

for a constant positive K, the right hand side of (4.5.1) is

$$(4.5.2) \qquad O(10^{-K \cdot 10^N} \operatorname*{Max}_{n \leq N} \operatorname{cosec}^2 a_n \theta).$$

If this tends to zero, and if f tends to a limit when $r \to 1$ (*a fortiori* when $N \to \infty$), then the sum in (4.5.1) will tend to a limit when $N \to \infty$, which is obviously false what-

ever be θ. It is therefore sufficient for our purpose that (4.5.2) tends to zero for almost all θ; and *a fortiori* sufficient to prove that

$$(4.5.3) \qquad |\operatorname{cosec} 10^{10^n}\theta| = O(10^{\varepsilon 10^n})$$

for every positive ε and almost all θ.

That this is so results from some older work of our own on Diophantine approximation [1]). We know that, if m is the number of zeros in the first n places of the decimal

$$\frac{\theta}{2\pi} = \alpha_1 \alpha_2 \alpha_3 \ldots,$$

then

$$m = \frac{n}{10} + O(\sqrt{n}\log n)$$

for almost all θ. In this case there cannot be, among the first n figures, a block of consecutive zeros of order $\sqrt{n}(\log n)^2$. Hence

$$|\operatorname{cosec} 10^n\theta| = O(10^{\sqrt{n}(\log n)^2})$$

for almost all θ. When n is replaced by 10^n, this is much stronger than (4.5.3).

It does not seem to be possible to adapt this proof to case (A), the increase of a_n not being sufficiently rapid. Mr R. E. A. C. Paley has proved that $M_\lambda(v)$ is unbounded when

$$f(z) = \sum \varepsilon_n \frac{z^{a^n}}{1 - z^{2a^n}},$$

and ε_n is always ± 1, for 'almost all sequences' (ε_n); but we know of no proof valid for any particular sequence.

The function $\sum \frac{(-1)^n}{n}\frac{z^n}{1+z^n}$.

4.6. We consider last the function

$$(4.6.1) \qquad f(z) = \sum_1^\infty \frac{(-1)^n}{n}\frac{z^n}{1+z^n}.$$

We shall prove that (i) $M_\lambda(u)$ is bounded for $0 < \lambda < 1$, (ii) $M_\lambda(v)$ is not bounded for any positive λ, and (iii) u does not possess a boundary function U defined by radial approach. Our proofs are much less elementary, but the function (4.6.1) is much more important. In fact if we write momentarily q instead of z, we have, in the ordinary notation of the theory of elliptic functions [2]),

$$k = \frac{\vartheta_2^2(0,\tau)}{\vartheta_3^2(0,\tau)} = 4q^{\frac{1}{2}}\prod_1^\infty\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^4,$$

$$\log k = \log 4 + \frac{1}{2}\log q + 4\sum_1^\infty \frac{(-1)^n}{n}\frac{q^n}{1+q^n},$$

so that $f(q)$ is substantially $4\log k$.

The function (4.6.1) is one of the same character as that used by Littlewood (8, 12), and shown by him to possess the same peculiarities; but the analysis necessary is a good deal simpler. Littlewood used

1) Hardy and Littlewood (2), Theorem 1. 45. Khintchine (6) has proved a stronger (and a best possible) result.

2) We follow Tannery and Molk (15).

$$\bar{Q}(z) = \log k^2(\tau),$$

where z is (for example) $\dfrac{\tau - i}{\tau + i}$, while our z is $e^{\pi i \tau}$, or

$$\exp\left(-\pi \frac{1+z}{1-z}\right)$$

in his notation.

4.7. We prove first that $M_\lambda(u)$ is bounded for $\lambda < 1$; we may suppose $\lambda > \dfrac{1}{2}$. We use the notation and results of our paper 3 [1]), and the method of 'Farey dissection'.

We write

$$z = e^{\pi i \tau} = e^{\pi i (x+iy)} = e^{-\pi y} e^{\pi i x} = re^{i\theta},$$

$$r = e^{-\pi y}, \quad \theta = \pi x,$$

and we integrate round the circle $y = \dfrac{A}{n}$, where A, as in the sequel, is an absolute constant, using the Farey dissection of order $\nu = [\sqrt{n}]$.

The typical Farey arc $\xi_{p,q}$ is defined by

$$\frac{p}{q} - \frac{\alpha}{q\sqrt{n}} \leqq x \leqq \frac{p}{q} + \frac{\beta}{q\sqrt{n}},$$

where α and β lie between two A's. On $\xi_{p,q}$ we write

$$\mathsf{T} = \frac{c + d\tau}{a + b\tau},$$

with

$$a = p, \quad b = -q, \quad c = \pm p', \quad d = \mp q', \quad ad - bc = 1.$$

Then

$$\vartheta_\mu(0, \tau) = \frac{1}{\omega\sqrt{a + b\tau}} \vartheta_\varrho(0, \mathsf{T}),$$

where $|\omega| = 1$, μ is 2 or 3, and ϱ is one of 2, 3, 4; and

$$u = \frac{1}{2}\log\left|\frac{\vartheta_2(0,\tau)}{\vartheta_3(0,\tau)}\right| + O(1) = \frac{1}{2}\log\left|\frac{\vartheta_\varrho(0,\mathsf{T})}{\vartheta_\sigma(0,\mathsf{T})}\right| + O(1),$$

where each of ϱ and σ is 2, 3, or 4, and $\varrho \neq \sigma$.

We write [2])

$$Z = e^{\pi i \mathsf{T}} = e^{\pi i (X+iY)} = e^{-\pi Y} e^{\pi i X},$$

and consider the expansions of ϑ_ϱ, ϑ_σ in powers of Z. Of the possible ϑ_ϱ, ϑ_σ two begin with 1 and the third (ϑ_2) with $2Z^{\frac{1}{4}}$.

Let us assume that (as we shall prove in a moment) there is an A such that $|Z| \leqq A < 1$. Then, if both ϑ_ϱ and ϑ_σ begin with 1, we have

(4.7.1) $$u = O(1),$$

while if one begins with $2Z^{\frac{1}{4}}$ we have

(4.7.2) $$u = O(|\log|Z||) = O(Y).$$

Indeed (as we shall require to know later), u is actually of order Y.

[1]) But we do not use q for z (q being required otherwise).

[2]) Here again we diverge from 3, where Z is Q and Y is λ.

4.8. We have [1])

$$Y = \Im\Big(-\frac{1}{b(a+b\tau)}\Big) = \Im\Big(\frac{1}{q(p-q(x+iy))}\Big) = \frac{y}{(p-qx)^2+q^2y^2}.$$

Here $y = \frac{A}{n}$ and

$$|p-qx| = \Big|p - q\Big(\frac{p}{q}+\varphi\Big)\Big| = |q\varphi|,$$

where $0 \leqq |\varphi| < \frac{A}{q\sqrt{n}}$. Hence

$$\frac{\frac{A}{n}}{q^2\Big(\varphi^2+\frac{1}{n^2}\Big)} < Y < \frac{\frac{A}{n}}{q^2\Big(\varphi^2+\frac{1}{n^2}\Big)}. \tag{4.8.1}$$

From (4.8.1) it follows, first, that

$$Y > \frac{\frac{A}{n}}{q^2\Big(\frac{1}{q^2 n}+\frac{1}{n^2}\Big)} > \frac{An}{q^2} > A,$$

and so that $|Z| < A < 1$, and we may use (4.7.1) and (4.7.2).

Next we have, in either case,

$$u = O(Y) = O\left(\frac{1}{nq^2}\frac{1}{\varphi^2+\frac{1}{n^2}}\right),$$

$$\int_{\xi_{p,q}} |u|^\lambda d\theta = O\left(n^{-\lambda}q^{-2\lambda}\int_{-A/q\sqrt{n}}^{A/q\sqrt{n}} \frac{d\varphi}{\Big(\varphi^2+\frac{1}{n^2}\Big)^\lambda}\right);$$

and $\Big(\text{since } \lambda > \frac{1}{2}\Big)$ the integral here is

$$O\Big(n^{2\lambda-1}\int_0^\infty \frac{dt}{(t^2+1)^\lambda}\Big) = O(n^{2\lambda-1}).$$

Finally

$$\int |u|^\lambda\, d\theta = \sum \int_{\xi_{p,q}} |u|^\lambda\, d\theta = O(n^{-\lambda}\cdot n^{2\lambda-1}\cdot \sum_{q,p} q^{-2\lambda})$$

$$= O(n^{\lambda-1}\sum_{q=1}^{\nu} q^{1-2\lambda}) = O(n^{\lambda-1}(\sqrt{n})^{2-2\lambda}) = O(1).$$

We are unable to prove this result in any more elementary manner; but the proof is as simple as a proof based on 'Farey dissection' can well be.

4.9. It remains to prove that $M_\lambda(v)$ is not bounded for any positive λ. If $M_\lambda(v)$ is bounded, f, and *a fortiori* u, has a finite radial limit for almost all θ. It is therefore sufficient to prove that

$$\overline{\lim_{r\to 1}}\, |u| = \infty$$

in a set of θ of positive measure.

1) See 8, 227.

We suppose that $0 < x = \dfrac{\theta}{\pi} < 1$ and that

$$x = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots;$$

and that $\dfrac{p_n}{q_n}$ is the convergent to x which precedes a_{n+1}. We shall say that x is *favourable* if (i) $a_{n+1} \to \infty$ when $n \to \infty$ through an appropriate sequence n_i, and (ii) p_{n_i} is odd for an infinity of n_i. Our conclusion will be established if we prove (a) that $|u|$ is unbounded when x is favourable and (b) that the measure of the set of favourable x is positive.

To prove (a) we take, as in 3, p. 226,

$$a = \pm p_n, \quad b = \mp q_n, \ldots,$$

and (as on p. 227)

$$r = e^{-\pi y}, \quad y = \frac{1}{q_n q'_{n+1}}.$$

Then

$$Y = \frac{q'_{n+1}}{2q_n} > \frac{1}{2} a'_{n+1}$$

and tends to infinity if n tends to infinity through the values n_i. Also, if we examine the cases which, on p. 227, we called $1^0 - 6^0$, we find that p_n is odd in cases $1^0 - 4^0$; and these are just the cases in which one of the ϑ_ϱ and ϑ_σ of § 4.7 is ϑ_2 [1]). In these cases $|u|$ is actually of order Y, and accordingly tends to infinity.

It remains to prove (b). We denote by E the set of x in (0, 1) which satisfy condition (i), by E_1 the set of x which also satisfy condition (ii). By a well-known theorem of Borel and Bernstein, CE has measure zero.

Let

$$x' = \frac{1}{1+x} = \frac{1|}{|1} + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \cdots,$$

so that x' lies in $(\frac{1}{2}, 1)$; and let us define E' and E'_1 as the sets of x' which satisfy the conditions just imposed on the sets E, E_1 of x. The measure of CE', in $(\frac{1}{2}, 1)$, is zero. Hence, ignoring nul-sets, every number of $(\frac{1}{2}, 1)$ belongs both to E (considered as an x) and to E' (considered as an x'). Also the convergent to x' which precedes a_{n+1} is

$$\frac{q_n}{q_n + p_n}.$$

Since either p_n or q_n is odd, the measure of $E_1 + E'_1$ is $\frac{1}{2}$, and one at least of E_1 and E'_1 has measure not less than $\frac{1}{4}$.

We have thus proved what we stated at the beginning of § 4.6. In fact more is true; the set in which u tends to a limit is nul; but to prove this demands a rather more detailed analysis [2]).

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[1]) See Tannery and Molk (**15**, t. 2, 241 and 262, Tables XX and XLII).

[2]) Compare Littlewood (**6**, 516), where the full result is proved for $\bar{Q}(z)$.

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—— **13.** On a theorem of Zygmund, Journal London Math. Soc. **4** (1929), 305—307.

J. D. Tamarkine. **14.** Some points in the theory of conjugate functions (to appear shortly).

J. Tannery et J. Molk. **15.** Théorie des fonctions elliptiques, 1902.

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A. Zygmund. **17.** Sur les fonctions conjuguées, Fundamenta Math. **13** (1929), 284—303.

— **18.** Some points in the theory of conjugate functions (to appear shortly).

Eingegangen 11. November 1931.

CORRECTION

p. 409, *line 5 from below*. The upper limits in the summations on the right should be $n-1$.
p. 411, (2.6.3). The term $|u(r, \theta)|$ in the bracket should be multiplied by 2.
p. 411, *footnote*. For Theorem 33 read Theorem 32.
p. 412, *line* 5. For 2 read 3.
p. 413, *first line of statement of Theorem* 4. For $a > 1$ read $a > 0$.
p. 413, (3.1,2). For $\lambda < 1$ read $\lambda \leqslant 1$.

COMMENTS

An alternative proof of Theorem 2 has been given by A. E. Gwilliam, *Proc. London Math. Soc.* (2), 40 (1936), 353–64.

Some results relating to the theorems of this paper, involving one-sided conditions for $u(r,\theta)$, have been given by M. L. Cartwright, *Quart. J. of Math.* 4 (1933), 246–57, and C. N. Linden, *Proc. Camb. Phil. Soc.* 58 (1962), 26–37.

Some properties of fractional integrals. II.

By

G. H. Hardy in Oxford and J. E. Littlewood in Cambridge.

1. Introduction.

1.1. This paper does not demand much preliminary explanation, since it is essentially an extension to the complex field of our earlier paper (7)[1]) with the same title.

In 7 we were concerned with functions of a real variable belonging to one or other of the "Lebesgue classes" L^p, where $p \geqq 1$. Here our functions are analytic (or harmonic) functions of $z = re^{i\theta}$, where $0 \leqq r < 1$; and a function is said to belong to L^p when the mean M_p defined below is bounded. The scope of our analysis is then enlarged in two ways. In the first place we have to consider the range $p > 0$ instead of the range $p \geqq 1$. Secondly we have to consider not only when a mean M_p is bounded but also when it is of a given order $(1-r)^{-a}$. The chief novelties of the paper arise from the first extension, the case $p < 1$ being decidedly more difficult than the case $p \geqq 1$.

The contrast between these two cases becomes much more striking when we consider the analogous problems connected with a pair of conjugate harmonic functions u, v, since then it affects the results as well as the methods of proof. These problems we reserve for a separate paper.

The means M_p and the classes L^p.

1.2. Suppose that

$$f(z) = f(re^{i\theta}) \qquad (r \geqq 0)$$

1) See the list of references at the end. We number our theorems in succession to those of 7.

is an analytic function of z regular for $r = |z| < 1$; and that $p > 0$. Then we write

$$M_p = M_p(f) = M_p(r) = M_p(r, f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta\right)^{\frac{1}{p}}.$$

The following properties of these means are well known.

(1) M_p is an increasing function of p.

(2) $\log M_p^p$ is a convex function of p.

(3) M_p tends to

$$M = M(f) = M(r) = M(r, f),$$

the maximum modulus of $f(z)$, when $p \to \infty$.

(4) M_p tends to

$$M_0 = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})| \, d\theta\right)$$

when $p \to 0$.

(5) M_p is an increasing function of r.

(6) $\log M_p$ is a convex function of $\log r$[2]).

If, for any p, $M_p(f) < K$, where K is independent of r, we say that $f(z)$ *belongs to the* (complex) *class* L^p or, more shortly, that $f(z)$ *is* L^p.

We may frame similar definitions for harmonic functions. Suppose that

$$u(z) = u(re^{i\theta}) = u(r, \theta)$$

is a complex harmonic function regular for $r < 1$. Then we write

$$M_p = M_p(u) = M_p(r) = M_p(r, u) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|^p \, d\theta\right)^{\frac{1}{p}},$$

and define the classes L^p as before. These means possess properties analogous to (1)—(6) when $p > 1$. We shall not be concerned in this paper with means of harmonic functions in which $p \leqq 1$.

1.3. We collect here, in the form of a comprehensive lemma, a number of known theorems concerning power series, Fourier series, and boundary functions.

We write

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}, \qquad Q(r, \theta) = \frac{2r\sin\theta}{1 - 2r\cos\theta + r^2}.$$

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[2]) For (1)–(4) see Littlewood (**11**), and Hausdorff's paper there referred to; for (5) and (6) see Hardy (**2**), Landau (**10**), Pólya und Szegö (**12**, p. 143, 329), F. Riesz (**13**, **14**).

Lemma A. (1) *If* $p>0$, $f(z)=f(re^{i\theta})$ *is regular for* $r<1$, *and* $M_p(f)\leqq C$, *then* $f(z)$ *tends radially to a limit*

$$f(e^{i\theta})=F(\theta)$$

for almost all θ, *and*

$$M_p(F)=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|F(\theta)|^p\,d\theta\right)^{\frac{1}{p}}\leqq C.$$

(2) *If further* $p\geqq 1$, *then* $f(z)$ *is the Poisson integral, and the Cauchy integral, of* $F(\theta)$; *i. e.*

$$f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}P(r,\varphi-\theta)\,F(\varphi)\,d\varphi=\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{F(\varphi)\,de^{i\varphi}}{e^{i\varphi}-z}.$$

(3) *If* $p>0$, $F(\theta)$ *is a function of* L^p *whose Fourier series is a Fourier power series*

$$\sum_0^\infty c_n e^{ni\theta},$$

and $f(z)=\sum c_n z^n$, *then* $M_p(f)\leqq M_p(F)$; *and, when* $p\geqq 1$, $f(z)$ *and* $F(\theta)$ *are related as in* (1) *and* (2) *above.*

(4) *If* $p>1$, $u(z)=u(r,\theta)$ *is a (real or complex) harmonic function regular for* $r<1$, *and* $M_p(u)\leqq C$, *then* $u(r,\theta)$ *tends radially to a limit*

$$u(1,\theta)=U(\theta)$$

for almost all θ, *and* $M_p(U)\leqq C$. *Also* $u(r,\theta)$ *is the Poisson integral of* $U(\theta)$.

(5) *If* $p>1$, $U(\theta)$ *is a function of* L^p *whose Fourier series is*

$$\sum_{-\infty}^{\infty}c_n e^{ni\theta}\quad\left[or\ \ \frac{1}{2}a_0+\sum_1^\infty(a_n\cos n\theta+b_n\sin n\theta)\right],$$

and

$$u(r,\theta)=\sum_{-\infty}^{\infty}c_n r^{|n|}e^{ni\theta}\left[or\ \ \frac{1}{2}a_0+\sum_1^\infty(a_n\cos n\theta+b_n\sin n\theta)\,r^n\right],$$

then $M_p(u)\leqq M_p(U)$, *and* $u(r,\theta)$ *and* $U(\theta)$ *are related as in* (4) *above.*

(6) *If* $p>1$ *and* u *is a real harmonic function for which* $M_p(u)\leqq C$, *then there are positive harmonic functions* u_1 *and* u_2 *such that*

$$u=u_1-u_2,\qquad M_p(u_1)\leqq C,\qquad M_p(u_2)\leqq C.$$

For proofs and explanations of these propositions we may refer to F. and M. Riesz (**15**), F. Riesz (**13**), M. Riesz (**16**), Hardy and Littlewood (**5**). It is to be observed that here, and throughout the paper, we use C (as

opposed to K) for a number which preserves its identity throughout a theorem and its proof.

The various clauses of Lemma A indicate important differences between the ranges $p>1$ and $p \leqq 1$. The most striking of these appears in the well known theorem of M. Riesz. If $p>1$, u is L^p, and v is a conjugate of u, then v and $f(z)=u+iv$ are L^p [3]). It is familiar that this is false for $p=1$ [4]), and there are simple examples which show that it is also false for $p<1$. When $p<1$, however, the proposition is false in a much more comprehensive sense, to which we shall return in the paper mentioned at the end of § 1.1.

We shall also make much use of

Lemma B [5]). *If $p>0$, $f(z)$ is regular for $r<1$, and $M_p(f) \leqq C$, then*

$$f=f_1+f_2,$$

where (a) f_1 and f_2 are regular for $r<1$, (b) f_1 and f_2 have no zeros for which $r<1$, and (c)

$$M_p(f_1) \leqq 2C, \quad M_p(f_2) \leqq 2C.$$

The result is true in the limiting case $p=\infty$, M_p being replaced by M.

2. Some theorems concerning the means M_p.

2.1. Theorem 27. *If $p>0$, $a \geqq 0$,*

$$M_p(f) \leqq C(1-r)^{-a}, \tag{2.11}$$

and $q>p$, then

$$M_q(f) \leqq KC(1-r)^{-a-\frac{1}{p}+\frac{1}{q}}, \tag{2.12}$$

$$M(f) \leqq KC(1-r)^{-a-\frac{1}{p}}, \tag{2.13}$$

where $K=K(p,a)$.

If $a=0$, then also

$$M_q(f)=o\left((1-r)^{-\frac{1}{p}+\frac{1}{q}}\right), \tag{2.14}$$

$$M(f)=o\left((1-r)^{-\frac{1}{p}}\right), \tag{2.15}$$

when $r \to 1$.

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3) M. Riesz, **16**.

4) The example being

$$u=P(r,\theta), \quad v=Q(r,\theta).$$

5) See **4**, p. 207.

When $p \geqq 1$, all this is proved in our paper 8 [6]), so that we need only consider the case $p < 1$.

(i) Suppose first that $f(z)$ has no zeros in the unit circle. Then we may write $f = \varphi^k$, where $k = k(p)$, $kp \geqq 1$; so that

$$M_{kp}(\varphi) \leqq C^{\frac{1}{k}}(1-r)^{-\frac{a}{k}},$$

and therefore

$$M_{kq}(\varphi) \leqq K C^{\frac{1}{k}}(1-r)^{-\frac{a}{k}-\frac{1}{kp}+\frac{1}{kq}},$$

where $K = K(kp, a, k) = K(p, a)$. This is equivalent to (2.12); and (2.13), (2.14), and (2.15) follow similarly.

(ii) Passing to the general case, suppose that

$$0 < r < 1, \qquad \varrho = \frac{1}{2}(1+r).$$

Then

$$M_p(f) \leqq C(1-\varrho)^{-a} = C_\varrho$$

for $r < \varrho$. By Lemma B, we can find f_1 and f_2 so that $f = f_1 + f_2$, $f_1 \neq 0$, $f_2 \neq 0$, and

$$M_p(f_1) \leqq 2\,C_\varrho, \qquad M_p(f_2) \leqq 2\,C_\varrho \tag{2.16}$$

for $r < \varrho$ [7]). Applying the result of (i) to $f_1(z) = F_1(\varrho\zeta)$ in the circle $|\zeta| < 1$, we obtain

$$M_q(f_1) \leqq 2\,K\,C_\varrho\left(1-\frac{r}{\varrho}\right)^{-\frac{1}{p}+\frac{1}{q}}$$

$$= 2\,K\,C(1-\varrho)^{-a}\left(1-\frac{r}{\varrho}\right)^{-\frac{1}{p}+\frac{1}{q}} \leqq K\,C(1-r)^{-a-\frac{1}{p}+\frac{1}{q}}.$$

There is a similar inequality for f_2; and, since

$$M_q(f) \leqq K\,M_q(f_1) + K\,M_q(f_2),$$

we deduce

$$M_q(f) \leqq K\,C(1-r)^{-a-\frac{1}{p}+\frac{1}{q}},$$

which is (2.12). The proofs of the other results follow the same line.

It should however be observed that there is a difference between the logic of the arguments for the cases $a > 0$ and $a = 0$. Our argument

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[6]) Theorem 2, pp. 623–625. The result is stated in "O" form, but an examination of the argument shows at once that K depends on p and a only.

[7]) f_1 and f_2 depend upon ϱ and are regular only for $r < \varrho$. Here and elsewhere we use the obvious extension of Lemma B to circles of any radius.

under (ii) shows in effect that "it is sufficient to prove the theorem for a function which has no zeros in the unit circle"; and this although we cannot assert that a function satisfying the conditions of the theorem is necessarily the sum of two functions which also satisfy them and have no zeros in the circle. When $a=0$ the matter is simpler, since we can apply Lemma B directly to the unit circle, this being indeed essential, since the proof when $p\geqq 1$ involves the boundary function. Arguments of the same type will recur frequently in the sequel, and we shall not always write them out in full.

2.2. Theorem 28. *If*

$$f(z)=\sum a_n z^n$$

satisfies (2.11), *then*

$$a_n = O\left(n^{a+\frac{1}{p}-1}\right) \qquad (p\leqq 1), \tag{2.21}$$

$$a_n = O(n^a) \qquad (p>1). \tag{2.22}$$

If $a=0$, we may replace O by o.

If Γ is the circle $r=e^{-\frac{1}{n}}$, we have

$$|a_n| = \left|\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z^{n+1}}\,dz\right| \leqq e M_1\left(e^{-\frac{1}{n}}\right).$$

If $p\leqq 1$ the results follow from (2.12) and (2.14). If $p>1$, we have only to observe that $M_1\leqq M_p$.

The examples

$$f(z)=(1-z)^{-a-\frac{1}{p}} \qquad (p\leqq 1),$$

$$f(z)=\sum 2^{an} z^{2^n} \qquad (p>1),$$

show that the indices in (2.21) and (2.22) are the best possible.

Returning to Theorem 27, we observe that when $p\leqq 2$ we can assert more than is asserted by (2.13) and (2.15).

Theorem 29. *If $0<p\leqq 2$, $a\geqq 0$, and $f(z)$ satisfies* (2.11), *then*

$$\mathfrak{F}(r)=\sum |a_n|\, r^n \leqq K\,C(1-r)^{-a-\frac{1}{p}}. \tag{2.23}$$

If $a=0$, then

$$\mathfrak{F}(r)=o\left((1-r)^{-\frac{1}{p}}\right). \tag{2.24}$$

This is proved for $1<p\leqq 2$ in our paper 8 [8]) (for $p>2$ it is false). If $p\leqq 1$, it is an immediate corollary of Theorem 28.

[8]) p. 625.

3. The fractional integrals of an analytic function.

3.1. Suppose that

$$f(z)=\sum a_n z^{n+c}=z^c\sum a_n z^n=z^c g(z), \tag{3.11}$$

where c is real[9]), z^c has its principal value

$$z^c=e^{c\log z}\qquad(-\pi<\Im\log z\leqq\pi),$$

and $g(z)$ is regular for $r<1$; and that α is any real number. Then we define $f_\alpha(z)$, the integral of $f(z)$ of order α, by the formula

$$f_\alpha(z)=\sum\frac{\Gamma(n+1+c)}{\Gamma(n+1+c+\alpha)}a_n z^{n+c+\alpha}, \tag{3.12}$$

where $z^{c+\alpha}$ has its principal value. The definition obeys the formal laws

$$(f_\alpha)_\beta=(f_\beta)_\alpha=f_{\alpha+\beta},$$

and $f_\alpha(z)$ reduces to a derivative of $f(z)$ when α is a negative integer. It is the definition adopted by Hadamard[10]).

In particular, when $c=0$, so that $f(z)$ is regular, we have

$$f_\alpha(z)=\sum\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}a_n z^{n+\alpha}, \tag{3.13}$$

$$f_{-m}(z)=\sum_m^\infty n(n-1)\dots(n-m+1)\,a_n z^{n-m}=\left(\frac{d}{dz}\right)^m f(z). \tag{3.14}$$

It is with this case that we shall usually be concerned. It is then sometimes convenient to write

$$f_\alpha(z)=z^\alpha g_{(\alpha)}(z), \tag{3.15}$$

so that $g_{(\alpha)}(z)$ is regular.

We may extend the definitions of the means M_p and the classes L^p, given in § 1.2, to the functions $f_\alpha(z)$ and $g_{(\alpha)}(z)$. The means of f_α and $g_{(\alpha)}$ differ only by a factor r^α and are for our purposes equivalent.

If $\alpha>0$ then

$$f_\alpha(z)=\frac{1}{\Gamma(\alpha)}\int_0^z(z-u)^{\alpha-1}f(u)\,du, \tag{3.16}$$

the path of integration being rectilinear and $(z-u)^{\alpha-1}$ having its principal value at the origin. This is the natural extension to analytic functions of Liouville's definition of the fractional integral.

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[9]) There is no reason for this restriction except that real values of c are the only values relevant here.

[10]) Hadamard, 1, p. 56.

3.2. In 7 we used two definitions of the integral $F_\alpha(\theta)$, where $\alpha > 0$, of an integrable function $F(\theta)$, viz. Liouville's and Weyl's, which proved to be for our purposes equivalent. It is the second, which requires that $a_0 = 0$ or

$$\int_{-\pi}^{\pi} F(\theta)\, d\theta = 0,$$

which is relevant here. Then Weyl's definition is

$$F_\alpha(\theta) = \int_{-\infty}^{\theta} (\theta - u)^{\alpha-1} F(u)\, du,$$

and $F_\alpha(\theta)$ is integrable, periodic, and has mean-value 0. Further if, as here, $F(\theta)$ is of power series type, we have

$$i^\alpha F_\alpha(\theta) \sim \sum_1^\infty \frac{a_n}{n^\alpha} e^{ni\theta}\ {}^{11)};$$

and $i^\alpha F_\alpha(\theta)$ is, by Lemma A (3), the boundary function of the analytic function

$$\gamma_\alpha(z) = \sum_1^\infty \frac{a_n}{n^\alpha} z^n.$$

The factors i^α here, and z^α in (3.15), correspond to the difference between integration with respect to θ and with respect to $z = re^{i\theta}$.

The functions $g_{(\alpha)}(z)$ and $\gamma_\alpha(z)$ are (when, as here, $\alpha > 0$) equivalent for our present purposes. For if $M_1(f) \leqq C$ then

$$|a_n| \leqq \lim_{r\to 1} r^{-n} M_1(f) \leqq C;$$

and

$$\left| \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)} - \frac{1}{n^\alpha} \right| < \frac{K}{n^{1+\alpha}},$$

where $K = K(\alpha)$; so that

$$|g_{(\alpha)} - \gamma_\alpha| \leqq |a_0| + K \sum_1^\infty \frac{|a_n| r^n}{n^{1+\alpha}} \leqq KC.$$

We are thus able to translate many of the theorems of 7 into our present language.

Suppose in particular that $p > 1$ and

$$M_p(f) \leqq C.$$

[11]) See Weyl, 17. Weyl is concerned with general Fourier series.

Then $F(\theta)$ belongs to L^p and

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|F(\theta)|^p d\theta\right)^{\frac{1}{p}} \leqq C.$$

If now $0 < \alpha < \frac{1}{p}$ and

$$q = \frac{p}{1-p\alpha} > p, \quad \alpha = \frac{1}{p} - \frac{1}{q},$$

we have, by Theorem 4 of 7

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|F_\alpha(\theta)|^q d\theta\right)^{\frac{1}{q}} \leqq K\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|F(\theta)|^p d\theta\right)^{\frac{1}{p}} \leqq KC,$$

where $K = K(p,\alpha) = K(p,q)$. It follows, by Lemma A (3), that

$$M_q(\gamma_\alpha) \leqq KC,$$

$$M_q(g_{(\alpha)}) \leqq KC,$$

$$M_q(f_\alpha) = r^\alpha M_q(g_{(\alpha)}) \leqq KC.$$

It is plain that a slight modification of the argument will prove that

$$M_q(f_\alpha) \leqq K M_p(f).$$

We thus obtain the following theorem.

Theorem 30. *If*

$$p > 1, \quad 0 < \alpha < \frac{1}{p}, \quad q = \frac{p}{1-p\alpha},$$

and

$$M_p(f) \leqq C,$$

then

$$M_q(f_\alpha) \leqq K M_p(f) \leqq KC,$$

where $K = K(p,\alpha)$.

We shall return to this theorem later and extend it to all positive values of p. The case $p = 2$, $q > 2$ is required in the proof of Theorem 31.

4. The means M_p (continued).

4.1. Theorem 31. *Suppose that*

$$0 < p < q, \quad \alpha = \frac{1}{p} - \frac{1}{q}, \quad l \geqq p, \tag{4.11}$$

and

$$M_p(r) = M_p(r,f) \leqq C. \tag{4.12}$$

Then

(4.13) $$\int_0^1 (M_q(r))^l (1-r)^{l\alpha-1}\, dr \leqq K C^l,$$

(4.14) $$\int_0^1 (M(r))^l (1-r)^{\frac{l}{p}-1}\, dr \leqq K C^l.$$

In particular

(4.15) $$\int_0^1 (M_q(r))^{\frac{pq}{q-p}}\, dr \leqq K C^{\frac{pq}{q-p}},$$

(4.16) $$\int_0^1 (M(r))^p\, dr \leqq K C^p,$$

(4.17) $$\int_0^1 (M_q(r))^p (1-r)^{-\frac{p}{q}}\, dr \leqq K C^p,$$

(4.18) $$\int_0^1 M_q(r)(1-r)^{\alpha-1}\, dr \leqq K C \qquad (p \leqq 1),$$

(4.19) $$\int_0^1 M(r)(1-r)^{\frac{1}{p}-1}\, dr \leqq K C \qquad (p \leqq 1).$$

Here $K = K(p, q, l)$, $K = K(p, q)$, *or* $K = K(p)$, *according to the parameters which appear in the conclusion.*

The equations (4.14), (4.16), and (4.19) correspond to $q = \infty$.

4.2. (i) Whether q is finite or infinite, we can reduce the general case $l \geqq p$ to the special case $l = p$. For (supposing for example q finite) we have, by Theorem 27,

$$M_q(r) \leqq K C (1-r)^{-\frac{1}{p}+\frac{1}{q}}$$

and so

$$M_q^l(r)(1-r)^{l\alpha-1} \leqq K C^{l-p} M_q^p(r)(1-r)^{-\frac{p}{q}}.$$

This reduction is also valid for $q = \infty$.

(ii) If $f(z)$ has no zeros in the unit circle, we write

$$k = \frac{2q}{p} > 2, \qquad f = g^{\frac{2}{p}},$$

so that

$$M_2^2(g) = M_p^p(f) \leqq C^p,$$

and

(4.21) $$\int_0^1 M_q^p(r, f)(1-r)^{-\frac{p}{q}}\, dr = \int_0^1 M_k^2(r, g)(1-r)^{-\frac{2}{k}}\, dr.$$

Suppose now that

$$g(z) = \sum c_n z^n = \sum \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}-\frac{1}{k}\right)} b_n z^n, \qquad h(z) = \sum b_n z^n,$$

so that

$$g(z) = z^{-\frac{1}{2}+\frac{1}{k}} h_{\frac{1}{2}-\frac{1}{k}}(z).$$

Taking $p = 2$, $q = k$ in Theorem 30, we find

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(r,\theta)|^k d\theta \leqq K\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |h(r,\theta)|^2 d\theta\right)^{\frac{1}{2}k},$$

or

$$M_k^k(r,g) \leqq K\left(\sum |b_n|^2 r^{2n}\right)^{\frac{1}{2}k}.$$

Hence

$$\int_0^1 M_k^2(r,g)(1-r)^{-\frac{2}{k}} dr \leqq K\sum |b_n|^2 \int_0^1 r^{2n}(1-r)^{-\frac{2}{k}} dr$$

$$\leqq K\sum\left(\frac{\Gamma\left(n+\frac{3}{2}-\frac{1}{k}\right)}{\Gamma(n+1)}\right)^2 \frac{\Gamma(2n+1)}{\Gamma\left(2n+2-\frac{2}{k}\right)} |c_n|^2$$

$$\leqq K\sum |c_n|^2 = K M_2^2(1,g) \leqq K C^p;$$

and this and (4.21) give (4.17).

We can now extend the result to general f by Lemma B, writing $f = f_1 + f_2$, where

$$M_p(r,f_1) \leqq KC, \quad M_p(r,f_2) \leqq KC,$$

applying (4.17) to f_1 and f_2, and observing finally that

$$M_q^p(r,f) \leqq K M_q^p(r,f_1) + K M_q^p(r,f_2).$$

(iii) There remains the case $q = \infty$. If $f(z)$ has no zeros, we define $g(z)$ as before, when

$$M(f) = M^{\frac{2}{p}}(g),$$

$$\int_0^1 M^p(r,f)\,dr = \int_0^1 M^2(r,g)\,dr \leqq K C^p \text{ [12]}.$$

The extension to the general case is effected as before.

[12]) In fact

$$\int_0^1 \mathfrak{G}^2(r)\,dr \leqq K C^p,$$

$\mathfrak{G}$ being the majorant $\sum |c_n| r^n$. See Theorem 15 of our paper **4**. The result here is a direct corollary of Hilbert's double series theorem.

4.3. Before going further we require some additional preliminaries.

Theorem 32. (1) *If $f(z)$ is regular for $r<1$, $p>0$,*

$$M_p(f) \leqq C,$$

and

$$\mathfrak{F}(\theta) = \overline{\underset{r<1}{\text{bound}}} \, |f(r e^{i\theta})|,$$

then

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathfrak{F}^p(\theta)\, d\theta\right)^{\frac{1}{p}} \leqq K C,$$

where $K = K(p)$.

The same result is true if $\mathfrak{F}(\theta)$ is the upper bound of $|f|$ in the region $S_\omega(\theta)$ defined by [13])

$$|\arg(e^{i\theta} - z)| \leqq \omega < \frac{1}{2}\pi, \qquad 0 \leqq |e^{i\theta} - z| \leqq 1,$$

but in this case with $K = K(p, \omega)$.

(2) *If*

$$\mathfrak{F}(r, \theta) = \underset{\varrho \leqq r}{\text{Max}} \, |f(\varrho e^{i\theta})|,$$

then

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathfrak{F}^p(r, \theta)\, d\theta\right)^{\frac{1}{p}} \leqq K M_p(f).$$

(3) *When $p > 1$, the corresponding results for harmonic and subharmonic functions are also true.*

Whe shall sometimes refer shortly to this theorem as "Max". It is merely a restatement of the principal function-theoretic theorems of our paper **9** (Theorems 17, 24, 25, 26, 27).

4.4. Suppose that $f(z)$ is regular for $r<1$, and that $\alpha > 0$. We write

$$f_\alpha^*(z) = f_\alpha^*(r e^{i\theta}) = f_\alpha^*(r, \theta) = \frac{1}{\Gamma(\alpha)}\int_0^r (r-\varrho)^{\alpha-1} |f(\varrho e^{i\theta})|\, d\varrho.$$

Plainly

$$|f_\alpha(z)| \leqq f_\alpha^*(z).$$

Similarly, if

$$u(z) = u(r e^{i\theta}) = u(r, \theta) = \frac{1}{2}a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta) r^n$$

[13]) $S_\omega(\theta)$ is a kite-shaped region of fixed size and shape pointing inwards from the point $e^{i\theta}$ and with a boundary which does not touch the unit circle. It diminishes when ω decreases, and it reduces to the radius vector when $\omega = 0$.

is a (real or complex) harmonic function regular for $r < 1$, we write

$$u_\alpha(z) = u_\alpha(r, \theta) = \frac{1}{\Gamma(\alpha)} \int_0^r (r - \varrho)^{\alpha - 1} u(\varrho, \theta)\, d\varrho,$$

$$u_\alpha^*(z) = u_\alpha^*(r, \theta) = \frac{1}{\Gamma(\alpha)} \int_0^r (r - \varrho)^{\alpha - 1} |u(\varrho, \theta)|\, d\varrho,$$

and again

$$|u_\alpha(z)| \leqq u_\alpha^*(z).$$

We define the means

$$M_p(r, f_\alpha^*), \quad M_p(r, u_\alpha), \quad M_p(r, u_\alpha^*)$$

in a way similar to that in which we defined $M_p(r, f)$ and $M_p(r, u)$ in § 1.2.

4.5. We come now to the principal theorems of the paper.

Theorem 33. *If*

$$p > 0, \quad 0 < \alpha < \frac{1}{p}, \quad q = \frac{p}{1 - p\alpha},$$

and

$$M_p(f) \leqq C,$$

then

$$M_q(f_\alpha) \leqq KC,$$

where

$$K = K(p, \alpha) = K(p, q).$$

Theorem 34. *Under the same conditions*

$$M_q(f_\alpha^*) \leqq KC.$$

Theorem 35. *If*

$$p > 1, \quad 0 < \alpha < \frac{1}{p}, \quad q = \frac{p}{1 - p\alpha},$$

and

$$M_p(u) \leqq C,$$

then

$$M_q(u_\alpha) \leqq KC.$$

Theorem 36. *Under the conditions of Theorem* 35

$$M_q(u_\alpha^*) \leqq KC.$$

We shall sometimes refer to Theorems 33 and 34 as

$$\text{“}p \to q\text{”}, \quad \text{“}p \xrightarrow{*} q\text{”}$$

respectively. It is plain that Theorem 34 is stronger than and includes Theorem 33, and that, when $p > 1$, Theorems 35 and 36 include Theorems 33 and 34 respectively.

Proof of Theorems 33—36.

4.61. (1) *The case* $1 < p < q$. When $p > 1$, all the theorems are easy deductions from Theorem 30 (and so of Theorem 4 of 7).

In the first place, Theorem 33 reduces to Theorem 30.

Next, suppose that u is harmonic and real and that $M_p(u) \leqq C$, and let v be the conjugate of u which vanishes at the origin, and $f = u + iv$. By M. Riesz's theorem,

$$M_p(f) \leqq KC,$$

where $K = K(p)$, and so, by Theorem 33,

$$M_p(f_\alpha) \leqq KC.$$

Since

$$f_\alpha(r, \theta) = \frac{1}{\Gamma(\alpha)} \int_0^r (r-\varrho)^{\alpha-1} (u(\varrho, \theta) + iv(\varrho, \theta))\, d\varrho = u_\alpha(r, \theta) + iv_\alpha(r, \theta),$$

we have

$$|u_\alpha| \leqq |f_\alpha|,$$

$$M_p(u_\alpha) \leqq M_p(f_\alpha) \leqq KC.$$

This proves Theorem 35 when u is real, and the general form of the theorem follows by considering the real and imaginary parts separately.

Next, suppose that $M_p(u) \leqq C$, and that $U(\theta) = u(e^{i\theta})$ is the boundary function of u. If $W(r, \theta)$ is the harmonic function which has the boundary values $|U(\theta)|$, then

$$M_p(W) \leqq C.$$

Also $|u(r, \theta)|$ is subharmonic, and therefore, by F. Riesz's fundamental theorem on subharmonic functions [14]),

$$|u(r, \theta)| \leqq W(r, \theta);$$

and therefore

$$u_\alpha^*(r, \theta) = \frac{1}{\Gamma(\alpha)} \int_0^r (r-\varrho)^{\alpha-1} |u(\varrho, \theta)|\, d\varrho \leqq W_\alpha(r, \theta),$$

$$M_q(u_\alpha^*) \leqq M_q(W_\alpha) \leqq KC,$$

by Theorem 35. This proves Theorem 36 (and so Theorem 34, when $p > 1$).

Alternatively, we may argue as follows. Assuming u to be real, we have, by Lemma A (6), $u = u_1 - u_2$, where u_1 and u_2 are harmonic and positive and

$$M_p(u_1) \leqq C, \quad M_p(u_2) \leqq C.$$

[14]) F. Riesz, **14**.

Then

$$u_\alpha^* \leqq (u_1)_\alpha + (u_2)_\alpha,$$

$$M_q(u_\alpha^*) \leqq M_q((u_1)_\alpha) + M_q((u_2)_\alpha) \leqq KC,$$

by Theorem 35.

This argument is more elementary, since it avoids any appeal to the theory of sub-harmonic functions; but we shall have to refer to the first argument later.

4.62. (2) *The case* $p \leqq 1 < q$. By the integral form of Minkowski's inequality, we have

$$M_q(f_\alpha^*) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f_\alpha^{*\,q}\, d\theta\right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \left(\frac{1}{\Gamma(\alpha)}\int_0^r (r-\varrho)^{\alpha-1} |f(\varrho e^{i\theta})|\, d\varrho\right)^q\right)^{\frac{1}{q}}$$

$$\leqq \frac{1}{\Gamma(\alpha)}\int_0^r (r-\varrho)^{\alpha-1}\, d\varrho \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\varrho e^{i\theta})|^q\, d\theta\right)^{\frac{1}{q}}$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^r (r-\varrho)^{\alpha-1} M_q(\varrho, f)\, d\varrho$$

$$= \frac{r^\alpha}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1} M_q(rt, f)\, dt$$

$$\leqq \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1} M_q(t, f)\, dt \leqq KC,$$

by Theorem 31 (4.18).

It should be observed that the proof of Theorem 31 depends upon Theorem 30 (*i. e.* upon the case $p > 1$ of Theorem 33).

4.63. (3) *The case* $p < q \leqq 1$. We give two proofs, each of which has interesting features.

(i) Suppose that $0 < r < 1$ and

$$r_n = (1 - 2^{-n})\, r \qquad (n = 0, 1, 2, \ldots),$$

so that $r_0 = 0$ and $r_n \to r$ when $n \to \infty$. Then

$$r - r_n = r_n - r_{n-1} = 2^{-n} r,$$

so that the ratio of two factors of the form $(r - r_\nu)^c$, where ν is $n-1$, n, or $n+1$, lies between positive bounds depending on c only.

In these circumstances we have

$$f_\alpha^*(r,\theta) = K\int_0^r (r-\varrho)^{\alpha-1}|f(\varrho,\theta)|\,d\varrho$$

$$= K\sum_1^\infty \int_{r_{n-1}}^{r_n} (r-\varrho)^{\alpha-1}|f(\varrho,\theta)|\,d\varrho = K\sum_1^\infty J_n,$$

say. Also

$$J_n \leqq K(r_n - r_{n-1})(r-r_n)^{\alpha-1}\mu_n \leqq K\,2^{-n\alpha} r^\alpha \mu_n,$$

where

$$\mu_n = \mu_n(\theta) = \max_{r_{n-1}\leqq\varrho\leqq r_n} |f(\varrho,\theta)|;$$

and so (since $q \leqq 1$)

$$f_\alpha^{*q} \leqq K\sum J_n^q \leqq K\sum 2^{-nq\alpha} r^{q\alpha}\mu_n^q,$$

$$M_q^q(f_\alpha^*) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\alpha^{*q}\,d\theta \leqq K\sum_1^\infty 2^{-nq\alpha} r^{q\alpha}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mu_n^q\,d\theta$$

$$\leqq K\sum_1^\infty 2^{-nq\alpha} r^{q\alpha} M_q^q(r_n, f),$$

by "Max" (Theorem 32). Hence

$$M_q^q(f_\alpha^*) \leqq K\sum_1^\infty (r_{n+1}-r_n)(r-r_{n+1})^{q\alpha-1} M_q^q(r_n, f)$$

$$\leqq K\sum_0^\infty \int_{r_n}^{r_{n+1}} (r-\varrho)^{q\alpha-1} M_q^q(\varrho, f)\,d\varrho$$

$$= K\int_0^r (r-\varrho)^{q\alpha-1} M_q^q(\varrho)\,d\varrho = K r^{q\alpha}\int_0^1 (1-t)^{q\alpha-1} M_q^q(rt)\,dt$$

$$\leqq K\int_0^1 (1-t)^{q\alpha-1} M_q^q(t)\,dt \leqq K C^q,$$

by Theorem 31 (4.13).

The proof as stated depends on the deep theorem "Max", and it may be interesting to notice that a logical simplification is possible. We may use, instead of Theorem 32, the theorem:

If

$$r_2 < r_1 < r < 1$$

and

$$0 < A_1 < \frac{r-r_1}{r_1-r_2} < A_2,$$

where the A are absolute constants; and if

$$\nu = \nu(\theta) = \underset{r_2 \leqq \varrho \leqq r_1}{\mathrm{Max}} |f(\varrho, \theta)|,$$

then

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \nu^p \, d\theta\right)^{\frac{1}{p}} \leqq K M_p(r, f).$$

This theorem is included in Theorem 32 but is considerably easier, since the proof does not depend upon the maximal principle involved in that of Theorem 32.

To apply it for our present purpose, we take r_2, r_1, r to be r_{n-1}, r_n, r_{n+1}, when we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mu_n^q \, d\theta \leqq K M_q^q(r_{n+1}, f).$$

The changes in the subsequent argument owing to the occurrence of $n+1$ in place of n are trivial.

4. 64. (ii) We may suppose that $f(z)$ has no zeros in the unit circle, the general case being plainly reducible to this one by Lemma B.

We can choose s so that

$$s > 1, \quad \frac{q}{q-p+pq} < s < \frac{q}{q-p}.$$

Then, if

$$\tau = \frac{sp}{sq-q},$$

we have

$$\tau > 1,$$

$$q\tau' = \frac{q\tau}{\tau-1} = \frac{spq}{sp-sq+q} > 1$$

and

$$\frac{p}{q\tau} + \frac{1}{s} = \frac{s-1}{s} + \frac{1}{s} = 1.$$

If

$$f = g^s$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g|^{sp} \, d\theta \leqq C^p.$$

Also

$$f_\alpha^* = K\int_0^r |g(\varrho e^{i\theta})|^s (r-\varrho)^{\alpha-1} d\varrho$$

$$\leqq K \mathfrak{G}^{s-1} \int_0^r |g(\varrho e^{i\theta})| \, (r-\varrho)^{\alpha-1} d\varrho,$$

where

$$\mathfrak{G} = \mathfrak{G}(r, \theta) = \operatorname*{Max}_{\varrho \leqq r} |g(\varrho e^{i\theta})|;$$

so that

$$f_\alpha^{*q} \leqq K \mathfrak{G}^{(s-1)q} g_\alpha^{*q}.$$

Hence, by Hölder's inequality,

$$(4.641) \qquad \int f_\alpha^{*q} d\theta \leqq K\left(\int \mathfrak{G}^{(s-1)q\tau} d\theta\right)^{\frac{1}{\tau}} \left(\int g_\alpha^{*q\tau'} d\theta\right)^{\frac{1}{\tau'}}$$

$$= K\left(\int \mathfrak{G}^{sp} d\theta\right)^{\frac{1}{\tau}} \left(\int g_\alpha^{*q\tau'} d\theta\right)^{\frac{1}{\tau'}}.$$

Now

$$(4.642) \qquad \left(\int \mathfrak{G}^{sp} d\theta\right)^{\frac{1}{\tau}} \leqq K\left(\int |g|^{sp} d\theta\right)^{\frac{1}{\tau}} \leqq K C^{\frac{p}{\tau}},$$

by Theorem 32. Also

$$q\tau' = \frac{spq}{sp - sq + q} = \frac{sp}{1 - sp\alpha}.$$

Hence[15])

$$(4.643) \qquad \left(\int g_\alpha^{*q\tau'} d\theta\right)^{\frac{1}{\tau'}} \leqq K\left(\int |g|^{sp} d\theta\right)^{\frac{q}{sp}} \leqq K C^{\frac{q}{s}}.$$

From (4.641), (4.642) and (4.643) we deduce

$$M_q(f_\alpha^*) \leqq K\left(K C^{\frac{p}{\tau}}\right)^{\frac{1}{q}} (KC)^{\frac{1}{s}} = KC;$$

which proves the theorem.

Additional remarks on the propositions $p \to q$ and $p \overset{*}{\to} q$.

4.7. (1) It follows immediately from the definitions that $p \to q$ is transitive, *i. e.* that $p \to q$ and $q \to r$ imply $p \to r$, while this relation is not immediate for $p \overset{*}{\to} q$. On the other hand we can show very easily (without being able to prove the proposition itself) that the latter proposition is convex in q, *i. e.* that $p \overset{*}{\to} q_1$ and $p \overset{*}{\to} q_2$, where $p < q_1 < q_2$, imply $p \overset{*}{\to} q$ for $q_1 < q < q_2$. In fact if

$$q_1 = \frac{p}{1 - p\alpha_1}, \quad q_2 = \frac{p}{1 - p\alpha_2}, \quad \alpha_1 = \frac{1}{p} - \frac{1}{q_1}, \quad \alpha_2 = \frac{1}{p} - \frac{1}{q_2},$$

and

$$\alpha = \delta\alpha_1 + (1 - \delta)\alpha_2,$$

[15]) By one or other of the cases of the theorem proved already; the case $1 < p < q$ or $p \leqq 1 < q$ according as $sp > 1$ or $sp \leqq 1$.

where $0 < \delta < 1$, so that

$$\frac{1}{q} = \frac{\delta}{q_1} + \frac{1-\delta}{q_2},$$

we have at once, from Hölder's inequality,

$$f_\alpha^* = K \int_0^r (r-\varrho)^{\alpha-1} \, | f(\varrho e^{i\theta}) | \, d\varrho \leqq K f_{\alpha_1}^{*\,\delta} f_{\alpha_2}^{*\,1-\delta},$$

$$\int f_\alpha^{*q} \, d\theta \leqq K \left(\int f_{\alpha_1}^{*q_1} \, d\theta \right)^{\frac{\delta q}{q_1}} \left(\int f_{\alpha_2}^{*q_2} \, d\theta \right)^{\frac{(1-\delta) q}{q_2}} \leqq K \left(\int | f |^p \, d\theta \right)^{\frac{q}{p}}.$$

There is no such immediate proof of the convexity of $p \to q$.

(2) The proof that $p \to q$ is true generally depends on a rather formidable chain of theorems. There are simpler proofs of a number of special cases of the theorem which are of some intrinsic interest, and it may be worth while to notice one or two of them explicitly. In the remarks which follow we suppose, for simplicity of writing, that $a_0 = 0$ and that our series begin from $n = 1$, and we state our theorems as convergence theorems instead of as inequalities.

(i) The theorem $1 \to 2$ asserts that

$$\sum \frac{|a_n|^2}{n} < \infty$$

for any $f(z)$ of L. Since $a_n = o(1)$, this is included in the stronger theorem (Theorem 16 of our paper **4**[16]), with $\lambda = 1$) that

$$\sum \frac{|a_n|}{n} < \infty. \tag{4.71}$$

(ii) Similarly the theorem $p \to 2$, where $p < 1$, asserts that

$$\sum n^{-\frac{2}{p}+1} |a_n|^2 < \infty$$

for any $f(z)$ of L^p. Since, by Theorem 28, $a_n = o\left(n^{\frac{1}{p}-1}\right)$, this is included in the stronger theorem (Theorem 16 of **4**) that

$$\sum n^{p-2} |a_n|^p < \infty.$$

An intermediate theorem is that

$$\sum n^{-\frac{1}{p}} |a_n| < \infty. \tag{4.72}$$

We stated this in our original note **3** on fractional integrals, but do not seem to have ever published a proof.

[16]) Some simplifications of the argument of **4** are indicated in **6**.

(iii) We can deduce the theorem $p \xrightarrow{*} 1$, where $p < 1$, from $p \to 2$, as follows. We have to prove that

$$\int f^*_{\frac{1}{p}-1}\, d\theta,$$

where

$$f^*_{\frac{1}{p}-1} = K\int_0^r (r-\varrho)^{\frac{1}{p}-2}\,|f(\varrho e^{i\theta})|\,d\varrho,$$

is bounded for any $f(z)$ of L^p. It will be sufficient, after Lemma B, to prove the result on the assumption that $f(z)$ has no zeros in the unit circle. Writing then

$$f = \varphi^2, \qquad \varphi = \sum c_n z^n,$$

so that φ is L^{2p}, we have

$$\int f^*_{\frac{1}{p}-1}\, d\theta = K\int_0^r (r-\varrho)^{\frac{1}{p}-2}\, d\varrho \int |\varphi(\varrho e^{i\theta})|^2\, d\theta$$

$$= K\int_0^r (r-\varrho)^{\frac{1}{p}-2}\left(\sum |c_n|^2 \varrho^{2n}\right) d\varrho$$

$$= K\sum \frac{\Gamma\left(\frac{1}{p}-1\right)\Gamma(2n+1)}{\Gamma\left(2n+\frac{1}{p}\right)}\,|c_n|^2 r^{2n+\frac{1}{p}-1}$$

$$\leqq K\sum n^{1-\frac{1}{p}}\,|c_n|^2 r^{2n} \leqq K\int \left|\varphi_{\frac{1}{2p}-\frac{1}{2}}\right|^2 d\theta < K,$$

by $2p \to 2$.

(iv) If $p < 1$ and f is L^p, and we assume $p \to 1$, then $f_{\frac{1}{p}-1}$ is L. The convergence of the series (4.72) then follows from (4.71). This is however not an economical line of proof.

(3) We can if we please prove a still stronger proposition than $p \xrightarrow{*} q$, which we may denote by $p \xrightarrow{**} q$; viz.

Theorem 37. *If the conditions of Theorem* 33 *are satisfied then*

$$M_q(f^{**}_\alpha) \leqq KC,$$

where

$$f^{**}_\alpha = \frac{1}{\Gamma(\alpha)}\int_0^r (r-\varrho)^{\alpha-1}\,\underset{t\leqq\varrho}{\operatorname{Max}}\,|f(t e^{i\theta})|\,d\varrho.$$

We shall be content to sketch the proof of this theorem, which depends on a lemma of more interest than the theorem itself.

Lemma C. *If* $\lambda > 0$, *and*

$$\mathfrak{F} = \mathfrak{F}(r, \theta) = \operatorname*{Max}_{\varrho \leqq r} |f(\varrho e^{i\theta})|$$

is the function of Theorem 32, *then* $\mathfrak{F}^\lambda$ *is sub-harmonic.*

We suppose that P is the point $r e^{i\theta}$, and write

$$\mathfrak{F}(P) = \mathfrak{F}(r e^{i\theta}) = |f(t e^{i\theta})|,$$

where $t = t(r, \theta)$ gives the maximum of $|f(t e^{i\theta})|$. We have to show that

$$\mathfrak{F}_P^\lambda \leqq \operatorname*{Av.}_{\Gamma} \mathfrak{F}_{P'}^\lambda, \tag{4.73}$$

where "Av." denotes an average over points P' of a sufficiently small circle Γ with centre P.

There are three possibilities as regards t, viz. (a) $t = 0$, (b) $t = r$, and (c) $0 < t < r$. Since $\mathfrak{F} \geqq |f(0)|$, (4.73) is certainly true in case (a). In case (b) $\mathfrak{F}(P) = |f(P)|$, and, since $|f|^\lambda$ is sub-harmonic,

$$\mathfrak{F}^\lambda(P) = |f(P)|^\lambda \leqq \operatorname*{Av.}_{\Gamma} |f(P')|^\lambda \leqq \operatorname*{Av.}_{\Gamma} \mathfrak{F}^\lambda(P').$$

Finally, in case (c), we denote the point $t e^{i\theta}$ by Q and construct the circle γ which has its centre at Q and is in similitude with Γ about the origin. If Q' is the point of γ in similitude with P', then $|f(Q')| \leqq \mathfrak{F}(P')$, and so, since corresponding arcs of γ and Γ are proportional,

$$\mathfrak{F}^\lambda(P) = |f(Q)|^\lambda \leqq \operatorname*{Av.}_{\gamma} |f(Q')|^\lambda \leqq \operatorname*{Av.}_{\Gamma} \mathfrak{F}^\lambda(P').$$

It may be verified by direct calculation that, in case (c),

$$\nabla^2 \mathfrak{F}^\lambda = \frac{t^2}{r^2} \nabla^2 |f(t e^{i\theta})|^\lambda + \frac{\lambda |f|^{\lambda-2}}{r^2} \frac{K^2 + L^2}{K},$$

where ∇^2 is Laplace's operator,

$$K = -\frac{1}{2} t^2 \frac{\partial^2}{\partial t^2} |f(t e^{i\theta})|^2 \geqq 0, \qquad L = u \frac{\partial^2 v}{\partial \theta^2} - v \frac{\partial^2 u}{\partial \theta^2},$$

and u and v are the components of $f(t e^{i\theta})$.

It follows from the lemma that, if u^* is the harmonic function which has the boundary values $\mathfrak{F}(1, \theta)$, then $\mathfrak{F} \leqq u^*$ inside the unit circle. The proof of Theorem 37 for $p > 1$, and also the subsequent stages of the proof, then run parallel to the corresponding stages of the proof of Theorem 34, if we appeal to Theorem 32 at the appropriate places.

An extension of Theorem 33.

4.8. Theorem 38. *If*

$$p > 0, \quad a > 0, \quad 0 < \alpha < \frac{1}{p}, \quad q = \frac{p}{1 - p\alpha}$$

and

$$M_p(f) \leqq C(1 - r)^{-a}, \tag{4.81}$$

then

(4.82) $$M_q(f_\alpha) \leqq KC(1-r)^{-a},$$

where

$$K = K(p, a, \alpha), = K(p, a, q).$$

This is a corollary of Theorem 33. We write

$$\varrho = \frac{1}{2}(1+r), \quad \varphi(z) = f(\varrho z).$$

Then

$$M_p(f) \leqq C(1-\varrho)^{-a}$$

for $r < \varrho$, or

$$M_p(\varphi) \leqq C(1-\varrho)^{-a}$$

for $r < 1$; so that, by Theorem 33

$$M_q(\varphi_\alpha) \leqq KC(1-\varrho)^{-a}$$

for $r < 1$. But

$$f_\alpha(\varrho z) = \varrho^\alpha \varphi_\alpha(z)$$

and so

$$M_q(f_\alpha(\varrho z)) \leqq KC(1-\varrho)^{-a} \leqq KC(1-\varrho r)^{-a}$$

for $r < 1$. This is equivalent to (4.82).

5. The order of the integrals and derivatives of $f(z)$.

5.1. The theorems proved in this section are not difficult, but they require some preliminaries.

We write

$$f^\beta(z) = f_{-\beta}(z),$$

so that

$$f^\beta(z) = \left(\frac{d}{dz}\right)^\beta f(z)$$

when β is a positive integer.

Suppose that $\beta > 0$. Then

$$f^\beta(z) = \sum \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} a_n z^{n-\beta}.$$

But

$$\frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} = \frac{\Gamma(1+\beta)}{2\pi i}\int_C u^n (u-1)^{-1-\beta}\, du,$$

where C is a loop from $u=0$ round $u=1$ in the positive direction, and $(u-1)^{-1-\beta}$ has its principal value at the positive point of C. Hence, provided $|zu| < 1$ at all points of C (a condition we can certainly secure

if $|z| < 1$) we have

$$f^{\beta}(z) = \frac{\Gamma(1+\beta)}{2\pi i} z^{-\beta} \int_C f(zu)(u-1)^{-1-\beta} du$$

$$= \frac{\Gamma(1+\beta)}{2\pi i} \int_{C_z} f(w)(w-z)^{-1-\beta} dw,$$

where C_z is a loop from $w = 0$ round $w = z$.

When β is an integer, we can replace C_z by a circle (so obtaining the ordinary formula for $f^{\beta}(z)$). In general we may replace it by the contour Γ_z formed by the circle

$$|w| = \varrho \qquad (r < \varrho < 1)$$

and a double lacet from $w = 0$ to the point on this circle opposite to z, viz. $-\varrho e^{i\theta}$. We call the circle and the two edges of the lacet $\gamma_z, \Gamma_{1,z}, \Gamma_{2,z}$.

Theorem 39. *If $a > 0$, $a + \beta > 0$, then the hypotheses*

$$f(z) = O((1-r)^{-a}) \tag{5.11}$$

and

$$f^{\beta}(z) = O((1-r)^{-a-\beta}) \tag{5.12.}$$

are equivalent. If $a = 0$, $a + \beta > 0$ then (5.11) *implies* (5.12); *and if $a > 0$, $a + \beta = 0$ then* (5.12) *implies* (5.11); *the converse implication being in either case untrue.*

(i) Suppose that $a \geqq 0$, $\beta > 0$, and that (5.11) is true; and take

$$\varrho = \frac{1}{2}(1+r).$$

Then $f^{\beta}(z)$ is a constant multiple of

$$\int_{\Gamma_z} \frac{f(w)}{(w-z)^{1+\beta}} dw = \int_{\gamma_z} + \int_{\Gamma_{1,z}} + \int_{\Gamma_{2,z}}.$$

The two last integrals are

$$O\left(\int_0^{\varrho} (1-t)^{-a} dt\right),$$

and so negligible. The first is

$$O\left\{(1-r)^{-a} \int_{-\pi+\theta}^{\pi+\theta} \frac{d\varphi}{|\varrho e^{i\varphi} - r e^{i\theta}|^{1+\beta}}\right\}$$

$$= O((1-r)^{-a}(\varrho - r)^{-\beta}) = O((1-r)^{-a-\beta}).$$

(ii) Suppose that $a > 0$, $\beta > 0$, and that (5.12) is true, and write

$$f^{\beta} = g, \quad f = g_{\beta}.$$

Then

$$f(z)=\frac{1}{\Gamma(\beta)}\int_0^z g(w)(z-w)^{\beta-1}dw=O\left(\int_0^r(1-t)^{-\alpha-\beta}(r-t)^{\beta-1}dt\right).$$

If we write $x=\frac{1-t}{1-r}$, the last integral becomes

$$(1-r)^{-\alpha}\int_1^{\frac{1}{1-r}}x^{-\alpha-\beta}(x-1)^{\beta-1}dx<(1-r)^{-\alpha}\int_1^\infty=K(1-r)^{-\alpha}.$$

Hence

$$f(z)=O((1-r)^{-\alpha}).$$

A moment's consideration shows that (i) and (ii) cover all the positive assertions of the theorem. The negative assertion is established by the trivial example

$$f=\log\frac{1}{1-z},\quad f'=\frac{1}{1-z}.$$

It may be worth observing (in view of our later Theorem **46**) that $f(z)=O(1)$ does not necessarily imply $f^{\beta}(z)=o((1-r)^{-\beta})$ for $\beta>0$: consider, for example, the function

$$f(z)=(1-z)^i.$$

Lipschitz conditions.

5.2. Suppose that $0\leqq k\leqq 1$. We say that $f(z)$ satisfies a Lipschitz condition of order k in the unit circle, or belongs to the complex class Lip k, if

$$|f(z')-f(z)|\leqq K|z'-z|^k, \tag{5.21}$$

where K is independent of z and z', for $|z|\leqq 1$, $|z'|\leqq 1$. If

$$f(z')-f(z)=o(|z'-z|^k),$$

when $z'\to z$, uniformly in z, we say that $f(z)$ belongs to Lip* k. The classes Lip 0 and Lip* 0 are those of bounded and continuous functions. In the first case we assert (5.21) in the first instance only for $|z|<1$, $|z'|<1$; it is then satisfied automatically for those z and z' on the circle for which the boundary function exists.

Theorem 40. *In order that* $F(\theta)=f(e^{i\theta})$ *should belong to the real class* Lip k, *where* $0<k\leqq 1$, *it is necessary and sufficient that*

$$f'(z)=O((1-r)^{k-1}). \tag{5.22}$$

This is Theorem 4 of **8**. As we did not prove it explicitly there, and the proof is short, we give it here.

The real class Lip 1 is the same as the class of integrals of bounded functions. Also, if $f(e^{i\theta})$ belongs to the real class Lip k, it follows from the principle of the maximum modulus, applied to the function

$$\varphi(z) = f(ze^{ih}) - f(z),$$

that

$$|f(re^{i(\theta+h)}) - f(re^{i\theta})| \leqq Kh^k \tag{5.21}$$

uniformly for $0 \leqq r < r + h < 1$ and all θ, *i. e.* that $f(re^{i\theta})$, considered as a function of θ, belongs to Lip k, uniformly in r. If in particular $k = 1$, $f'(re^{i\theta})$ is uniformly bounded. This proves the theorem when $k = 1$. We may therefore suppose that $k < 1$.

(i) Suppose that $F(\theta)$ belongs to Lip k. We have, as on p. 626 of 8[17]),

$$f'(re^{i\theta}) = \frac{e^{-i\theta}}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\psi}}{(e^{i\psi}-r)^2}(F(\theta+\psi) - F(\theta))\,d\psi = O\left(\int_{-\pi}^{\pi}\frac{|\psi|^k\,d\psi}{|e^{i\psi}-r|^2}\right)$$
$$= O((1-r)^{k-1}).$$

(ii) Suppose that (5.22) is true. It is plain that $f(z)$ tends uniformly to a limit when $r \to 1$, and that $F(\theta)$ is continuous. Also, if $h > 0$,

$$F(\theta+h) - F(\theta) = \int_{e^{i\theta}}^{e^{i(\theta+h)}} f'(z)\,dz = \left(\int_{(1)} + \int_{(2)} + \int_{(3)}\right) f'(z)\,dz = I_1 + I_2 + I_3;$$

the paths of integration in I_1, I_2 and I_3 being respectively (1) the radius from $e^{i\theta}$ to $(1-h)e^{i\theta}$, (2) the circle $r = 1 - h$ from $(1-h)e^{i\theta}$ to $(1-h)e^{i(\theta+h)}$, and (3) the radius from $(1-h)e^{i(\theta+h)}$ to $e^{i(\theta+h)}$. It is plain that

$$I_2 = O(h \cdot h^{k-1}) = O(h^k);$$

and I_1 and I_3 are each

$$O\left(\int_{1-h}^{1}(1-r)^{k-1}\,dr\right) = O(h^k).$$

5.3. Theorem 41. *If $F(\theta) = f(e^{i\theta})$ belongs to the real class* Lip k, *then $f(z)$ belongs to the complex class* Lip k.

We may suppose $k > 0$, the result being classical when $k = 0$. It is plainly enough to prove (5.21) and

$$|f((r+h)e^{i\theta}) - f(re^{i\theta})| \leqq Kh^k, \tag{5.31}$$

uniformly for $0 \leqq r < r + h < 1$ and all θ.

---

17) The notation is now different.

In proving (5.31), we may take $\theta = 0$ without real loss of generality. We have then, by Theorem 40,

$$|f(r+h) - f(r)| \leqq \int_r^{r+h} |f'(t)|\, dt \leqq K \int_r^{r+h} (1-t)^{k-1}\, dt.$$

If $r + h \leqq \frac{1}{2}(1+r)$, then $h \leqq \frac{1}{2}(1-r)$ and $1 - t \geqq \frac{1}{2}(1-r)$; and

$$\int_r^{r+h} (1-t)^{k-1}\, dt < Kh(1-r)^{k-1} < Kh \cdot h^{k-1} = Kh^k.$$

If $r + h \geqq \frac{1}{2}(1+r)$ then $h \geqq \frac{1}{2}(1-r)$ and

$$\int_r^{r+h} (1-t)^{k-1}\, dt < K(1-r)^k < Kh^k.$$

Hence (5.31) is true in either case.

5.4. Theorem 42. *If* $0 < k \leqq 1$ *and*

$$f(re^{i\theta}) - f(r'e^{i\theta}) = O((r-r')^k),$$

uniformly for $0 \leqq r' < r \leqq 1$ *and all* θ, *then* $f(z)$ *belongs to* Lip k.

In other words, if $f(z)$ satisfies a Lipschitz condition uniformly on every radius, it satisfies one in the circle. We do not actually need this, theorem but it has a certain interest and is in a sense a partial converse of Theorem 41.

When $k = 1$, the hypothesis and the conclusion are each equivalent to the assertion that $f'(z)$ is bounded. We may therefore suppose $k < 1$.

If $0 < h < 1$, we have

$$f(re^{i\theta}) - f((r-hr)e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \bigl(f(e^{i\varphi}) - f((1-h)e^{i\varphi})\bigr) P(r, \varphi - \theta)\, d\varphi.$$

Differentiating with respect to θ, we obtain

$$rf'(re^{i\theta}) - (r-hr) f'((r-hr)e^{i\theta}) = \frac{e^{-i\theta}}{2\pi i} \int_{-\pi}^{\pi} \bigl(f(e^{i\varphi}) - f((1-h)e^{i\varphi})\bigr) \frac{\partial P}{\partial \theta}\, d\varphi.$$

But

$$\left|\frac{\partial P}{\partial \theta}\right| = \left|\frac{2r(1-r^2)\sin(\varphi-\theta)}{(1-2r\cos(\varphi-\theta)+r^2)^2}\right| \leqq \frac{AP}{1-r},$$

where A is an absolute constant. Hence

$$(5.41) \qquad rf'(re^{i\theta}) - (r-hr) f'((r-hr)e^{i\theta}) = O\left(\frac{h^k}{1-r}\right).$$

We now take

$$r_n = 1 - 2^{-n}, \quad r_n(1 - h_n) = r_{n-1},$$

so that

$$h_n = \frac{2^{-n}}{1 - 2^{-n}} = O(2^{-n});$$

and write

$$r = r_n, \quad h = h_n \qquad (n = 2, 3, \ldots, N)$$

in (5.41). Adding the resulting equations, we obtain

$$\begin{aligned} r_N f'(r_N e^{i\theta}) &= O(2^{N(1-k)} + 2^{(N-1)(1-k)} + \ldots) + O(1) \\ &= O(2^{N(1-k)}) = O((1 - r_N)^{k-1}). \end{aligned}$$

If we observe that $1 - r$ varies only by a factor 2 in the interval (r_{N-1}, r_N), and use the principle of the maximum modulus, we obtain

$$f'(z) = O((1 - r)^{k-1}),$$

and the conclusion then follows from Theorem 40.

5.5. Theorem 43. *If* $0 < k < 1$, $\beta > k$, *then a necessary and sufficient condition that* $f(z)$ *should belong to* Lip k *is that*

$$f^{\beta}(z) = O((1 - r)^{k-\beta}).$$

After Theorem 39, we may suppose that $\beta = 1$, and the conclusion then follows from Theorems 40 and 41.

Theorem 44. *If* $0 < k < 1$, $0 < k - \beta < 1$, *then a necessary and sufficient condition that* $f(z)$ *should belong to* Lip k *is that* $f^{\beta}(z)$ *should belong to* Lip $(k - \beta)$.

Suppose, for example, that $\beta > 0$. In order that $f(z)$ should belong to Lip k, it is necessary and sufficient (by Theorems 40 and 41) that

$$f'(z) = O((1 - r)^{k-1});$$

and therefore (by Theorem 39) that

$$f^{\beta+1}(z) = O((1 - r)^{k-\beta-1});$$

and therefore (by Theorems 40 and 41) that $f^{\beta}(z)$ should belong to Lip $(k - \beta)$.

There are partial implications corresponding to the cases $k = 0$, $k - \beta = 0$. There are also theorems corresponding to Theorems 40–44 in which "Lip" is replaced by "Lip*" and "O" by "o". These we may leave to the reader.

6. Further theorems concerning the means M_p.

6.1. Theorem 45. *Suppose that* $p > 0$, $a \geqq 0$, *and*

$$M_p(f) \leqq C(1 - r)^{-a}.$$

If $\beta > -a - \frac{1}{p}$, *then*

$$|f^{\beta}(z)| \leqq KC(1-r)^{-a-\beta-\frac{1}{p}}.$$

If $-a-1-\frac{1}{p} < \beta < -a-\frac{1}{p}$, *then* $f^{\beta}(z)$ *belongs to*

$$\mathrm{Lip}\left(-\alpha-\frac{1}{p}-\beta\right).$$

If $a = 0$, *we may replace* O *by* o *and* Lip *by* Lip* *in the conclusions.*

This theorem follows by combination of Theorem 27 with those of the preceding section.

The special case

$$a = 0, \quad \beta = -\alpha < 0, \quad p > 1, \quad \frac{1}{p} < \alpha < 1 + \frac{1}{p},$$

for example, gives a theorem substantially equivalent to Theorem 12 of 7.

6.2. Theorem 46. *If*

$$p > 0, \quad a \geqq 0, \quad a + \beta > 0,$$

and

$$M_p(f) \leqq C(1-r)^{-a}, \tag{6.21}$$

then

$$M_p(f^{\beta}) \leqq KC(1-r)^{-a-\beta}, \tag{6.22}$$

where $K = K(p, a, \beta)$. *If* $a = 0$, $\beta > 0$, *then also*

$$M_p(f^{\beta}) = o((1-r)^{-\beta}). \tag{6.23}$$

(i) Suppose first that $p \geqq 1$. We distinguish the sub-cases

(ia) $\quad p \geqq 1, \quad \beta > 0;$

(ib) $\quad p \geqq 1, \quad \beta = -\alpha, \quad 0 < \alpha < a.$

When $\beta = 0$ there is naturally nothing to prove.

(ia) In this case we have

$$f^{\beta}(z) = \frac{\Gamma(1+\beta)}{2\pi i}\int_{\Gamma_z} \frac{f(u)\,du}{(u-z)^{1+\beta}} \tag{6.24}$$

$$= \frac{\Gamma(1+\beta)}{2\pi i}\left(\int_{\gamma_z} + \int_{\Gamma_{1,z}} + \int_{\Gamma_{2,z}}\right) = F_1 + F_2 + F_3.$$

We take ϱ, the radius of Γ_z, to be $\frac{1}{2}(1+r)$. Then plainly

$$|F_2| \leqq K\int_0^{\varrho} |f(te^{i\theta+i\pi})|\,dt \leqq KC\int_0^{\varrho}(1-t)^{-a-\frac{1}{p}}\,dt,$$

by Theorem 27, (2.13); and F_2 is bounded if $a+\frac{1}{p}<1$, of order

$$(1-\varrho)^{1-a-\frac{1}{p}}=K(1-r)^{1-a-\frac{1}{p}}$$

in the contrary case. It is plain in either case that

$$|F_2|\leqq KC(1-r)^{-a}, \qquad M_p(F_2)\leqq KC(1-r)^{-a},$$

and *a fortiori*

$$M_p(F_2)\leqq KC(1-r)^{-a-\beta}. \tag{6.25}$$

Similarly for F_3.

Next

$$\int_{-\pi}^{\pi}|F_1|^p\,d\theta\leqq K\int_{-\pi}^{\pi}d\theta\left(\int_{-\pi}^{\pi}\frac{|f(\varrho e^{i\varphi+i\theta})|\,d\varphi}{|\varrho e^{i\varphi}-r|^{1+\beta}}\right)^p. \tag{6.26}$$

If $p>1$, we can choose β_1 and β_2 so that

$$\beta_1+\beta_2=1+\beta, \qquad p\beta_1>1, \qquad p'\beta_2>1.$$

Then, by Hölder's inequality,

$$\left(\int_{-\pi}^{\pi}\frac{|f(\varrho e^{i\varphi+i\theta})|\,d\varphi}{|\varrho e^{i\varphi}-r|^{1+\beta}}\right)^p\leqq\int_{-\pi}^{\pi}\frac{|f(\varrho e^{i\varphi+i\theta})|^p\,d\varphi}{|\varrho e^{i\varphi}-r|^{p\beta_1}}\left(\int_{-\pi}^{\pi}\frac{d\varphi}{|\varrho e^{i\varphi}-r|^{p'\beta_2}}\right)^{p-1};$$

and the second factor is not greater than

$$K(\varrho-r)^{(1-p'\beta_2)(p-1)}\leqq K(1-r)^{p-1-p\beta_2}.$$

Also

$$\int_{-\pi}^{\pi}d\theta\int_{-\pi}^{\pi}\frac{|f(\varrho e^{i\varphi+i\theta})|^p\,d\varphi}{|\varrho e^{i\varphi}-r|^{p\beta_1}}=\int_{-\pi}^{\pi}\frac{d\varphi}{|\varrho e^{i\varphi}-r|^{p\beta_1}}\int_{-\pi}^{\pi}|f(\varrho e^{i\varphi+i\theta})|^p\,d\theta$$

$$\leqq KC^p(1-\varrho)^{-pa}\int_{-\pi}^{\pi}\frac{d\varphi}{|\varrho e^{i\varphi}-r|^{p\beta_1}}$$

$$\leqq KC^p(1-\varrho)^{-pa}(\varrho-r)^{1-p\beta_1}\leqq KC^p(1-r)^{1-pa-p\beta_1}.$$

Hence

$$\int_{-\pi}^{\pi}|F_1|^p d\theta\leqq KC^p(1-r)^{p-1-p\beta_2}(1-r)^{1-pa-p\beta_1}\leqq KC^p(1-r)^{-pa-p\beta},$$

$$M_p(F_1)\leqq KC(1-r)^{-a-\beta}; \tag{6.27}$$

and (6.22) follows from (6.24), (6.25) and (6.27).

The proof is simpler when $p=1$, no use of Hölder's inequality being required.

We have still to prove (6.23) when $a = 0$. In this case $f(z)$ has, by Lemma A(1), a boundary function $f(e^{i\theta})$ of L^p, and we may take $\varrho = 1$ in (6.24). The contributions of F_2 and F_3 are $O(1)$ and *a fortiori* $o((1-r)^{-\beta})$.

We choose δ so that

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} |f(e^{i\varphi+i\theta})|^p \, d\varphi < \varepsilon$$

for all θ, and write (6.26) in the form

$$\int_{-\pi}^{\pi} |F_1|^p \, d\theta \leqq K \int_{-\pi}^{\pi} (J_1 + J_2)^p \, d\theta \leqq K\left(\int_{-\pi}^{\pi} J_1^p \, d\theta + \int_{-\pi}^{\pi} J_2^p \, d\theta\right),$$

where

$$J_1 = \int_{-\delta}^{\delta} \frac{|f(e^{i\varphi+i\theta})|}{|e^{i\varphi} - r|^{1+\beta}} \, d\varphi, \qquad J_2 = \int_{|\varphi|>\delta} \frac{|f(e^{i\varphi+i\theta})|}{|e^{i\varphi} - r|^{1+\beta}} \, d\varphi.$$

Plainly J_2 is bounded, and our former argument, applied to J_1, will give

$$\int J_1^p \, d\theta \leqq K \varepsilon C^p (1-r)^{-p\beta}.$$

We thus obtain (6.23).

6.3. (i*b*) If $p \geqq 1$, $\beta = -\alpha$, $0 < \alpha < a$, we have

$$|f_\alpha(re^{i\theta})| \leqq K \int_0^r (r-\varrho)^{\alpha-1} |f(\varrho e^{i\theta}| \, d\varrho,$$

$$\left(\int_{-\pi}^{\pi} |f_\alpha(re^{i\theta})|^p \, d\theta\right)^{\frac{1}{p}} \leqq K \int_0^r (r-\varrho)^{\alpha-1} \, d\varrho \left(\int_{-\pi}^{\pi} |f(\varrho e^{i\theta})|^p \, d\theta\right)^{\frac{1}{p}}$$

$$\leqq K \int_0^r (r-\varrho)^{\alpha-1} (1-\varrho)^{-a} \, d\varrho$$

$$= K(1-r)^{\alpha-a} \int_1^{\frac{1}{1-r}} (x-1)^{\alpha-1} x^{-a} \, dx < K(1-r)^{\alpha-a} \int_1^{\infty} = K(1-r)^{\alpha-a},$$

which proves the theorem.

It may be observed that in this argument we do not use the regularity of $f(z)$; the argument is valid if, for example, $f(z)$ is an analytic function with branch points and $|f(z)|$ is one-valued.

It is also to be observed that we have really proved more than is asserted by (6.22), and in fact that

$$(6.31) \qquad M_p(f_\alpha^*) = O((1-r)^{\alpha-a}).$$

We shall make use of these remarks later.

6.4. (ii) We now suppose $p<1$. It is plainly sufficient to prove the results (ii$a$) when $\beta=1$ and (ii$b$) when $\beta=-\alpha$ and $0<\alpha<a$.[18])

(ii$a$) Suppose that $p<1$, $\beta=1$, and, in the first instance, that $f(z)$ has no zeros in the unit circle. If we choose k so that $pk>1$, and write $f=g^k$, then

$$M_{pk}(g) \leqq C^{\frac{1}{k}}(1-r)^{-\frac{a}{k}};$$

and so, from (ia) above,

$$M_{pk}(g') \leqq K C^{\frac{1}{k}}(1-r)^{-\frac{a}{k}-1}$$

But

$$\int |f'|^p d\theta = K\int |g^{k-1}g'|^p d\theta$$

$$\leqq K\left(\int |g|^{pk}d\theta\right)^{\frac{1}{k'}}\left(\int |g'|^{pk}d\theta\right)^{\frac{1}{k}}$$

$$\leqq K C^p\left((1-r)^{-\frac{a}{k}}\right)^{\frac{pk}{k'}}\left((1-r)^{-\frac{a}{k}-1}\right)^{\frac{pk}{k}} = K C^p(1-r)^{-ap-p},$$

which is (6.22). Plainly (6.23) may be proved similarly.

In the general case we may argue as in § 2.1 (ii). Taking r and ϱ as there, determining f_1 and f_2 so as to satisfy (2.16), and applying what we have proved to $f_1(z)=F_1(\varrho z)$, we obtain

$$M_p(f_1') \leqq 2KC_\varrho\left(1-\frac{r}{\varrho}\right)^{-1} \leqq KC(1-r)^{-a-1};$$

and the proof may be completed as before.

6.5. (ii$b$) There remains the case $p<1$, $\beta=-\alpha$, $0<\alpha<a$. We give two proofs, by arguments analogous to those of §§ 4.63 and 4.64.

---

[18]) We might have used this remark in case (i), and simplified the analysis of § 6.2 by supposing $\beta=1$. There, however, there is no great gain in simplicity, whereas here the simplification is forced upon us, since the argument used under (ii$a$) depends essentially on the fact that β is an integer.

(i) Using the same notation as in § 4.63, and the transformation used already in § 5.1, we have

$$
\begin{aligned}
M_p^p(f_\alpha^*) &\leqq K \sum_1^\infty 2^{-np\alpha} r^{p\alpha} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mu_n^p \, d\theta \\
&\leqq K \sum_1^\infty 2^{-np\alpha} r^{p\alpha} M_p^p(r_n, f) \\
&\leqq K \int_0^r (r-\varrho)^{p\alpha-1} M_p^p(\varrho, f) \, d\varrho \\
&\leqq K C^p \int_0^r (r-\varrho)^{p\alpha-1} (1-\varrho)^{-pa} \, d\varrho \\
&= K C^p (1-r)^{p\alpha-pa} \int_1^{\frac{1}{1-r}} (x-1)^{p\alpha-1} x^{-pa} \, dx < K C^p (1-r)^{p\alpha-pa} \int_1^\infty \\
&= K C^p (1-r)^{p\alpha-pa},
\end{aligned}
$$

which proves the theorem.

(ii) Suppose first that $\alpha < ap$. We can then choose s so that

$$\frac{1}{p} < s < \frac{a}{\alpha},$$

and we write

$$f = g^s.$$

In general g is not regular in the unit circle, having branch points at the zeros of f; but $|g(z)|$ is one-valued, and we can therefore apply (6.31) to g.

We have now

$$M_{sp}(g) \leqq C^{\frac{1}{s}} (1-r)^{-\frac{a}{s}},$$

and so, from (6.31),

(6.51) $$M_{sp}(g_\alpha^*) \leqq K C^{\frac{1}{s}} (1-r)^{\alpha - \frac{a}{s}}.$$

Now

$$
\begin{aligned}
f_\alpha^*(re^{i\theta}) &= K \int_0^r |g(\varrho e^{i\theta})|^s (r-\varrho)^{\alpha-1} \, d\varrho \\
&\leqq K \mathfrak{G}^{s-1}(r, \theta) \int_0^r |g(\varrho e^{i\theta})| (r-\varrho)^{\alpha-1} \, d\varrho \\
&= K \mathfrak{G}^{s-1}(r, \theta) \, g_\alpha^*(r, \theta),
\end{aligned}
$$

where

$$\mathfrak{G}(r,\theta)=\underset{\varrho\leqq r}{\mathrm{Max}}\,|g(\varrho e^{i\theta})|;$$

so that

$$\int f_\alpha^{*p}\,d\theta \leqq K\int \mathfrak{G}^{(s-1)p} g_\alpha^{*p}\,d\theta$$

$$\leqq K\Big(\int \mathfrak{G}^{sp}\,d\theta\Big)^{\frac{1}{s'}}\Big(\int g_\alpha^{*sp}\,d\theta\Big)^{\frac{1}{s}}.$$

But

$$\int \mathfrak{G}^{sp}\,d\theta=\int \mathfrak{F}^{p}\,d\theta \leqq K\int |f|^p\,d\theta \leqq KC^p(1-r)^{-pa},$$

by Theorem 32; and

$$\int g_\alpha^{*sp}\,d\theta \leqq KC^p(1-r)^{sp\alpha-pa},$$

by (6.51). Hence

$$\int f_\alpha^{*p}\,d\theta \leqq KC^p(1-r)^{-\frac{pa}{s'}}(1-r)^{p\alpha-\frac{pa}{s}}=K(1-r)^{p\alpha-pa},$$

which proves (6.22).

We have thus proved that (6.21) implies

$$M_p(f_{\alpha_1})\leqq KC(1-r)^{\alpha_1-a}$$

for $0<\alpha_1<pa$. A repetition of the argument shows that

$$M_p(f_{\alpha_2})\leqq KC(1-r)^{\alpha_2-a}$$

for $0<\alpha_2-\alpha_1<p(a-\alpha_1)$, so that (6.22) holds for $0<\alpha<pa+(1-p)pa$. Another repetition shows that it holds for $\alpha<pa+(1-p)pa+(1-p)^2pa$; and some finite number of repetitions that it holds for any α less than

$$(1+(1-p)+(1-p)^2+\dots)pa=a.$$

This completes the proof of Theorem 46.

6.6. We can now obtain a very general theorem by combination of the theorems proved already.

Theorem 47. *If*

(6.61) $$0<p\leqq q,\quad a>0,\quad \beta>-a-\frac{1}{p}+\frac{1}{q},$$

or

(6.62) $$0<p\leqq q,\quad a=0,\quad \beta\geqq-\frac{1}{p}+\frac{1}{q},$$

and

(6.63) $$M_p(f)\leqq C(1-r)^{-a},$$

then

(6.64) $$M_q(f^\beta)\leqq KC(1-r)^{-a-\beta-\frac{1}{p}+\frac{1}{q}},$$

where $K = K(p, q, a, \beta)$. *The result is not necessarily true when* $a > 0$, $\beta = -a - \frac{1}{p} + \frac{1}{q}$. *But when* $a = 0$, $\beta > -\frac{1}{p} + \frac{1}{q}$, *we have also*

$$M_q(f^\beta) = o\left((1-r)^{-\beta - \frac{1}{p} + \frac{1}{q}}\right). \tag{6.65}$$

(i) Suppose first that $a > 0$, $\beta = -\alpha$, so that

$$\alpha < a + \frac{1}{p} - \frac{1}{q}.$$

If

$$\gamma = \frac{1}{p} - \frac{1}{q}, \quad q = \frac{p}{1 - p\gamma}$$

we have

$$M_q(f_\gamma) \leqq KC(1-r)^{-a}$$

by Theorem 38, and therefore

$$M_q(f_\alpha) \leqq KC(1-r)^{\alpha - \gamma - a}$$

by Theorem 46 (6.22). This is equivalent to (6.64).

To prove the falsity of the result when

$$\alpha = a + \frac{1}{p} - \frac{1}{q}$$

take

$$f(z) = (1-z)^{-a - \frac{1}{p}}.$$

Then (6.63) is satisfied, but $f_\alpha(z)$ behaves like a constant multiple of $(1-z)^{-\frac{1}{q}}$, and $M_q(f_\alpha) \to \infty$.

(ii) Next suppose that $a = 0$. If

$$\alpha = \frac{1}{p} - \frac{1}{q}$$

the theorem reduces to Theorem 33. If

$$\alpha < \frac{1}{p} - \frac{1}{q}$$

the results follow from Theorem 33 and Theorem 46 (6.23).

7. Integrated Lipschitz conditions.

7.1. In 7 we defined the class Lip (k, p), where $p \geqq 1$, $0 < k \leqq 1$, as the class of functions $F(\theta)$ for which

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(\theta + h) - F(\theta - h)|^p \, d\theta \leqq C^p h^{pk},$$

where $h > 0$ and C is independent of h; and the class $\mathrm{Lip}^*(k, p)$, where $p \geqq 1$, $0 < k < 1$, as the class for which

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(\theta+h) - F(\theta-h)|^p \, d\theta = o(h^{pk}),$$

when $h \to 0$. It proved indifferent which of the differences

$$F(\theta+h) - F(\theta-h), \qquad F(\theta+h) - F(\theta), \qquad F(\theta) - F(\theta-h)$$

we selected.

We considered only the case $p \geqq 1$, and we do not propose to go beyond this case here, so that we have nothing essentially new to prove. There are however certain theorems which are required for an application which we propose to make of the results of this paper to Parseval's Theorem.

We may say that $f(z)$ belongs to $\mathrm{Lip}(k, p)$ if it belongs to L^p and its boundary function $f(e^{i\theta})$ belongs to $\mathrm{Lip}(k, p)$; or, what is the same thing, if

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(r e^{i\theta+ih}) - f(r e^{i\theta-ih})|^p \, d\theta \leqq C^p h^{pk},$$

where C is independent of h and r.

Theorem 48. *A necessary and sufficient condition that* $f(z)$ *should belong to* $\mathrm{Lip}(k, p)$ *is that*

(7.11) $$M_p(f') = O((1-r)^{k-1}).$$

This is Theorem 3 of 8 [19]).

[19]) There is one point in the proof of this theorem where we were perhaps not sufficiently explicit. We took for granted on p. 626 the existence of the limit function $f(e^{i\theta})$ of an $f(z)$ satisfying (7.11). If (7.11) is satisfied we have (using Minkowski's inequality as in § 6.3, and assuming as we may that $f(0) = 0$),

$$\begin{aligned} M_p(f) &\leqq K\left(\int d\theta \left(\int_0^r |f'(\varrho e^{i\theta})|\, d\varrho\right)^p\right)^{\frac{1}{p}} \\ &\leqq K\int_0^r d\varrho \left(\int |f'(\varrho e^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \leqq K\int_0^r M_p(f')\, d\varrho \\ &= O\left(\int_0^r \frac{d\varrho}{(1-\varrho)^{1-k}}\right) = O(1), \end{aligned}$$

so that f has a limit function of L^p.

7.2. Theorem 49. *If* $f(z)$ *belongs to* $\mathrm{Lip}(k, p)$, *and*

$$-1 + k < \beta < k,$$

then $f^{\beta}(z)$ *belongs to* $\mathrm{Lip}(k - \beta, p)$.

This theorem, which is substantially equivalent to Theorems 25 and 26 of 7, follows at once from Theorems 46 and 48. We have in this way an alternative proof of the theorems referred to.

We conclude by stating without proof the theorem for the class $\mathrm{Lip}(k, p)$ which corresponds to Theorem 41.

Theorem 50. *If* $f(z)$ *belongs to* $\mathrm{Lip}(k, p)$, *then* $f(r e^{i\theta})$, *regarded as a function of* r *on the radius* $0 \leqq r \leqq 1$, *belongs to* $\mathrm{Lip}(k, p)$, *and uniformly in* θ; *that is to say*

$$\int_h^1 |f(r e^{i\theta}) - f((r-h) e^{i\theta})|^p \, dr = O(h^{pk}),$$

uniformly in θ.

The converse inference, which would correspond to Theorem 50 as Theorem 42 corresponds to Theorem 41, is false. The function

$$f(z) = \sum 2^{-kn} z^{2^n} \qquad \left(0 < k < k + \frac{1}{p} < 1\right)$$

belongs to $\mathrm{Lip}\left(k + \frac{1}{p}, p\right)$ on a radius, but on the circle only to $\mathrm{Lip}(k, p)$.

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Addendum.

We take this opportunity of correcting a mistake in 7 pointed out to us by Miss M. L. Cartwright. The proof of Theorem 24 in the case $p=1$ (p. 604–605) is incomplete until the restriction on the set e is removed.

The set E in which φ_n does not tend to f is of measure zero. We define f^* as follows: in the complementary set CE, $f^*=f$; in E

$$f^*(x)=\overline{\lim}\, f(z)$$

when $z\rightarrow x$ through points of CE. Then f^* is equivalent to f, and therefore belongs to Lip$(1,1)$. The argument of the text, applied to f^*, shows that

$$\sum_e |f^*(\eta)-f^*(\xi)|\leqq C$$

for any set e whose extremities lie in CE. But if x lies in E, $f^*(x)$ is a limit of $f^*(z)$ for values of z lying in CE and tending to x, so that a continuity argument is sufficient to remove the restriction on e.

CORRECTIONS

p. 409, *footnote* 10). For p. 56 read p. 156.

p. 411, *line* 5. For Theorem 4 read Theorem 5.

p. 414, *line* 19. For Whe read We.

p. 416, *line* 2. For Theorem 4 read Theorem 5.

p. 419, *line* 16. The last fraction should be $q/(q-p)$.

p. 419, *line* 21. For $\dfrac{spq}{sp-sq+q} > 1$ read $\dfrac{spq}{sp-sq+q} > \max\{1, sp\}$.

p. 425, *Theorem* 39. The argument of part (ii) of the proof should be applied to $f(z)-c_0-\ldots-c_\nu z^\nu$, where ν is the integral part of β. For a general f the integral on p. 426, line 2, may diverge at 0 when β is fractional and greater than 1.

pp. 429–30, *Theorem* 45. The inequality for $|f^\beta(z)|$ holds only for $0 < \delta \leqslant r < 1$. A similar remark applies also to (6.22) in Theorem 46 and to (6.64) in Theorem 47.

pp. 430–1, *Theorem* 46. The proof of (6.25) requires a slight modification when $a+1/p = 1$.

p. 438, *reference* **3**. For xxxii read xxxvii.

p. 438, *reference* **15**. For 1920 read 1916.

COMMENTS

p. 409. The formal law $(f_\alpha)_\beta = (f_\beta)_\alpha = f_{\alpha+\beta}$ is satisfied when $\alpha > -1$, $\beta > -1$, $\alpha+\beta > -1$, but there are difficulties of interpretation in other cases, e.g. when α is a negative integer and $-1 < \beta < 0$. These difficulties concern the fractional derivative only, and can be avoided by a slight alteration of Hadamard's definition of fractional derivative (see T. M. Flett, *Pac. J. Math.* 25 (1968), 463–94).

p. 411. Alternative proofs of Theorem 31 independent of Theorem 30 have been given by Hardy and Littlewood in 1941, 1, and by T. M. Flett, *loc. cit.*

p. 415. The case $q \geqslant 1$ of Theorem 33 has been generalized by Hardy and Littlewood in 1937, 3 and 1941, 1. Alternative proofs of Theorem 33 have been given by A. Zygmund (Z II, pp. 140–2), and T. M. Flett, *loc. cit.*

p. 430. Alternative proofs of the case $\beta > 0$ of Theorem 46 have been given by Hardy and Littlewood in 1941, 1, and by T. M. Flett, *loc. cit.* Flett (*loc. cit.*) has also given a simpler proof of the case $\beta < 0$, $p < 1$.

p. 437. The result of Theorem 48 has been extended to the case $0 < p < 1$ by A. E. Gwilliam, *Proc. London Math. Soc.* (2), 40 (1936), 353–64.

THEOREMS CONCERNING CESÀRO MEANS OF POWER SERIES

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1. *Introduction.*

1.1. We suppose throughout this paper that $z = re^{i\theta}$, where $r \leqslant 1$, that

$$f(z) = \Sigma c_n z^n = \sum_0^\infty c_n z^n$$

is a power series convergent for $r < 1$, that $\lambda > 0$, and that

$$M_\lambda = M_\lambda(r) = M_\lambda(f) = M_\lambda(r, f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^\lambda \, d\theta\right)^{1/\lambda}.$$

If $M_\lambda(r)$ is bounded, we say that $f(z)$ belongs to the (complex) class L^λ. It is well known that $f(z)$ then possesses a "boundary function" $f(e^{i\theta})$, of the (ordinary) Lebesgue class L^λ, with which it has certain standard relations. These are (i) that

$$f(z) = f(re^{i\theta}) \to f(e^{i\theta}),$$

when $r \to 1$, for almost all θ; (ii) that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta}) - f(e^{i\theta})|^\lambda \, d\theta \to 0,$$

i.e. that $f(re^{i\theta})$ "converges strongly" to $f(e^{i\theta})$; and (iii) that

$$M_\lambda(r) \to M_\lambda(1) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^\lambda \, d\theta\right)^{1/\lambda}.$$

When $\lambda > 1$, all this is true also (with the appropriate changes of notation) for harmonic functions $u(r, \theta)$.

1.2. We denote by $s_n^k(f, z)$ the Cesàro sum, of order $k > -1$, formed from the series $\Sigma c_n z^n$, and by $\sigma_n^k(f, z)$ the corresponding mean. The

1934, 1 (with J. E. Littlewood) *Proceedings of the London Mathematical Society* (2), 36, 516–31.

formulae defining s_n^k and σ_n^k are

$$\sigma_n^k(f, z) = \frac{s_n^k(f, z)}{A_n^k} = \frac{1}{A_n^k} \sum_{\nu=0}^{n} A_{n-\nu}^k c_\nu z^\nu,$$

where
$$A_n^k = \binom{n+k}{n} = \frac{\Gamma(n+k+1)}{\Gamma(n+1)\,\Gamma(k+1)}$$

(so that
$$A_n^k \sim \frac{n^k}{\Gamma(k+1)}$$

when $n \to \infty$). We may also define $s_n^k(f, z)$ by

$$\Sigma\, s_n^k(f, z)\, u^n = \frac{f(zu)}{(1-u)^{k+1}},$$

where $\rho = |u| < 1$. We may omit either or both of the arguments of s_n^k or σ_n^k when no ambiguity results.

The properties of the means σ_n^k are very well known when $\lambda \geqslant 1$ and $k > 0$. In particular it is known (i) that

$$\sigma_n^k(e^{i\theta}) \to f(e^{i\theta}), \tag{1.2.1}$$

when $n \to \infty$, for almost all θ; (ii) that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\sigma_n^k(e^{i\theta})|^\lambda\, d\theta \leqslant A M_\lambda^\lambda(1), \tag{1.2.2}$$

where A depends only on λ and k; (iii) that $\sigma_n^k(e^{i\theta})$ converges strongly to $f(e^{i\theta})$, *i.e.* that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\sigma_n^k(e^{i\theta}) - f(e^{i\theta})|^\lambda\, d\theta \to 0; \tag{1.2.3}$$

and (iv) that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\sigma_n^k(e^{i\theta})|^\lambda\, d\theta \to M_\lambda^\lambda(1). \tag{1.2.4}$$

No one seems to have attacked the corresponding problems for $\lambda < 1$, and we propose to consider them here.

The problems raised by (1.2.2), (1.2.3), and (1.2.4) we solve completely; we show that these results are true when

$$k > \frac{1}{\lambda} - 1, \tag{1.2.5}$$

but false (for appropriate f) when

$$k \leqslant \frac{1}{\lambda} - 1. \tag{1.2.6}$$

We cannot give so definite a solution of the remaining problem. We show that (1.2.1) is true when

$$k > \left[\frac{1}{\lambda}\right],$$

but it is obviously unlikely that this is the best possible result†.

1.3. We may clear the ground by three preliminary remarks:

(i) Whatever λ may be, (1.2.4) is an elementary corollary of (1.2.3). In fact, (1.2.4) follows from (1.2.3) by Minkowski's inequality

$$\left(\int |g+h|^\lambda dx\right)^{1/\lambda} \leqslant \left(\int |g|^\lambda dx\right)^{1/\lambda} + \left(\int |h|^\lambda dx\right)^{1/\lambda}$$

when $\lambda \geqslant 1$, and by the associated inequality

$$\int |g+h|^\lambda dx \leqslant \int |g|^\lambda dx + \int |h|^\lambda dx$$

when $\lambda < 1$.

(ii) We may prove all our theorems on the assumption that $f(z)$ is *wurzelfrei*, *i.e.* has no zeros inside the unit circle. Any f of L^λ is the sum of two *wurzelfrei* functions f_1 and f_2, for which‡

$$M_\lambda(f_s) \leqslant 2M_\lambda(f) \quad (s = 1, 2),$$

and (i), (ii), and (iii), for f, follow by combination of the corresponding results for f_1 and f_2.

(iii) When $\lambda > 1$ there are corresponding results for harmonic functions $u(r, \theta)$, $f(e^{i\theta})$ being replaced by $u(1, \theta)$ and $M_\lambda{}^\lambda(1)$ by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(1, \theta)|^\lambda d\theta.$$

All this breaks down when $0 < \lambda < 1$. A harmonic function of L^λ (*i.e.* a function for which

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(r, \theta)|^\lambda d\theta \tag{1.3.1}$$

† The difference between the two bounds for k is sometimes small and at other times nearly 1.

‡ See Hardy and Littlewood (2), 207.

is bounded) has generally no "boundary function"; the integral (1.3.1) does not usually tend to a limit; and it is improbable that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\sigma_n^k(1,\theta)|^\lambda\,d\theta,$$

the integral corresponding to (1.2.2), is bounded†.

2. *The boundedness of* $\int|\sigma_n^k|^\lambda\,d\theta$.

2.1. THEOREM 1. *Suppose that* $0<\lambda<1$, *that* $f(z)$ *is* L^λ, *and that*

(2.1.1) $$k>\frac{1}{\lambda}-1.$$

Then

(2.1.2) $$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\sigma_n^k(e^{i\theta})|^\lambda\,d\theta\leqslant AM_\lambda^\lambda(1)=\frac{A}{2\pi}\int_{-\pi}^{\pi}|f(e^{i\theta})|^\lambda\,d\theta,$$

where $A=A(\lambda,k)$ *depends only on* λ *and* k.

We may suppose (as we explained in § 1.3) that f is *wurzelfrei*. We take

$$r=1,\quad z=e^{i\theta},\quad u=\rho e^{i\phi},\quad \rho<1.$$

We define an integer l by

(2.1.3) $$\frac{1}{l}<\lambda\leqslant\frac{1}{l-1},$$

so that $$l=\left[\frac{1}{\lambda}\right]+1\geqslant 2,$$

and a by

(2.1.4) $$k=l-1+la,\quad a=\frac{k-l+1}{l}.$$

It is plain that $a>-1$. Also $a<0$ if

$$k<l-1=\left[\frac{1}{\lambda}\right],$$

† In view of the breakdown, for $\lambda<1$, of the orthodox relations between pairs of conjugate harmonic functions. See our paper 5.

which is always true if k is near enough to the lower bound assigned by (2.1.1). Hence

$$-1 < a < 0$$

in the cases with which we are primarily concerned.

We write
$$f = g^l;$$

since f is *wurzelfrei*, g is regular. Then g is L^p, where

(2.1.5) $$p = l\lambda > 1.$$

From
$$\frac{f(zu)}{(1-u)^{k+1}} = \left\{\frac{g(zu)}{(1-u)^{a+1}}\right\}^l$$

we deduce

(2.1.6) $$s_n{}^k(f) = \sum_{\nu_1+\nu_2+\ldots+\nu_l=n} s^a_{\nu_1}(g)\, s^a_{\nu_2}(g) \ldots s^a_{\nu_l}(g),$$

and it is on this identity that all our work depends.

2.2. We write

(2.2.1) $$S_\nu = |s_\nu{}^a(g)|.$$

It follows from (2.1.6) and Hölder's inequality that

$$|s_n{}^k(f)| \leqslant \Sigma S_\nu{}^l,$$

where the summation is $(l-1)$-fold, ν having one of the values ν_1, ν_2, ... and the summation being then over $\nu_1+\nu_2+\ldots = n$†. When we allow for this, we obtain

(2.2.2) $$|s_n{}^k(f)| \leqslant An^{l-2} \sum_{\nu=0}^{n} S_\nu{}^l = An^{l-2} P,$$

say, where A (as always in the sequel) is a number of the type prescribed by the theorem.

It follows from (2.1.3) and (2.1.5) that

(2.2.3) $$1 < p \leqslant 1+\lambda < 2$$

and that

(2.2.4) $$l \leqslant \frac{l\lambda}{l\lambda-1} = \frac{p}{p-1} = p'.$$

† Thus, for example, $\Sigma 1$ is asymptotic to a multiple of n^{l-1}, and not to a multiple of n.

Hence, by Hölder's inequality,

$$\sum_0^n S_\nu^{\,l} \leqslant n^{(p'-l)/p'} \left(\sum_0^n S_\nu^{p'}\right)^{l/p'}. \tag{2.2.5}$$

Also S_ν is the modulus of the coefficient of u^ν in

$$\frac{g(zu)}{(1-u)^{\alpha+1}} = \frac{g(\rho e^{i\phi+i\theta})}{(1-\rho e^{i\phi})^{\alpha+1}},$$

and (since $p < 2$) we may apply Hausdorff's theorem to this function. We thus obtain

$$(\Sigma S_\nu^{p'} \rho^{p'\nu})^{p/p'} \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|g(\rho e^{i\phi+i\theta})|^p}{|1-\rho e^{i\phi}|^{(\alpha+1)p}}\, d\phi = J(\theta), \tag{2.2.6}$$

say.

2.3. Taking

$$\rho = 1 - \frac{1}{n}$$

in (2.2.6), we obtain

$$\left(\sum_0^n S_\nu^{p'}\right)^{p/p'} \leqslant AJ(\theta).$$

Combining this with (2.2.5), and observing that $p = l\lambda$ and that

$$\frac{p'-l}{p'}\lambda = \lambda\left(1-l+\frac{l}{p}\right) = -l\lambda+\lambda+1,$$

we obtain

$$P^\lambda = \left(\sum_0^n S_\nu^{\,l}\right)^\lambda \leqslant A n^{-l\lambda+\lambda+1} J(\theta).$$

Hence

$$\begin{aligned} \int_{-\pi}^{\pi} P^\lambda\, d\theta &\leqslant A n^{-l\lambda+\lambda+1} \int_{-\pi}^{\pi} J(\theta)\, d\theta \\ &\leqslant A n^{-l\lambda+\lambda+1} \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} \frac{|g(\rho e^{i\phi+i\theta})|^{l\lambda}}{|1-\rho e^{i\phi}|^{(\alpha+1)l\lambda}}\, d\phi \\ &= A n^{-l\lambda+\lambda+1} \int_{-\pi}^{\pi} \frac{d\phi}{|1-\rho e^{i\phi}|^{(k+1)\lambda}} \int_{-\pi}^{\pi} |g(\rho e^{i\phi+i\theta})|^{l\lambda}\, d\theta \\ &\leqslant A n^{-l\lambda+\lambda+1} M_\lambda^{\lambda}(1) \int_{-\pi}^{\pi} \frac{d\phi}{|1-\rho e^{i\phi}|^{(k+1)\lambda}}. \end{aligned} \tag{2.3.1}$$

Since $(k+1)\lambda > 1$, in virtue of (2.1.1), the integral here is less than

$$A n^{(k+1)\lambda - 1}.$$

It follows that

$$(2.3.2) \quad \int_{-\pi}^{\pi} P^{\lambda}\,d\theta \leqslant A n^{-l\lambda+\lambda+1} M_{\lambda}^{\lambda}(1)\, n^{(k+1)\lambda-1} = A n^{k\lambda-(l-2)\lambda} M_{\lambda}^{\lambda}(1).$$

Finally, combining this with (2.2.2), we obtain

$$\int_{-\pi}^{\pi} |s_n^k(f)|^{\lambda}\,d\theta \leqslant A n^{k\lambda} M_{\lambda}^{\lambda}(1),$$

which is equivalent to (2.1.2)

3. *Strong convergence.*

3.1. Theorem 2. *If the conditions of Theorem* 1 *are satisfied, then*

$$(3.1.1) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sigma_n^k(e^{i\theta})-f(e^{i\theta})|^{\lambda}\,d\theta \to 0.$$

Theorem 3. *Under the same conditions,*

$$(3.1.2) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sigma_n^k(e^{i\theta})|^{\lambda}\,d\theta \to M_{\lambda}^{\lambda}(1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^{\lambda}\,d\theta.$$

The second of these theorems is, as we remarked in § 1.3 (i), an elementary corollary of the first, and we need not refer to it further.

We have two proofs of Theorem 2, the first of which is direct and proceeds on the same lines as those of Theorem 1. The second is shorter (when Theorem 1 has been proved), but its basis lies rather deeper, and we postpone it to § 4.

We begin by observing that

$$s_n^k(f, z) - A_n^k f(z)$$

is the coefficient of u^n in

$$\frac{g^l(zu)-g^l(z)}{(1-u)^{k+1}} = \prod_{\omega}\left\{\frac{g(zu)-\omega g(z)}{(1-u)^{\alpha+1}}\right\},$$

where ω runs through

$$\omega_1 = 1, \quad \omega_2 = e^{2\pi i/l}, \quad \ldots,$$

the l-th roots of unity. Hence

$$s_n^k(f, z) - A_n^k f(z) = \sum_{\nu_1+\ldots+\nu_l=n} \{s_{\nu_1}^{\alpha}(g) - \omega_1 A_{\nu_1}^{\alpha} g\} \ldots \{s_{\nu_l}^{\alpha}(g) - \omega_l A_{\nu_l}^{\alpha} g\},$$

and so

$$|s_n^k(f)-A_n^k f| \leqslant (\Sigma|s_\nu^\alpha(g)-A_\nu^\alpha g|^l)^{1/l} \prod_{\omega\neq 1} (\Sigma|s_\nu^\alpha(g)-\omega A_\nu^\alpha g|^l)^{1/l}.$$

In each of these sums the range of summation is multiple, as in § 2.2. When we allow for this, as there, we obtain

$$|s_n^k(f)-A_n^k(f)| \leqslant An^{l-2} P_0^{1/l} P_1^{1/l} \dots P_{l-1}^{1/l}, \tag{3.1.3}$$

where

$$P_0 = \sum_{\nu=0}^{n} |s_\nu^\alpha(g)-A_\nu^\alpha g|^l, \tag{3.1.4}$$

and each of $P_1, P_2, \dots, P_{l-1}$ is majorized by

$$Q = \sum_{\nu=0}^{n} \{|s_\nu^\alpha(g)|+|A_\nu^\alpha||g|\}^l. \tag{3.1.5}$$

We shall prove that

$$\int_{-\pi}^{\pi} Q^\lambda \, d\theta \leqslant An^{k\lambda-(l-2)\lambda} M_\lambda^\lambda(1), \tag{3.1.6}$$

and that

$$\int_{-\pi}^{\pi} P_0^\lambda \, d\theta = o(n^{k\lambda-(l-2)\lambda}) \tag{3.1.7}$$

when $n\to\infty$. These results correspond to (2.3.2); and it is plain after § 2.3 that, when they are proved, we can deduce

$$\int_{-\pi}^{\pi} |s_n^k(f)-A_n^k f|^\lambda \, d\theta = o(n^{k\lambda}),$$

which is (3.1.1).

3.2. Now

$$\begin{aligned}\int_{-\pi}^{\pi} Q^\lambda \, d\theta &= \int_{-\pi}^{\pi} \left\{\sum_0^n (S_\nu+|A_\nu^\alpha||g|)^l\right\}^\lambda d\theta \\ &\leqslant A\int_{-\pi}^{\pi} \left\{\sum_0^n (S_\nu^l+|A_\nu^\alpha|^l|g|^l)\right\}^\lambda d\theta \\ &\leqslant A\int_{-\pi}^{\pi} \left(\sum_0^n S_\nu^l\right)^\lambda d\theta + A\left(\sum_0^n |A_\nu^\alpha|^l\right)^\lambda \int_{-\pi}^{\pi} |g|^{l\lambda} d\theta \\ &\leqslant An^{k\lambda-(l-2)\lambda} M_\lambda^\lambda(1) + AM_\lambda^\lambda(1)\left(\sum_0^n |A_\nu^\alpha|^l\right)^\lambda,\end{aligned}$$

by (2.3.1). Also $$A_\nu^\alpha = O(\nu^\alpha)$$

and $$l\alpha = k-l+1 > \frac{1}{\lambda}-l \geqslant -1,$$

by (2.1.3). Hence

$$\left(\sum_0^n |A_\nu^\alpha|^l\right)^\lambda = O(n^{\alpha l\lambda+\lambda}) = O(n^{k\lambda-(l-2)\lambda});$$

and this completes the proof of (3.1.5).

It remains only to prove (3.1.4). For this, we observe that, since $s_\nu^\alpha(g)-A_\nu^\alpha g$ is the coefficient of u^ν in

$$\frac{g(zu)-g(z)}{(1-u)^{\alpha+1}},$$

the same argument which led to (2.3.1) in § 2.3 will lead here to

$$(3.2.1)\quad \int_{-\pi}^{\pi} P_0^\lambda\, d\theta \leqslant An^{-l\lambda+\lambda+1}\int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} \frac{|g(\rho e^{i\phi+i\theta})-g(e^{i\theta})|^{l\lambda}}{|1-\rho e^{i\phi}|^{(k+1)\lambda}}\, d\phi.$$

The integral here is

$$(3.2.2)\quad \int_{|\phi|\geqslant\delta} \frac{d\phi}{|1-\rho e^{i\phi}|^{(k+1)\lambda}} \int_{-\pi}^{\pi} |g(\rho e^{i\phi+i\theta})-g(e^{i\theta})|^{l\lambda}\, d\theta + \int_{|\phi|\leqslant\delta} \dots d\phi \int_{-\pi}^{\pi} \dots d\theta$$
$$= I_1+I_2,$$

say. It is plain that I_1 is bounded, for any fixed δ, when $n\to\infty$ and $\rho\to 1$; and *a fortiori*

$$(3.2.3)\quad I_1 = o(n^{(k+1)\lambda-1}).$$

The inner integral in I_2 does not exceed

$$A\int_{|\phi|\leqslant\delta} |g(\rho e^{i\phi+i\theta})-g(\rho e^{i\theta})|^{l\lambda}\, d\theta + A\int_{-\pi}^{\pi} |g(\rho e^{i\theta})-g(e^{i\theta})|^{l\lambda}\, d\theta.$$

The second term tends to zero when $\rho\to 1$, and the first tends to zero, uniformly in ρ, when $\delta\to 0$†. It follows that

$$(3.2.4)\quad I_2 < \epsilon n^{(k+1)\lambda-1}$$

if δ is sufficiently small and n sufficiently large. Hence

$$(3.2.5)\quad I_1+I_2 = o(n^{(k+1)\lambda-1});$$

† See F. Riesz (7), 95.

and (3.1.4) follows from (3.2.1), (3.2.2), and (3.2.5). This completes the proof of Theorem 2.

4. *An alternative proof of Theorem 2.*

4.1. Our second proof of Theorem 2 is formally simpler, but depends on theorems one of which lies rather deeper while others are presumably not the best of their kind.

THEOREM 4. *If*

$$k > \left[\frac{1}{\lambda}\right] = l-1, \tag{4.1.1}$$

then

$$\sigma_n^k(f, e^{i\theta}) \to f(e^{i\theta}) \tag{4.1.2}$$

for almost all θ.

If f is *wurzelfrei*, we have, by (2.1.6),

$$\sigma_n^k(f) = \frac{1}{A_n^k} \sum_{\nu_1+\ldots+\nu_l=n} s_{\nu_1}^a(g) \ldots s_{\nu_l}^a(g). \tag{4.1.3}$$

Now $k > l-1$ and so $a > 0$. Hence

$$\sigma_\nu^a(g) \to g(e^{i\theta}),$$

or

$$s_\nu^a(g) \sim \frac{\nu^a}{\Gamma(a+1)} g(e^{i\theta}),$$

for almost all θ. It follows from well known elementary theorems† that

$$\Sigma s_{\nu_1}^a(g) \ldots s_{\nu_l}^a(g) \sim \frac{g^l}{\{\Gamma(a+1)\}^l} \Sigma \nu_1^a \ldots \nu_l^a$$

$$\sim \frac{n^{la+l-1}}{\Gamma(la+l-1)} g^l = \frac{n^k}{\Gamma(k+1)} f,$$

and $\sigma_n^k(f) \to f$, for the same θ.

The extension to general f follows in the usual manner.

THEOREM 5. *If k satisfies* (4.1.1), *and*

$$\sigma^* = \sigma^*(f, \theta) = \operatorname*{Max}_n |\sigma_n^k(f, e^{i\theta})|, \tag{4.1.4}$$

† See Knopp (**6**).

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma^{*\lambda}\,d\theta \leqslant AM_\lambda^\lambda(1) = \frac{A}{2\pi}\int_{-\pi}^{\pi}|f(e^{i\theta})|^\lambda\,d\theta. \tag{4.1.5}$$

It follows from (2.1.6), when we allow again for the multiple summation, that

$$|\sigma_n^k(f)| \leqslant An^{-k}\,.\,n^{l-1}\,.\operatorname*{Max}_{\nu\leqslant n}|s_\nu^a(g)|^l$$

$$\leqslant An^{-k+l-1+la}\operatorname*{Max}_{\nu\leqslant n}|\sigma_\nu^a(g)|^l = A\operatorname*{Max}_{\nu\leqslant n}|\sigma_\nu^a(g)|^l.$$

Since the right-hand side increases with n, this involves

$$\sigma_n^* = \operatorname*{Max}_{\nu\leqslant n}|\sigma_\nu^k(f)| \leqslant A\operatorname*{Max}_{\nu\leqslant n}|\sigma_\nu^a(g)|^l;$$

and so
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma_n^{*\lambda}\,d\theta \leqslant \frac{A}{2\pi}\int_{-\pi}^{\pi}\operatorname*{Max}_{\nu\leqslant n}|\sigma_\nu^a(g)|^{l\lambda}\,d\theta$$

$$\leqslant AM_\lambda^\lambda(1),$$

by Theorem 21 of our paper **3**. Since σ_n^* increases with n, this in its turn involves (4.1.5).

The proofs of Theorems 4 and 5 demand that a shall be positive, and so that k shall satisfy (4.1.1). We can none the less deduce Theorem 2 (with the better bound for k) by combining them with Theorem 1.

Suppose first that k satisfies (4.1.1). Then

$$|\sigma_n^k(f)-f|^\lambda$$

is majorized by
$$(\sigma^* + |f|)^\lambda,$$

an integrable function independent of n; and $\sigma_n^k(f)-f\to 0$ for almost all θ. It follows by Lebesgue's criterion that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\sigma_n^k(f)-f|^\lambda\,d\theta \to 0$$

for *some* k (in fact, for $k > l-1$). Hence there is a polynomial $P(e^{i\theta})$ satisfying

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f-P|^\lambda\,d\theta < \epsilon; \tag{4.1.6}$$

for $\sigma_n^k(f)$ is such a polynomial when n is sufficiently large.

We now suppose k subject only to (1.2.5). Applying Theorem 1 to $f-P$, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\sigma_n{}^k(f)-\sigma_n{}^k(P)|^\lambda \, d\theta < A\epsilon \tag{4.1.7}$$

for all n. From (4.1.6) and (4.1.7) we deduce

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\sigma_n{}^k(f)-f|^\lambda \, d\theta < A\epsilon+\frac{1}{2\pi}\int_{-\pi}^{\pi} |\sigma_n{}^k(P)-P|^\lambda \, d\theta.$$

Since $\sigma_n{}^k(P)\to P$ uniformly, this proves Theorem 2.

5. *Negative results.*

5.1. We have to show finally that our bound (2.1.1) for k cannot be improved.

THEOREM 6. *If*

$$k \leqslant \frac{1}{\lambda}-1, \tag{5.1.1}$$

then the result of Theorem 1 *is sometimes false.*

The proof is easy when

$$k < \frac{1}{\lambda}-1. \tag{5.1.2}$$

The function $$f(z) = (1-z)^{-(1/\lambda)+\delta}$$

is L^λ if $\delta > 0$, and c_n is (exactly) of order

$$\frac{1}{\lambda}-1-\delta.$$

We shall prove that, if (2.1.2) is true, then

$$c_n = O(n^k). \tag{5.1.3}$$

This will give a contradiction if k satisfies (5.1.2) and δ is sufficiently small.

Suppose, first, that

$$k \neq \frac{1}{\lambda}-p$$

for any integral p. Since

$$s_n^{k+1} = s_0{}^k+s_1{}^k+\dots+s_n{}^k,$$

the truth of (2.1.2) for $k=l$ involves its truth for $k=l+1$. It is, therefore, true for a k for which

$$\frac{1}{\lambda}-2<k<\frac{1}{\lambda}-1, \tag{5.1.4}$$

and we may suppose that k satisfies this condition.

Taking $z=e^{i\theta}$, as in §2.1, we have

$$\Sigma c_n z^n u^n = (1-u)^{k+1}\Sigma s_n^{k}(z)\, u^n,$$

and so

$$c_n z^n = \sum_0^n (-1)^\nu \binom{k+1}{\nu} s^k_{n-\nu}(z).$$

Hence
$$\int |c_n z^n|^\lambda\, d\theta \leqslant \sum_0^n \left|\binom{k+1}{\nu}\right|^\lambda \int |s^k_{n-\nu}(z)|^\lambda\, d\theta.$$

But
$$\binom{k+1}{\nu} = O(\nu^{-k-2}),$$

and so

$$|c_n|^\lambda = \frac{1}{2\pi}\int |c_n z^n|^\lambda\, d\theta = O\left\{n^{k\lambda}\sum_0^n \nu^{-(k+2)\lambda}\right\} = O(n^{k\lambda}),$$

which is (5.1.3).

A slight change is required when k differs from $1/\lambda$ by an integer. We may then suppose that

$$k=\frac{1}{\lambda}-2.$$

We start from
$$s_n(z) = \sum_0^n (-1)^\nu \binom{k}{\nu} s^k_{n-\nu}(z)$$

and show, as above, that

$$\int |s_n(z)|^\lambda\, d\theta = O(n^{k\lambda}).$$

From this and
$$c_n z^n = s_n(z) - s_{n-1}(z)$$

we deduce (5.1.3).

5.2. The case of equality in (5.1.1) is rather more delicate. We use an example suggested by one which we have used for a somewhat similar purpose elsewhere†. We shall pass rather lightly over some details of

† Hardy and Littlewood (**4**).

the argument which are a little tedious but do not involve any point of principle.

We take

$$f(z) = (N+1)^{-(p\lambda-1)/\lambda}\left(\frac{1-z^{N+1}}{1-z}\right)^p = (N+1)^{-\beta}\left(\frac{1-z^{N+1}}{1-z}\right)^p,$$

where $p\lambda > 1$, $p > 2$, and N is large. Then

$$\int_{-\pi}^{\pi} |f|^\lambda\, d\theta \leqslant (N+1)^{-p\lambda+1}\int_{-\pi}^{\pi}\left|\frac{\sin\frac{1}{2}(N+1)\theta}{\sin\frac{1}{2}\theta}\right|^{p\lambda} d\theta$$

$$< AN^{-p\lambda+1}\int_0^{1/N} N^{p\lambda}\, d\theta + AN^{-p\lambda+1}\int_{1/N}^{\pi} \theta^{-p\lambda}\, d\theta < A,$$

where the A are independent of N.

The Cesàro sum formed from $f(z)$, with $z = e^{i\theta}$, of order

$$k = \frac{1}{\lambda} - 1$$

and rank N, is

$$s_N{}^k = A(N+1)^{-\beta}\sum_0^N (-1)^n \binom{N-n+k}{N-n}\binom{-p}{n} e^{ni\theta}.$$

We shall replace the second binomial coefficient by its asymptotic value $(-1)^n An^{p-1}$, and the first by

$$A(N-n)^k;$$

the second substitution amounts to passing from a Cesàro sum to the corresponding Rieszian sum. The substitutions may be justified by arguments similar to those which we use below to justify replacing a sum by the corresponding integral. Thus $s_N{}^k$ is replaced by

$$t_N{}^k = A(N+1)^{-\beta}\sum_0^N (N-n)^k n^{p-1} e^{ni\theta}.$$

We shall show that, if $\delta > 0$ is chosen appropriately,

$$(5.2.1) \qquad \int_{N^{-1}}^{N^{-1+\delta}} |t_N{}^k|^\lambda\, d\theta > A\delta N^{k\lambda}\log N = A\delta N^{1-\lambda}\log N.$$

We replace $t_N{}^k$ by the corresponding integral

$$u_N{}^k = A(N+1)^{-\beta}\int_0^N (N-x)^k x^{p-1} e^{xi\theta}\, dx.$$

To justify this we observe that, when $n < x < n+1$,

$$(N-x)^k x^{p-1} e^{xi\theta} - (N-n)^k n^{p-1} e^{ni\theta}$$

is the sum of two terms of orders

$$O(N^{k+p-1}\theta), \quad O(N^{k+p-2})$$

respectively. The contributions of these are of orders

$$O(N^{\sigma}), \quad O(N^{\tau}),$$

where
$$\sigma = 1-2\lambda+\delta(1+\lambda),$$
$$\tau = 1-2\lambda+\delta.$$

Each of these is less than $1-\lambda$ if δ is small enough, and the error involved in replacing series by integral is then negligible.

Now

$$u_N^k = A(N+1)^{-\beta} N^{k+p} e^{Ni\theta} \int_0^1 y^k (1-y)^{p-1} e^{-Nyi\theta}\, dy.$$

The asymptotic behaviour of the last integral, for large positive $N\theta$, is known; the integral behaves like

$$A\omega(N\theta)^{-k-1},$$

where $|\omega| = 1$†. Hence, finally, we obtain

$$\int_{N^{-1}}^{N^{-1+\delta}} |u_N^k|^{\lambda}\, d\theta > AN^{\varpi} \int_{N^{-1}}^{N^{-1+\delta}} \frac{d\theta}{\theta} = A\delta N^{\varpi} \log N,$$

where

$$\varpi = \lambda(-\beta+k+p-k-1) = \lambda(-\beta+p-1) = 1-\lambda.$$

Replacing u_n first by t_n and then by s_n, we conclude that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |s_n^k|^{\lambda}\, d\theta > A\delta N^{1-\lambda} \log N = A\delta N^{k\lambda} \log N;$$

and this completes the proof.

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† See, for example, Hardy and Littlewood (1), 216–217. There is a second asymptotic term of order

$$(N\theta)^{-p},$$

here negligible because $p > 1/\lambda = k+1$.

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CORRECTIONS

p. 524, *line* 5. For (3.1.5) read (3.1.6).

p. 524, *line* 6 and *p.* 525, *line* 1. For (3.1.4) read (3.1.7).

COMMENTS

The results of this paper have been extended by a number of authors, and the following results are now known. The 'complex class L^λ' is, of course, the Hardy class H^λ.

(A) *Let $f \in H^\lambda$, where $\lambda > 0$, and let $\sigma_n^k(z)$ be defined as above.*

(i) *If $0 < \lambda < 1$ and $k \geqslant 1/\lambda - 1$, then $\sigma_n^k(e^{i\theta}) \to f(e^{i\theta})$ as $n \to \infty$, for almost all θ.*

(ii) *If $0 < \lambda \leqslant 1$ and $k > 1/\lambda - 1$, then*

$$\int_{-\pi}^{\pi} \sup_n |\sigma_n^k(e^{i\theta})|^\lambda \, d\theta \leqslant A(k,\lambda) M_\lambda^\lambda(1). \tag{1}$$

(iii) *If $0 < \lambda \leqslant 1$, and $k = 1/\lambda - 1$, then*

$$\int_{-\pi}^{\pi} \sup_n \left\{ \frac{|\sigma_n^k(e^{i\theta})|^\lambda}{\log(n+1)} \right\} d\theta \leqslant A(\lambda) M_\lambda^\lambda(1).$$

(iv) *If $0 < \lambda < 1$, $0 < \mu < 1$, and $k = 1/\lambda - 1$, then*

$$\int_{-\pi}^{\pi} \sup_n |\sigma_n^k(e^{i\theta})|^{\lambda\mu} \, d\theta \leqslant A(\lambda,\mu) M_\lambda^{\lambda\mu}(1).$$

(B) *If $f \in H^\lambda \log^+ H$, where $0 < \lambda \leqslant 1$, and $k = 1/\lambda - 1$, then*

$$\int_{-\pi}^{\pi} \sup_n |\sigma_n^k(e^{i\theta})|^\lambda \, d\theta \leqslant A(\lambda) \int_{-\pi}^{\pi} |f(e^{i\theta})|^\lambda \log^+ |f(e^{i\theta})| \, d\theta + A(\lambda).$$

(C) *If u is the real part of f, and $0 < \lambda \leqslant 1$, $k > 1/\lambda - 1$, then*

$$\int_{-\pi}^{\pi} |\mathrm{re}\, \sigma_n^k(e^{i\theta})|^\lambda \, d\theta \leqslant A(k,\lambda) \sup_{0 \leqslant \rho < 1} \left\{ \int_{-\pi}^{\pi} |u(\rho,\theta)|^\lambda \, d\theta \right\}$$

whenever the supremum on the right is finite.

Here A (i) completes Theorem 4, and A (ii) completes Theorem 5 and contains Theorem 1. Theorem 6 shows that (1) is false when $k = 1/\lambda - 1$, and A (iii), (iv) and B are substitutes for this case.

The result of A (i) is due to A. Zygmund, *Proc. London Math. Soc.* (2), 47 (1942), 326–50. Previously, A. E. Gwilliam, *J. London Math. Soc.* 10 (1935), 248–53, had proved the case $k > 1/\lambda - 1$.

The result of A (ii) is due to G. I. Sunouchi, *Tôhoku Math. J.* (2), 2 (1950–1), 71–88 (for $\lambda = 1$) and ibid. 7 (1955), 96–109 (for $0 < \lambda < 1$). Alternative proofs have been given by T. M. Flett, *Proc. London Math. Soc.* (3), 7 (1957), 113–41 and 211–18, and *Tôhoku Math. J.* (2), 12 (1960), 34–46, and by E. M. Stein, *Ann. of Math.* 68 (1958), 584–603.

The case $\lambda = 1$ of A (iii) is due to A. Zygmund, *Fund. Math.* 30 (1938), 170–96, and the case $0 < \lambda < 1$ to G. I. Sunouchi, *Tôhoku Math. J.* (2), 7 (1955), 96–109 and 8 (1956), 125–46. Alternative proofs have been given by E. M. Stein and G. Weiss, *Tôhoku Math. J.* (2), 9 (1957), 318–39 and by T. M. Flett, *Quart. J. of Math.* (2), 10 (1959), 179–201.

The cases $0 < \lambda \leqslant \frac{1}{2}$ of A (iv) and B are due to G. I. Sunouchi, *Tôhoku Math. J.* (2), 7 (1955), 96–109, and A. Zygmund, *Proc. Nat. Acad. Sci.* 42 (1956), 208–12, and the cases $\frac{1}{2} < \lambda < 1$ to E. M. Stein, loc. cit. Alternative proofs have been given by T. M. Flett, *Tôhoku Math. J.* (2), 12 (1960), 34–46.

The result C is due to A. E. Gwilliam, *Proc. London Math. Soc.* (2), 40 (1936), 345–52. It is not quite the 'improbable' result mentioned at the end of § 1.3, but is nevertheless surprising.

NOTES ON THE THEORY OF SERIES (XX): GENERALIZATIONS OF A THEOREM OF PALEY

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1. In this note we are concerned primarily with power series, such as

$$f(z) = f(re^{i\theta}) = \sum_0^\infty a_n z^n,$$

convergent for $r < 1$. We write

$$M_p(f) = M_p(f,r) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p}.$$

Then $M_p(f,r)$ increases, as $r \to 1$, to a limit

$$M_p(f,1) = \lim_{r\to 1} M_p(f,r),$$

which may be $+\infty$.

If $M_p(f,r)$ is bounded or, what is the same thing, if $M_p(f,1) < \infty$, we say that $f(z)$ belongs to the (complex) class L^p. In this case there is a 'boundary function' $F(\theta) = f(e^{i\theta})$, which is both a radial limit of $f(re^{i\theta})$ for almost all θ, and a 'strong limit' with index p, and

$$M_p(f,1) = M_p(F) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(\theta)|^p \, d\theta\right)^{1/p}.$$

If
$$g(z) = \sum b_n z^n$$
is also regular for $r < 1$, then so is

$$h(z) = \sum a_n b_n z^n.$$

We shall say that 'the sequence (b_n) transforms f into h'.

A sequence (b_n) may have the property of transforming every function f of some given class into a function h of some other class. In particular it may transform every f of L^p into an h of L^q, where $q \geqslant p$. In this case we shall call (b_n) a '$(p \to q)$-sequence'. We have proved elsewhere* that if $q > p$ and

$$\alpha = \frac{1}{p} - \frac{1}{q}$$

then $(n^{-\alpha})$ is a $(p \to q)$-sequence. In particular, the sequence $(n^{-\frac{1}{2}})$ is a $(1 \to 2)$-sequence.

* Hardy and Littlewood (**3**, II), 415, Theorem 33.

1937, 3 (with J. E. Littlewood) *Quarterly Journal of Mathematics*, 8, 161–71.

2. There are $(1 \to 2)$-sequences of less regular types, one of which was found by Paley in the paper to which our title refers.* Paley proved that if f is L (i.e. L^1), λ_m is integral, and

$$\frac{\lambda_{m+1}}{\lambda_m} \geqslant c > 1, \tag{2.1}$$

then
$$\sum |a_{\lambda_m}|^2 < \infty, \tag{2.2}$$

so that
$$h(z) = \sum a_{\lambda_m} z^{\lambda_m}$$

is L^2. In particular
$$h(z) = \sum a_{2^m} z^{2^m}$$

is L^2. In other words, the sequence $b_n = 1$ $(n = 2^m)$, $b_n = 0$ $(n \neq 2^m)$ is a $(1 \to 2)$-sequence. The theorem is essentially one on power-series, the analogue for general Fourier series being false†; and this adds to its interest.

3. Our first object here is to prove a theorem which shall contain both Paley's theorem and the special theorem of our own referred to at the end of § 1. We shall, however, go further than is necessary for this particular purpose.

Our main result is

THEOREM 1. *Suppose that*

$$p \geqslant 1, \qquad s \geqslant 1, \qquad \frac{1}{p}+\frac{1}{s} > 1, \tag{3.1}$$

$$\frac{1}{\lambda} = \frac{1}{p}+\frac{1}{s}-1, \tag{3.2}$$

and
$$p \leqslant 2 \leqslant \lambda < \infty. \tag{3.3}$$

Suppose further‡ *that*
$$a_0 = 0, \tag{3.4}$$

$$M_p(f) \leqslant B, \tag{3.5}$$

and
$$M_s(g') \leqslant \frac{C}{1-r}. \tag{3.6}$$

Then
$$M_\lambda(h) \leqslant A(p,s)BC; \tag{3.7}$$

so that (3.6) *is a sufficient condition that* (b_n) *should be a* $(p \to \lambda)$-*sequence.*

* Paley (**6**).

† The series
$$\sum a_n \cos n\theta = \sum_{2}^{\infty} \frac{\cos n\theta}{(\log n)^c}$$
is a Fourier series for every positive c (by a well-known theorem of W. H. Young), but $\sum |a_{2^m}|^2$ diverges if $c \leqslant \frac{1}{2}$.

‡ This assumption avoids a number of trivial complications. It is natural because a_0 affects only the constant term $a_0 b_0$ of h and (3.6) does not involve b_0.

Here $A(p, s)$ denotes some positive number depending on p and s alone. Generally $A(\alpha, \beta, \ldots)$ is a positive number depending on $\alpha, \beta, \ldots$ alone; so that A, without argument, is a positive absolute constant.

We may observe that (3.1) and (3.2) imply that $p \leqslant \lambda < \infty$, which is part of (3.3). The condition $\lambda \geqslant 2$ is equivalent to

$$p \geqslant \frac{2s}{3s-2}.$$

If $s = 1$, this is $p \geqslant 2$; and 2 is then the only admissible value of p. If $s = 2$, it is $p \geqslant 1$, and if $s > 2$ it is weaker, so that in these cases it is a consequence of (3.1) and may be dropped.

Suppose in particular that $s = 2$. The condition (3.6) is then equivalent to

$$\sum_1^n \nu^2 |b_\nu|^2 = O(n^2).$$

This is plainly true when $b_n = n^{-\frac{1}{2}}$. If $b_n = 1$ when $n = \lambda_m$, $b_n = 0$ otherwise, λ_m satisfies (2.1), and λ_M is the largest λ_m not exceeding n, then

$$\sum_1^n \nu^2 |b_\nu|^2 = \sum_0^M \lambda_m^2 = O(\lambda_M^2) = O(n^2).$$

Hence Theorem 1 includes the two special theorems to which we have referred.

Lemmas

4. Lemma 1.* *Suppose that* $p \geqslant 1$, $q \geqslant 2$; *that* $a_0 = 0$; *that* ϑ *is the operator*

$$\vartheta = z\frac{d}{dz};$$

and that

$$J_1 = \left(\int_0^1 \frac{1}{r}\log\frac{1}{r}\, M_q^2(\vartheta f, r)\, dr\right)^{\frac{1}{2}},$$

$$J_2 = \left(\int_0^1 \frac{1}{r}\left(\log\frac{1}{r}\right)^3 M_q^2(\vartheta^2 f, r)\, dr\right)^{\frac{1}{2}}.$$

Then

$$M_q^2(f, 1) \leqslant A(q) J_1^2 \leqslant A(q) J_2^2. \tag{4.1}$$

We may suppose f regular for $r \leqslant 1$. If the inequalities have been

* Lemmas 1 and 2 can both be deduced from theorems of Littlewood and Paley (**5**); but the direct proofs of the lemmas given here are much simpler.

proved for such f, they may be extended to general f by standard processes of approximation.*

It is known† that

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}M_q^q(f,r)\right) = \frac{q^2}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^{q-2}|f'(re^{i\theta})|^2\,d\theta.$$

Hence, if we write for the moment

$$\chi(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\rho e^{i\theta})|^{q-2}|f'(\rho e^{i\theta})|^2\,d\rho,$$

we have
$$r\frac{d}{dr}M_q^q(f,r) = q^2\int_0^r \rho\chi(\rho)\,d\rho,$$

$$M_q^q(f,1) = q^2\int_0^1 \frac{dr}{r}\int_0^r \rho\chi(\rho)\,d\rho$$

$$= q^2\int_0^1 \rho\chi(\rho)\,d\rho\int_\rho^1 \frac{dr}{r} = q^2\int_0^1 \rho\log\frac{1}{\rho}\chi(\rho)\,d\rho.$$

But
$$\chi(\rho) \leqslant M_q^{q-2}(f,\rho)M_q^2(f',\rho),$$

* Suppose, for example, that the first inequality (4.1) has been proved when f is regular for $r \leqslant 1$, and apply it to

$$f_R(z) = f(Rz),$$

where $0 < R < 1$. Then

$$M_q^2(f,R) = M_q^2(f_R,1) \leqslant A(q)\int_0^1 \frac{1}{r}\log\frac{1}{r}M_q^2(\vartheta f_R, r)\,dr$$

$$= A(q)\int_0^1 \frac{1}{r}\log\frac{1}{r}M_q^2(\vartheta f, rR)\,dr$$

$$\leqslant A(q)\int_0^1 \frac{1}{r}\log\frac{1}{r}M_q^2(\vartheta f, r)\,dr,$$

for $0 < R < 1$ and therefore for $R = 1$.

† This identity was found by Hardy (1) and rediscovered by Stein, who used it in his proof of M. Riesz's theorem about conjugate functions. See Stein (7), or Zygmund (8), 149.

by Hölder's inequality. Hence

$$M_q^q(f,1) \leqslant q^2 \int_0^1 \rho \log\frac{1}{\rho}\, M_q^{q-2}(f,\rho) M_q^2(f',\rho)\, d\rho$$

$$\leqslant q^2 M_q^{q-2}(f,1) \int_0^1 \rho \log\frac{1}{\rho}\, M_q^2(f',\rho)\, d\rho.$$

Removing the common factor $M_q^{q-2}(f,1)$,* and observing that $f' = z^{-1}\vartheta f$, we obtain the first inequality (4.1), with $A(q) = q^2$. The inequality reduces to an identity when $q = 2$.

We may observe that

$$r \log\frac{1}{r} < 1-r$$

for $0 < r < 1$. Hence

$$M_q^2(f,1) \leqslant A(q) \int_0^1 (1-r) M_q^2(f',r)\, dr. \tag{4.2}$$

This inequality is formally a little simpler, but the form in (4.1) is a little stronger and serves our purpose better.

We now apply what we have proved to the function

$$\phi(z) = (\vartheta f(z))_R = Rzf'(Rz) \quad (0 < R < 1).$$

Then $\vartheta\phi(z) = z\dfrac{d}{dz}(Rzf'(Rz)) = Rz\dfrac{d}{dRz}(Rzf'(Rz)) = (\vartheta^2 f(z))_R$

and $\quad M_q^2(\phi,1) = M_q^2(\vartheta f, R), \qquad M_q^2(\vartheta\phi,1) = M_q^2(\vartheta^2 f, R).$

Hence

$$M_q^2(\vartheta f, R) \leqslant q^2 \int_0^1 \frac{1}{r}\log\frac{1}{r}\, M_q^2(\vartheta^2 f, Rr)\, dr$$

$$= q^2 \int_0^R \frac{1}{\rho}\log\frac{R}{\rho}\, M_q^2(\vartheta^2 f,\rho)\, d\rho.$$

Finally, replacing R by r, and integrating, we obtain

$$J_1^2 = \int_0^1 \frac{1}{r}\log\frac{1}{r}\, M_q^2(\vartheta f, r)\, dr$$

$$\leqslant q^2 \int_0^1 \frac{1}{r}\log\frac{1}{r}\, dr \int_0^r \frac{1}{\rho}\log\frac{r}{\rho}\, M_q^2(\vartheta^2 f,\rho)\, d\rho$$

* Finite because f is regular for $r \leqslant 1$. We ignore the trivial case $f = 0$.

$$= q^2 \int_0^1 \frac{1}{\rho} M_q^2(\vartheta^2 f, \rho)\, d\rho \int_\rho^1 \frac{1}{r} \log\frac{1}{r} \log\frac{r}{\rho}\, dr$$

$$= \frac{1}{6} q^2 \int_0^1 \frac{1}{\rho}\left(\log\frac{1}{\rho}\right)^3 M_q^2(\vartheta^2 f, \rho)\, d\rho,$$

and so the second inequality (4.1).

5. Lemma 2. *If* $1 \leqslant p \leqslant 2$, *then*

$$\int_0^1 (1-r) M_p^2(f', r)\, dr \leqslant A(p) M_p^2(f, 1) \tag{5.1}$$

We may suppose that f has no zeros in $r < 1$.* We may then write

$$f = \phi^k = (\sum \alpha_n z^n)^k,$$

where $k = 2/p \geqslant 1$; so that $f^p = \phi^2$ and ϕ is L^2.

Then
$$M_p^p(f') = A(p) \int |\phi^{k-1}\phi'|^p\, d\theta$$
$$\leqslant A(p)\Big(\int |\phi|^{kp}\, d\theta\Big)^{(k-1)/k}\Big(\int |\phi'|^{kp}\, d\theta\Big)^{1/k}$$
$$= A(p) M^{(k-1)p/k}(f, r)\Big(\int |\phi'|^2\, d\theta\Big)^{1/k};$$

throughout these formulae $|z| = r < 1$. Hence

$$M_p^2(f', r) \leqslant A(p) M_p^{(k-1)p}(f, 1) \int_{-\pi}^{\pi} |\phi'(re^{i\theta})|^2\, d\theta,$$

$$\int_0^1 (1-r) M_p^2(f', r)\, dr \leqslant A(p) M_p^{(k-1)p}(f, 1) \int_0^1 (1-r)\, dr \int_{-\pi}^{\pi} |\phi'(re^{i\theta})|^2\, d\theta.$$

The repeated integral is

$$2\pi \int_0^1 (1-r) \sum n^2 |\alpha_n|^2 r^{2n-2}\, dr = 2\pi \sum_1^\infty \frac{n^2}{(2n-1)2n} |\alpha_n|^2$$

$$\leqslant A \sum_1^\infty |\alpha_n|^2 = A \int_{-\pi}^{\pi} |\phi(e^{i\theta})|^2\, d\theta = A M_p^p(f, 1);$$

and (5.1) follows.†

* This is one more example of a general principle which we have used repeatedly. We may suppose $M_p(f,1)$ finite, since otherwise there is nothing to prove. If $M_p(f) \leqslant B$, then $f = f_1 + f_2$, where f_1 and f_2 are 'wurzelfrei', and $M_p(f_1) \leqslant AB$, $M_p(f_2) \leqslant AB$. We prove the result for f_1 and f_2, and the general result follows. See Hardy and Littlewood (**2**), 207.

† This inequality is a little stronger than

$$\int_0^1 \frac{1}{r} \log\frac{1}{r} M_p^2(\vartheta f, r)\, dr \leqslant A(p) M_p^2(f, 1).$$

LEMMA 3. $$M_\lambda(\vartheta^2 h, r^2) \leqslant M_p(\vartheta f, r) M_s(\vartheta g, r).$$

Since* $$\vartheta^2 h(r^2 e^{i\theta}) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \vartheta f(re^{i\psi})\vartheta g(re^{i\theta-i\psi})\,d\psi,$$

this is a case of a familiar theorem of W. H. Young.

Proof of Theorem 1

6. We may suppose, on grounds of homogeneity, that $B = C = 1$. We have then

$$M_\lambda^2(h,1) \leqslant A(p,s)\int\limits_0^1 \frac{1}{\rho}\left(\log\frac{1}{\rho}\right)^3 M_\lambda^2(\vartheta^2 h, \rho)\,d\rho$$

$$\leqslant A(p,s)\int\limits_0^1 \frac{1}{r}\left(\log\frac{1}{r}\right)^3 M_\lambda^2(\vartheta^2 h, r^2)\,dr$$

$$\leqslant A(p,s)\int\limits_0^1 \frac{1}{r}\left(\log\frac{1}{r}\right)^3 M_p^2(\vartheta f, r) M_s^2(\vartheta g, r)\,dr,$$

by Lemmas 1 and 3. Observing that

$$M_s^2(\vartheta g, r) = r^2 M_s^2(g', r) \leqslant \frac{r^2}{(1-r)^2},$$

that $$M_p^2(\vartheta f, r) \leqslant M_p^2(f', r),$$

and that $$r\left(\log\frac{1}{r}\right)^3 \leqslant A(1-r)^3$$

for $0 < r < 1$, we obtain

$$M_\lambda^2(h,1) \leqslant A(p,s)\int\limits_0^1 (1-r) M_p^2(f', r)\,dr;$$

and the conclusion follows from Lemma 2.

The case $p = 1$, $s = 2$

7. There is a much simpler proof when $p = 1, s = 2$ (a case which includes Paley's theorem).

We may suppose f 'wurzelfrei', and write

$$f = \phi^2 = \left(\sum \alpha_n z^n\right)^2,$$

so that $$a_n = \alpha_0\alpha_n + \alpha_1\alpha_{n-1} + \dots + \alpha_n\alpha_0$$

and $$\sum |\alpha_n|^2 = \mathbf{A} < \infty \tag{7.1}$$

* This formula is simpler than the corresponding formula for h'', and it is for this reason that we have worked throughout in terms of ϑ rather than with ordinary derivatives.

We have to prove, substantially*, that (7.1) and

$$\mathbf{B}_n = \sum_1^n \nu^2 |b_\nu|^2 = O(n^2) \tag{7.2}$$

imply

$$\sum |a_n|^2 |b_n|^2 < \infty. \tag{7.3}$$

It is plain that, in proving this, we may suppose α_n and b_n positive.

Now

$$a_n = \sum_{\mu+\nu=n} \alpha_\mu \alpha_\nu \leqslant 2 \sum_{\mu \leqslant \frac{1}{2}n} \alpha_\mu \alpha_{n-\mu} = 2p_n$$

say; and it is enough to prove that $\sum p_n^2 b_n^2$ is convergent. But

$$p_n^2 \leqslant \sum_{\mu \leqslant \frac{1}{2}n} \alpha_\mu^2 \sum_{\frac{1}{2}n \leqslant \nu \leqslant n} \alpha_\nu^2 \leqslant \mathbf{A} \sum_{\frac{1}{2}n}^{n} a_\nu^2,$$

$$\sum p_n^2 b_n^2 \leqslant \mathbf{A} \sum_1^\infty \left(n^2 b_n^2 \frac{1}{n^2} \sum_{\frac{1}{2}n}^{n} \alpha_\nu^2 \right),$$

which is convergent if

$$\sum_1^\infty \mathbf{B}_n \Delta \left(\frac{1}{n^2} \sum_{\frac{1}{2}n}^{n} \alpha_\nu^2 \right) \tag{7.4}$$

is convergent and $\mathbf{B}_n \sum_{\frac{1}{2}n}^{n} \alpha_\nu^2 = O(n^2)$. The second condition is satisfied, by (7.1) and (7.2). Also

$$\left| \Delta \left(\frac{1}{n^2} \sum_{\frac{1}{2}n}^{n} \alpha_\nu^2 \right) \right| \leqslant \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \sum_{\frac{1}{2}n}^{n} \alpha_\nu^2 + \frac{1}{n^2} (\alpha_{\frac{1}{2}n}^2 + \alpha_{n+1}^2),$$

where $\alpha_{\frac{1}{2}n}$ means 0 if n is odd. The second term contributes to (7.4) an amount

$$\sum \frac{O(n^2)}{n^2} (\alpha_{n+1}^2 + \alpha_{\frac{1}{2}n}^2) < \infty;$$

and the first contributes

$$\sum_n \frac{O(n^2)}{n^3} \sum_{\frac{1}{2}n}^{n} \alpha_\nu^2 = \sum_\nu \alpha_\nu^2 \sum_\nu^{2\nu} O\left(\frac{1}{n}\right) = \sum_\nu \alpha_\nu^2 O(1) < \infty.$$

8. We have expressed everything in terms of power series, and this is essential when $p = 1$. But when $p > 1$ we can extend the results to general Fourier series. Thus we have

THEOREM 2. *Suppose that $p > 1$, $s \geqslant 1$; that*

$$F(\theta) \sim \sum_{-\infty}^{\infty} a_n e^{ni\theta}$$

is L^p; that

$$u(r, \theta) = \sum_{-\infty}^{\infty} b_n r^{|n|} e^{ni\theta}$$

* In order to prove that (b_n) is a $(1 \to 2)$-sequence. If we assume that $a_0 = 0$, we can arrange the argument so as to prove the inequality (3.7).

satisfies $$M_s(u') = O\left(\frac{1}{1-r}\right),$$

where u' is one or other of $\dfrac{\partial u}{\partial r}$ *or* $\dfrac{\partial u}{\partial \theta}$;* *and that*

$$\frac{1}{\lambda} = \frac{1}{p}+\frac{1}{s}-1, \qquad 1 < p \leqslant 2 \leqslant \lambda.$$

Then $$H(\theta) \sim \sum_{-\infty}^{\infty} a_n b_n e^{ni\theta}$$

is L^λ.

This follows from Theorem 1 when we observe that (after M. Riesz's theorem)

$$\sum_{n\geqslant 0} c_n r^n e^{ni\theta}, \qquad \sum_{n<0} c_n r^{|n|} e^{|n|i\theta}$$

are power series of L^p, and that (after a theorem of our own†) the series

$$\sum_{n\geqslant 0} b_n r^n e^{ni\theta}, \qquad \sum_{n<0} b_n r^{|n|} e^{|n|i\theta}$$

satisfy (3.6).

9. The condition $p \geqslant 1$ in Theorem 1 is essential. If $p < 1$, $\delta > 0$, then

$$f(z) = \sum n^{\frac{1}{p}-1-\delta} z^n$$

is L^p, and $$g(z) = \sum z^{2^m}$$

satisfies (3.6) for every s. But

$$h(z) = \sum 2^{m\left(\frac{1}{p}+1-\delta\right)} z^{2^m}$$

is not L^λ for any λ if $$\frac{1}{p}-1 > \delta.$$

10. There is an analogue of Theorem 1 in which the condition (3.6) is replaced by

$$M_s(g') = O((1-r)^{k-1}), \tag{10.1}$$

where k is positive. We state this theorem here without proof, but the statement requires a few words of preliminary explanation.

Suppose first that $0 < k < 1$. Then the condition (10.1) is equivalent to the condition that $G(\theta) = g(e^{i\theta})$, the boundary function

* The two forms of the hypothesis are equivalent. See Hardy and Littlewood (**4**), 410, Theorem 1. It is not necessary for this that $s > 1$.

† The one referred to in the preceding footnote.

of $g(z)$, belongs to the class $\mathrm{Lip}(k,s)$, i.e. that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |G(\theta+h)-G(\theta-h)|^s\,d\theta = O(|h|^{ks}).*$$

The equivalence may be stated more generally, but requires an extension of the definition of the class $\mathrm{Lip}\,(k,s)$. We shall say that g is $\mathrm{Lip}(k,s)$, for any $k \geqslant 0$, if $[k] = \kappa$ and $g^{(\kappa)}$ is $\mathrm{Lip}(k-\kappa,s)$, i.e. if

$$M_s(g^{(\kappa+1)}) = O((1-r)^{k-\kappa-1}).\dagger$$

Then we have

THEOREM 3. *If* $p > 0$, $s \geqslant 1$, $k > 0$,

$$\frac{1}{p}-1 \leqslant k < \frac{1}{p}+\frac{1}{s}-1,$$

$$\frac{1}{\lambda} = \frac{1}{p}+\frac{1}{s}-k-1$$

(so that $p < \lambda < \infty$*),* f *is* L^p*, and* g *is* $\mathrm{Lip}(k,s)$*, then* h *is* L^λ.

In other words, (b_n) is a $(p \to \lambda)$-sequence whenever g is $\mathrm{Lip}(k,s)$.

It will be observed (i) that p can now be less than 1, and (ii) that the number 2 no longer appears in the conditions. There are indeed four cases of the theorem, viz.

(1) $0 < p < 1$, (2) $1 \leqslant p < \lambda \leqslant 2$,

(3) $1 \leqslant p \leqslant 2 < \lambda$, (4) $2 < p < \lambda$;

and Theorem 1 corresponds to case (3) only, the other cases falling out when $k = 0$. We proved in § 9 that (5) the conclusion of Theorem 1 becomes false when $p < 1$. We may add that (6) it also becomes false when $1 \leqslant p < \lambda < 2$ or when $2 < p < \lambda$; but the proof of this depends upon the most difficult of the theorems proved by Littlewood and Paley in their paper quoted on p. 163.‡

* Hardy and Littlewood (**3**, II), Theorem 48.

† In particular, the class $\mathrm{Lip}\,(0,s)$ is the class of g satisfying (3.6) for some C.

‡ Littlewood and Paley (**5**). We can prove cases (1) and (3) of Theorem 3 (like Theorem 1) without using any of the Littlewood-Paley theorems; the proofs are indeed a good deal easier than that of Theorem 1. The other cases of Theorem 3 require their Theorem 5. The most difficult of their theorems, viz. Theorem 7, is needed only in proving (6).

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COMMENTS

p. 162. An alternative proof of Theorem 1 is given in 1941, 1.

p. 170. A proof of the case $0 < k < 1, p > 1$ of Theorem 3 is given in 1941, 1. A generalization of the theorem has been given by T. M. Flett, *Pac. J. Math.* 25 (1968), 463–94.

It should be noted that parts of Theorems 1 and 3 of 1932, 5 correspond to the case $s = \infty$ of Theorem 3.

THEOREMS CONCERNING MEAN VALUES OF ANALYTIC OR HARMONIC FUNCTIONS

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Introduction

1. In this paper we complete (with one reservation stated at the end of this section) our account of a body of work which has occupied us at intervals since 1924.* This work has been a piecemeal growth, and the logical order in which it stands is in some ways odd and anomalous. We can now give a more unified account of the main results (omitting some extensions which have been rather side-issues).

The critical proofs will also be much *shorter*. We cannot claim without reservation that they are *simpler*, because we allow ourselves to appeal to certain theorems of Littlewood and Paley which were not available until recently;† and the proofs of these theorems (of one in particular) are difficult. We could avoid appealing to these theorems if we chose. Parts of our analysis are quite independent of them; in others we could substitute *ad hoc* arguments such as we used before; and in one part,‡ where we thought that we should be compelled to use them, we now find them unnecessary. On balance it seems best to take advantage of them where we can.

The paper is not a mere revision of old work: it contains proofs of theorems stated before without proof, and one entirely new theorem (Theorem 8). In one respect, however, we do less than in our earlier papers, since we suppose throughout that the parameter r is greater than 1.§

We begin by a statement of the chief theorems to which we shall appeal.

* See in particular our papers **3, 6, 7.**

† They were stated without proof in 1931, and the first proofs published in 1937: see Littlewood and Paley, **10** and **11.**

‡ §§ 24–7. See **7**, 170, footnote †.

§ In §§ 20–1 we suppose only that $r \geqslant 1$. But a good many of the theorems are true for all positive r (and were proved so in our former papers). Thus the proof of Theorem 3 of **7** remains incomplete: the case $0 < p < 1$ is still unaccounted for. Actually, this case is comparatively easy.

1941, 1 (with J. E. Littlewood) *Quarterly Journal of Mathematics*, 12, 221–56.

Theorems used

2. In what follows $$f(z) = \sum_0^\infty c_n z^n \tag{2.1}$$
is an analytic function of $z = \rho e^{i\theta}$ regular for $\rho < 1$. The indices p, q, r, s are finite and satisfy
$$1 < p \leqslant 2 \leqslant q, \qquad r > 1, \qquad s > 1 \tag{2.2}$$
(except in §§ 20–1, where we allow r to be 1).

If $\phi(\theta)$ is any function of θ, then
$$\mathfrak{M}_r(\phi) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\phi(\theta)|^r\,d\theta\right)^{1/r}. \tag{2.3}$$
In particular
$$\mathfrak{M}_r(f) = \mathfrak{M}_r(f,\rho) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\rho e^{i\theta})|^r\,d\theta\right)^{1/r}. \tag{2.4}$$

We use A, B, C as follows. A is, at each of its occurrences, a positive absolute constant; B a positive number depending only on the indices $p, q,\ldots$, or other parameters of the argument:* both A and B may differ at different occurrences even in the same formula. C is a positive number occurring in the hypothesis of a theorem and preserving its identity throughout the statement and proof of the theorem.

THEOREM A. *If*
$$\chi(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\theta-t)\psi(t)\,dt, \tag{2.5}$$
the Faltung $\qquad \mathfrak{F}(\phi,\psi)$

of ϕ *and* ψ, *and*
$$\alpha > 1, \qquad \beta > 1, \qquad \frac{1}{\alpha}+\frac{1}{\beta} > 1, \qquad \frac{1}{\gamma} = \frac{1}{\alpha}+\frac{1}{\beta}-1, \tag{2.6}$$
then $$\mathfrak{M}_\gamma(\chi) \leqslant \mathfrak{M}_\alpha(\phi)\mathfrak{M}_\beta(\psi). \tag{2.7}$$

This is a familiar inequality due to W. H. Young.† The inequality (and all others which we quote or prove) is to be interpreted as

* Which we shall indicate explicitly, by writing $B(p,\ldots)$, if there is any risk of confusion.

† See, for example, Hardy, Littlewood, and Pólya (**9**, 198–202) or Zygmund (**12**, 71).

meaning 'if the right-hand side is finite, then the left-hand side is also finite, and ...'.

THEOREM B. *If*
$$\mathfrak{M}_r(f) \leqslant C \tag{2.8}$$
for $\rho < 1$, *then* f *can be expressed in the form*
$$f = f_1 + f_2, \tag{2.9}$$
where (i) f_1 *and* f_2 *are regular,* (ii) $f_1 \neq 0$ *and* $f_2 \neq 0$, *and* (iii)
$$\mathfrak{M}_r(f_1) \leqslant AC, \qquad \mathfrak{M}_r(f_2) \leqslant AC \tag{2.10}$$
for $\rho < 1$.

This theorem (which is actually true for all positive r) is proved in our paper **2**.* It is a simple corollary of a theorem of F. Riesz.

THEOREM C. *Suppose that* $S(\theta)$ *is the region bounded by the two tangents from the point* $e^{i\theta}$ *to the circle* $\rho = \rho_0 < 1$, *and the more distant arc of the circle between the points of contact, and that* $F(\theta)$ *is the upper bound of* $|f(z)|$ *in* $S(\theta)$. *Then*
$$\mathfrak{M}_r(F) \leqslant B(r, \rho_0) \lim_{\rho \to 1} \mathfrak{M}_r(f, \rho). \tag{2.11}$$

This theorem (which also is true for all positive r) is proved in our paper **5**.†

The form in which we have written (2.11) requires a word of explanation. The inequality says nothing unless the limit on the right-hand side is finite, i.e. unless $\mathfrak{M}_r(f)$ is bounded. In this case $f(z)$ has a 'boundary function' $f(e^{i\theta})$ of the class L^r: $f(\rho e^{i\theta}) \to f(e^{i\theta})$, when $\rho \to 1$, for almost all θ, and
$$\mathfrak{M}_r\{f(\rho e^{i\theta}) - f(e^{i\theta})\} \to 0,$$
so that $f(e^{i\theta})$ is also a 'strong limit' of $f(\rho e^{i\theta})$. Also
$$\lim_{\rho \to 1} \mathfrak{M}_r(f, \rho) = \mathfrak{M}_r\{f(e^{i\theta})\} = \mathfrak{M}_r(f, 1),$$
and (2.11) may be written in the simpler form
$$\mathfrak{M}_r(F) \leqslant B(r, \rho_0) \mathfrak{M}_r(f, 1). \tag{2.12}$$
But $f(z)$ may have a boundary function of L^r, defined as a radial limit, even when $\mathfrak{M}_r(f, \rho)$ is not bounded, so that we could not state (2.11) unconditionally in the form (2.12).

* Hardy and Littlewood, **2**, 207.

† Hardy and Littlewood, **5**, 114. The definition of $S(\theta)$ there is slightly different.

THEOREM D. *Suppose that* $1 < p \leqslant 2 \leqslant q$,* *and that*

$$c_0 = f(0) = 0. \tag{2.13}$$

Then
$$\lim_{\rho \to 1} \mathfrak{M}_p^p(f, \rho) \leqslant B \int_0^1 \int_{-\pi}^{\pi} (1-\rho)^{p-1} |f'(\rho e^{i\theta})|^p \, d\rho d\theta, \tag{2.14}$$

$$\int_0^1 \int_{-\pi}^{\pi} (1-\rho)^{q-1} |f'(\rho e^{i\theta})|^q \, d\rho d\theta \leqslant B \lim_{\rho \to 1} \mathfrak{M}_q^q(f, \rho). \tag{2.15}$$

When the limits are finite, they are equal to $\mathfrak{M}_p^p(f, 1)$ and $\mathfrak{M}_q^q(f, 1)$. We suppose $c_0 = 0$ (as we shall do in a number of later theorems) in order to avoid trivial complications. It is obvious, for example, that (2.14) could not be true without some such restriction, since the right-hand side is independent of c_0.

THEOREM E. *If* $c_0 = 0$, $r > 1$, *and*

$$g(\theta) = \left\{ \int_0^1 (1-\rho) |f'(\rho e^{i\theta})|^2 \, d\rho \right\}^{\frac{1}{2}}, \tag{2.16}$$

then
$$\mathfrak{M}_r(g) \leqslant B \lim_{\rho \to 1} \mathfrak{M}_r(f, \rho), \qquad \lim_{\rho \to 1} \mathfrak{M}_r(f, \rho) \leqslant B \mathfrak{M}_r(g). \tag{2.17}$$

There is naturally the same gloss about $\mathfrak{M}_r(f, 1)$. Theorems D and E contain Theorems 5–7 of the second Littlewood-Paley paper (Theorems 1–3 of the first).

THEOREM F. *If*

$$u(z) = u(\rho, \theta) = \sum_{-\infty}^{\infty} c_n \rho^{|n|} e^{ni\theta} \tag{2.18}$$

is a harmonic function of z *regular for* $\rho < 1$, *and*

$$\mathfrak{M}_r(u) \leqslant C; \tag{2.19}$$

and if we write

$$u(\rho, \theta) = \sum_0^{\infty} + \sum_{-\infty}^{-1} = \sum_0^{\infty} c_n \rho^n e^{ni\theta} + \sum_1^{\infty} c_{-n} \rho^n e^{-ni\theta} = u_1(\rho, \theta) + u_2(\rho, \theta); \tag{2.20}$$

then
$$\mathfrak{M}_r(u_1) \leqslant BC, \qquad \mathfrak{M}_r(u_2) \leqslant BC. \tag{2.21}$$

This is equivalent to M. Riesz's theorem concerning conjugate functions of the class L^r.† We shall use the theorem once only (in § 17), and then u will be a (harmonic) polynomial. In this case

$$\mathfrak{M}_r(u, \rho) \leqslant \mathfrak{M}_r(u, 1),$$

* As in (2.2): but we repeat the inequalities for greater emphasis.

† See Hardy and Littlewood (**1**) for further explanations.

and we can take $C = \mathfrak{M}_r(u, 1)$, when

$$\mathfrak{M}_r(u_1, 1) = \lim_{\rho\to 1} \mathfrak{M}_r(u_1, \rho) \leqslant BC = B\mathfrak{M}_r(u, 1). \tag{2.22}$$

It is in this form that we shall actually use the theorem.

Some lemmas

3. We shall also require the following lemmas. Lemma γ, in particular, enables us to avoid many trivial complications, and we shall use it repeatedly.

LEMMA α. *If*

$$f_1(z) = \int_0^z f(u)\, du \tag{3.1}$$

and $k \geqslant 1$, then

$$\mathfrak{M}_k(f_1, \rho) \leqslant \rho\mathfrak{M}_k(f, \rho). \tag{3.2}$$

For
$$|f_1(\rho e^{i\theta})| = \left|e^{i\theta}\int_0^\rho f(\sigma e^{i\theta})\, d\sigma\right| \leqslant \int_0^\rho |f(\sigma e^{i\theta})|\, d\sigma,$$

$$\mathfrak{M}_k(f_1) \leqslant \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\left\{\int_0^\rho |f(\sigma e^{i\theta})|\, d\sigma\right\}^k\right]^{1/k} \leqslant \int_0^\rho d\sigma\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\sigma e^{i\theta})|^k\, d\theta\right\}^{1/k},$$

by 'Minkowski's inequality';* and this is

$$\int_0^\rho \mathfrak{M}_k(f, \sigma)\, d\sigma \leqslant \rho\mathfrak{M}_k(f, \rho),$$

because $\mathfrak{M}_k(f, \rho)$ increases with ρ.

LEMMA β. *If $k \geqslant 0$, $z^{-k}f(z)$ is regular at the origin, $r > 0$*† *and $0 < \lambda < 1$, then*

$$\mathfrak{M}_r(f, \rho) \leqslant \rho^{k(1-\lambda)}\mathfrak{M}_r(f, \rho^\lambda). \tag{3.3}$$

For $0 < \rho < \rho^\lambda < 1$ and

$$\rho^{-k}\mathfrak{M}_r(f, \rho) = \mathfrak{M}_r\{z^{-k}f(z)\}$$

increases with ρ, so that

$$\rho^{-k}\mathfrak{M}_r(f, \rho) \leqslant \rho^{-\lambda k}\mathfrak{M}_r(f, \rho^\lambda).$$

LEMMA γ. *Suppose that a and b are real and c and r positive, and that*

$$J = \int_0^1 (1-\rho)^a\rho^{-b}\mathfrak{M}_r^c(f, \rho)\, d\rho \tag{3.4}$$

* See Hardy, Littlewood, and Pólya, **9**, 148 (Theorem 202).

† We are concerned only with the case $r \geqslant 1$, but the proof is independent of this hypothesis.

is convergent at the origin. Then

$$J \leqslant B \int_0^1 (1-\rho)^a \mathfrak{M}_r^c(f,\rho)\, d\rho, \tag{3.5}$$

where B depends on a, b and c.

If $b \leqslant 0$, there is nothing to prove. If $b > 0$, let k be the least integer for which $b < ck+1$, and $0 < \lambda < 1$. Then $z^{-k}f(z)$ is regular at the origin, and

$$J \leqslant \int_0^1 (1-\rho)^a \rho^{ck(1-\lambda)-b} \mathfrak{M}_r^c(f,\rho^\lambda)\, d\rho,$$

by Lemma β. If we take

$$\lambda = 1-\frac{b}{ck+1}$$

(when $0 < \lambda < 1$), and put $\rho^\lambda = \sigma$, we obtain

$$J \leqslant \frac{1}{\lambda} \int_0^1 (1-\sigma^{1/\lambda})^a \mathfrak{M}_r^c(f,\sigma)\, d\sigma. \tag{3.6}$$

But
$$1 \leqslant \frac{1-\sigma^{1/\lambda}}{1-\sigma} \leqslant \frac{1}{\lambda},$$

so that (3.6) is equivalent to (3.5).

Lemma δ. *If $0 \leqslant \rho \leqslant 1$ and n is a positive integer greater than 1, then*

$$\int_{-\pi}^{\pi} \left|\frac{1-z^n}{1-z}\right| d\theta < A \log n.$$

For the integral increases with ρ, and therefore

$$\int_{-\pi}^{\pi} \left|\frac{1-z^n}{1-z}\right| d\theta \leqslant \int_{-\pi}^{\pi} \left|\frac{1-e^{ni\theta}}{1-e^{i\theta}}\right| d\theta = 2\int_0^{\pi} \frac{|\sin \frac{1}{2}n\theta|}{\sin \frac{1}{2}\theta}\, d\theta$$

$$\leqslant 2\int_0^{1/n} n\, d\theta + 2\int_{1/n}^{\pi} \frac{d\theta}{\sin\frac{1}{2}\theta} \leqslant 2+2\pi \int_{1/n}^{\pi} \frac{d\theta}{\theta} < 2+2\pi \log n.$$

The inequality is of course familiar.

Preliminary theorems

4. THEOREM 1.* *If $1 < r < s$ and*

$$\mathfrak{M}_r(f) \leqslant C, \tag{4.1}$$

then

$$\mathfrak{M}_s(f) \leqslant BC(1-\rho)^{-\left(\frac{1}{r}-\frac{1}{s}\right)} \tag{4.2}$$

and

$$\mathfrak{M}(f) = \mathfrak{M}_\infty(f) \leqslant BC(1-\rho)^{-1/r}. \tag{4.3}$$

Here $\mathfrak{M}(f)$ is the maximum modulus of $|f(z)|$ for $|z| = \rho$.

After Theorem B, we can write $f = f_1+f_2$, where $f_1 \neq 0$, $f_2 \neq 0$, $\mathfrak{M}_r(f_1) \leqslant AC$ and $\mathfrak{M}_r(f_2) \leqslant AC$ for $\rho < 1$. If we have proved (4.2) and (4.3) for f_1 and f_2, then

$$\mathfrak{M}(f) \leqslant \mathfrak{M}(f_1)+\mathfrak{M}(f_2) \leqslant BC(1-\rho)^{-1/r}$$

and

$$\mathfrak{M}_s(f) \leqslant \mathfrak{M}_s(f_1)+\mathfrak{M}_s(f_2) \leqslant BC(1-\rho)^{-\left(\frac{1}{r}-\frac{1}{s}\right)}$$

(the last by Minkowski's inequality†). It is therefore sufficient to prove the theorem for an f without zeros in $\rho < 1$.

If f has no zeros, we can write $f = \phi^{2/r}$, where ϕ is regular in $\rho < 1$. Then

$$\mathfrak{M}_2^2(\phi) = \mathfrak{M}_r^r(f) \leqslant C^r, \qquad \mathfrak{M}_s^s(f) = \mathfrak{M}_{2s/r}^{2s/r}(\phi).$$

If the theorem has been proved for $r = 2$, then (applying that case to ϕ)

$$\mathfrak{M}_s^s(f) = \mathfrak{M}_{2s/r}^{2s/r}(\phi) \leqslant BC^{\frac{r}{2}\cdot\frac{2s}{r}}(1-\rho)^{-\frac{2s}{r}\left(\frac{1}{2}-\frac{r}{2s}\right)} = BC^s(1-\rho)^{-\left(\frac{s}{r}-1\right)},$$

which is (4.2); and

$$\mathfrak{M}(f) = \mathfrak{M}^{2/r}(\phi) \leqslant BC(1-\rho)^{-\frac{1}{2}(2/r)} = BC(1-\rho)^{-1/r},$$

which is (4.3), It is therefore sufficient to prove the theorem for $r = 2$.

When $r = 2$, we have

$$\sum |c_n|^2 = \mathfrak{M}_2^2(f) \leqslant C^2;$$

so that

$$\mathfrak{M}(f) \leqslant \sum |c_n|\rho^n \leqslant (\sum |c_n|^2 \sum \rho^{2n})^{\frac{1}{2}} \leqslant C(1-\rho)^{-\frac{1}{2}}.$$

Also

$$\mathfrak{M}_s^s(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^s\,d\theta \leqslant \mathfrak{M}^{s-2}(f)\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2\,d\theta$$

$$\leqslant \{C(1-\rho)^{-\frac{1}{2}}\}^{s-2}C^2 = \left\{C(1-\rho)^{-\left(\frac{1}{2}-\frac{1}{s}\right)}\right\}^s.$$

These are the inequalities required.

* See Hardy and Littlewood (**4**, 623–5, and **6**, 406–7). We repeat the original proof, which works for $0 < r < s$.

† In one of its more usual forms, e.g. Theorem 198 of Hardy, Littlewood, and Pólya, **9**, 146.

We shall sometimes require variants of Theorem 1 in which the C of (4.1) is replaced by a $c(\rho)$ which tends steadily to infinity when $\rho \to 1$. If then we fix a σ between 0 and 1, and apply Theorem 1 to $f(\sigma z) = f(\sigma\rho e^{i\theta})$, we obtain, for example,

$$\mathfrak{M}(f, \sigma\rho) \leqslant B(1-\rho)^{-1/r}\mathfrak{M}_r(f, \sigma) \leqslant B(1-\rho)^{-1/r}c(\sigma),$$

and so $$\mathfrak{M}(f, \sigma^2) \leqslant B(1-\sigma)^{-1/r}c(\sigma) \leqslant B(1-\sigma^2)^{-1/r}c(\sigma).$$

Finally, replacing σ^2 by ρ, we obtain

THEOREM 2. *If* $\mathfrak{M}_r(f, \rho) \leqslant c(\rho)$, *then*

$$\mathfrak{M}_s(f) \leqslant B(1-\rho)^{-\left(\frac{1}{r}-\frac{1}{s}\right)}c(\rho^{\frac{1}{2}}), \qquad \mathfrak{M}(f) \leqslant B(1-\rho)^{-\frac{1}{r}}c(\rho^{\frac{1}{2}}). \quad (4.4)$$

The most important cases are those in which (i) $c(\rho)$ is $\mathfrak{M}_r(f, \rho)$ itself, and (ii) $c(\rho) = (1-\rho)^{-a}$, where $a > 0$. In the latter case we can replace $c(\rho^{\frac{1}{2}})$, in (4.4), by $c(\rho)$. Thus, if $\mathfrak{M}_r(f)$ is of order $(1-\rho)^{-a}$, $\mathfrak{M}(f)$ is of order $(1-\rho)^{-a-1/r}$ at most.

5. THEOREM 3.* *If* $\mathfrak{M}_r(f) \leqslant C$, *then*

$$\mathfrak{M}_r(f') \leqslant \frac{BC}{1-\rho}. \quad (5.1)$$

We have $$f'(z) = \frac{1}{2\pi i}\int\limits_{D(\theta,\rho)} \frac{f(u)}{(u-z)^2}\,du,$$

where $D(\phi, \rho)$ or $D(\theta)$ is a curve inside $\rho < 1$ and round $u = z = \rho e^{i\theta}$. We distinguish two cases.

(i) If $\rho < \frac{1}{4}$, we take $D(\theta)$ to be the circle $|u| = \frac{1}{2}$. Then $|u-z| > \frac{1}{4}$ on $D(\theta)$, and so

$$|f'(z)| \leqslant 16\mathfrak{M}_1(f, \tfrac{1}{2}) \leqslant 16\mathfrak{M}_r(f, \tfrac{1}{2}) \leqslant 16C,$$

and *a fortiori* $$\mathfrak{M}_r(f', \rho) \leqslant 16C \leqslant \frac{BC}{1-\rho}.$$

(ii) If $\rho \geqslant \frac{1}{4}$, we take $D(\theta)$ to be the circle whose centre is $u = z = \rho e^{i\theta}$ and which passes through $u = \rho^{\frac{1}{2}}e^{i\theta}$. The radius of this circle is $\rho^{\frac{1}{2}}-\rho$, and lies between two numbers $A(1-\rho)$, so that the circle is inside a region of the type of the $S(\theta)$ of Theorem C. Hence

$$|f'(z)| \leqslant \frac{AF(\theta)}{1-\rho}$$

* Actually $\mathfrak{M}_r(f') = o\{(1-r)^{-1}\}$, but we do not need this here.

Most of the content of Theorems 3–6 is to be found in Hardy and Littlewood, **6**, 430 et seq. The results proved there are in some ways a little less precise, but are proved for all positive r.

and
$$\mathfrak{M}_r(f') \leqslant \frac{B}{1-\rho}\mathfrak{M}_r(F) \leqslant \frac{BC}{1-\rho},$$
by Theorem C.

THEOREM 4. *If* $\mathfrak{M}_r(f) \leqslant c(\rho)$, *then*
$$\mathfrak{M}_r(f') \leqslant \frac{Bc(\rho^{\frac{1}{2}})}{\rho^{\frac{1}{2}}(1-\rho)}.* \tag{5.2}$$

Applying Theorem 3 to $f(\sigma z)$, we obtain
$$\mathfrak{M}_r\{\sigma f'(\sigma\rho e^{i\theta})\} \leqslant \frac{B\mathfrak{M}_r\{f(\sigma e^{i\theta})\}}{1-\rho} \leqslant \frac{Bc(\sigma)}{1-\rho}.$$

Hence
$$\mathfrak{M}_r(f',\rho^2) \leqslant \frac{Bc(\rho)}{\rho(1-\rho)},$$
which is equivalent to (5.2).

Fractional derivatives

6. We shall require similar results for $f^{\beta}(z)$, the βth derivative, of non-integral order, of $f(z)$. There are various definitions of these derivatives, which are not quite equivalent for our present purposes. We take first the definition (which we distinguish from our later definition by putting the β in brackets)
$$f^{(\beta)}(z) = \frac{\Gamma(1+\beta)}{2\pi i}\int\limits_{D(z)} (u-z)^{-1-\beta}f(u)\,du, \tag{6.1}$$
where $D(z)$ is a loop from the origin lying inside the unit circle and encircling $u = z$ in the positive direction,
$$\begin{aligned}(u-z)^{-1-\beta} &= \exp\{-(1+\beta)\log(u-z)\}\\ &= \exp[-(1+\beta)\{\log|u-z|+i\,\mathrm{am}(u-z)\}],\end{aligned}$$
and $\mathrm{am}(u-z) = \mathrm{am}\,z = \theta$ at the point where $D(z)$ cuts the radius vector through $u = z$.

The formula (6.1) defines $f^{(\beta)}(z)$ for all real (or complex) values of β except negative integral values. When β is negative, it may be replaced by
$$f^{(\beta)}(z) = \frac{1}{\Gamma(-\beta)}\int\limits_0^z (z-u)^{-1-\beta}f(u)\,du, \tag{6.2}$$
where the path of integration is rectilinear. This formula is significant

* The factor $\rho^{\frac{1}{2}}$ in the denominator is essential; for $c(\rho)$ can be small for small ρ if $f(0) = 0$, while $\mathfrak{M}_r(f')$ is usually not small.

except for $\beta = 0, 1, \ldots$ (so that the two together cover all values of β). In any case

$$f^{(\beta)}(z) = \sum \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} c_n z^{n-\beta}, \tag{6.3}$$

and $z^{\beta} f^{(\beta)}(z)$ is regular for $\rho < 1$. When β is a positive integer, $f^{(\beta)}(z)$ is an ordinary derivative of $f(z)$, and, when β is a negative integer,

$$f_{(-\beta)}(z) = f^{(\beta)}(z) \tag{6.4}$$

is an ordinary (repeated) integral of $f(z)$.

We can still define $f^{(\beta)}(z)$ by (6.1) or (6.2) when $f(z)$ is of the form

$$f(z) = \sum c_n z^{n+\lambda}$$

and $\lambda > -1$. We have then

$$f^{(\beta)}(z) = \sum \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-\beta)} c_n z^{n+\lambda-\beta},$$

instead of (6.3). The derivatives and integrals thus defined obey the ordinary operational laws

$$(f^{(\beta)})^{(\gamma)} = (f^{(\gamma)})^{(\beta)} = f^{(\beta+\gamma)}.$$

The important functions are $f^{(\beta)}(z)$ and $f_{(\beta)}(z)$ with $0 < \beta < 1$.

7. THEOREM 5. *If $\mathfrak{M}_r(f) \leqslant C$ and $\beta > 0$, then*

$$\mathfrak{M}_r(f^{(\beta)}) \leqslant \frac{BC}{\rho^{\beta}(1-\rho)^{\beta}}. \tag{7.1}$$

We use the formula (6.1), and again distinguish two cases.

(i) If $\rho < \frac{1}{4}$, we take $D(z)$ to be the circle through the origin whose centre is $u = z$. This circle lies within a region $S(\theta)$, and

$$|f^{(\beta)}(z)| \leqslant \frac{BF(\theta)}{\rho^{\beta}} \leqslant \frac{BF(\theta)}{\rho^{\beta}(1-\rho)^{\beta}}.$$

(ii) If $\frac{1}{4} \leqslant \rho < 1$, we take $D(z)$ to be the contour formed by (a) the circle used in the proof of case (ii) of Theorem 3, and (b) the straight line from the origin to the nearest point on the circle, described twice in opposite directions: this contour also lies within an $S(\theta)$. The modulus of the contribution of (a) does not exceed

$$\frac{BF(\theta)}{(\rho^{\frac{1}{2}}-\rho)^{\beta}} \leqslant \frac{BF(\theta)}{\rho^{\beta}(1-\rho)^{\beta}},$$

and that of (*b*) does not exceed

$$BF(\theta)\int\limits_0^{2\rho-\rho^{\frac{1}{2}}}\frac{d\sigma}{(\rho-\sigma)^{1+\beta}}\leqslant\frac{BF(\theta)}{(\rho^{\frac{1}{2}}-\rho)^{\beta}}\leqslant\frac{BF(\theta)}{\rho^{\beta}(1-\rho)^{\beta}}.$$

Hence, in any case,

$$|f^{(\beta)}(z)|\leqslant\frac{BF(\theta)}{\rho^{\beta}(1-\rho)^{\beta}};$$

and $$\mathfrak{M}_r(f^{(\beta)})\leqslant\frac{B\mathfrak{M}_r(F)}{\rho^{\beta}(1-\rho)^{\beta}}\leqslant\frac{BC}{\rho^{\beta}(1-\rho)^{\beta}},$$

by Theorem C.

THEOREM 6. *If $\mathfrak{M}_r(f)\leqslant c(\rho)$ and $\beta>0$, then*

$$\mathfrak{M}_r(f^{(\beta)})\leqslant\frac{Bc(\rho^{\frac{1}{2}})}{\rho^{\beta}(1-\rho)^{\beta}}. \tag{7.2}$$

Applying Theorem 5 to $f(\sigma z)$, we obtain

$$\mathfrak{M}_r\{\sigma^{\beta}f^{(\beta)}(\sigma\rho e^{i\theta})\}\leqslant\frac{Bc(\sigma)}{\rho^{\beta}(1-\rho)^{\beta}}.$$

Hence $$\mathfrak{M}_r\{f^{(\beta)}(\rho^2e^{i\theta})\}\leqslant\frac{Bc(\rho)}{\rho^{2\beta}(1-\rho)^{\beta}}\leqslant\frac{Bc(\rho)}{\rho^{2\beta}(1-\rho^2)^{\beta}},$$

which is equivalent to (7.2).

It will be observed that Theorems 3 and 4 are not included in Theorems 5 and 6, being sharper than the results of taking $\beta=1$ in those theorems. This is natural because $f^{(\beta)}(z)$ has generally a singularity at the origin, which disappears when $\beta=1$.

8. Our second definition of the derivative of order β applies only when

$$f(0)=c_0=0. \tag{8.1}$$

It is $$f^{\beta}(z)=i^{\beta}\sum_1^{\infty}n^{\beta}c_n z^n. \tag{8.2}$$

We also write $$f_{\beta}(z)=f^{-\beta}(z). \tag{8.3}$$

These definitions can be used for all real (or complex) β, and it is plain that

$$(f^{\beta})^{\gamma}=(f^{\gamma})^{\beta}=f^{\beta+\gamma}$$

for all β and γ.

The difference between the two systems of definitions is roughly that between differentiation (or integration) with respect to $z=\rho e^{i\theta}$ and with respect to θ. The definitions of this section, which require

c_0 to be 0, are the most convenient in the theory of Fourier series. If $\alpha = -\beta > 0$ and $\rho < 1$, then

$$\frac{1}{\Gamma(\alpha)} \int\limits_{-\infty}^{\theta} (\theta-t)^{\alpha-1} f(\rho e^{it})\, dt$$
$$= \frac{1}{\Gamma(\alpha)} \int\limits_{-\infty}^{\theta} (\theta-t)^{\alpha-1} \sum_1^\infty c_n \rho^n e^{nit}\, dt = \sum_1^\infty (ni)^{-\alpha} c_n \rho^n e^{ni\theta} = f_\alpha(\rho e^{i\theta});$$

and this equation can be extended, if properly interpreted, to the case $\rho = 1$. Thus $f_\alpha(z)$ is effectively the αth integral of $f(z)$, as defined by Weyl.

THEOREM 7. *If* $0 < \beta < 1$* *and*

$$\mathfrak{M}_r(f) \leqslant c(\rho), \tag{8.4}$$

then

$$\mathfrak{M}_r(f^\beta) \leqslant \frac{Bc(\rho^{\frac{1}{2}})}{(1-\rho)^\beta}. \tag{8.5}$$

Here there is no factor $\rho^{-\beta}$.

If we write for the moment

$$g = z^\beta f^{(\beta)} = \sum \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} c_n z^n,$$

then

$$\mathfrak{M}_r(g) \leqslant \frac{Bc(\rho^{\frac{1}{2}})}{(1-\rho)^\beta}, \tag{8.6}$$

by Theorem 6: and we shall prove that $i^{-\beta} f^\beta - g$ satisfies a similar inequality. Now

$$n^\beta = \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} + \frac{B}{n+1} \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} + u_n,$$

where

$$|u_n| \leqslant Bn^{\beta-2}.$$

Hence

$$i^{-\beta} f^\beta - g = B\phi + \psi, \tag{8.7}$$

where

$$\phi = \sum \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} \frac{c_n}{n+1} z^n, \qquad \psi = \sum u_n c_n z^n;$$

and it is sufficient to prove that $\mathfrak{M}_r(\phi)$ and $\mathfrak{M}_r(\psi)$ satisfy inequalities of the type (8.5).

Now ϕ is related to

$$\sum \frac{c_n}{n+1} z^n = \frac{1}{z} \int\limits_0^z f(u)\, du = \frac{f_1(z)}{z}.$$

* From this point onwards we confine our attention to this case.

as g is to f. Hence, applying (8.6) to f_1/z, and using Lemma α, we obtain

$$\mathfrak{M}_r(\phi) \leqslant \frac{B}{(1-\rho)^\beta}\mathfrak{M}_r\left(\frac{f_1}{z}, \rho^{\frac{1}{2}}\right) \leqslant \frac{B}{(1-\rho)^\beta}\mathfrak{M}_r(f, \rho^{\frac{1}{2}}) \leqslant \frac{Bc(\rho^{\frac{1}{2}})}{(1-\rho)^\beta}. \tag{8.8}$$

Finally, $$|\psi| \leqslant B\sum |c_n| n^{\beta-2}\rho^n.$$

But $$|c_n|\rho^n \leqslant \mathfrak{M}_1(f) \leqslant \mathfrak{M}_r(f);$$

and so (since $\beta-2 < -1$)

$$\mathfrak{M}_r(\psi) \leqslant \mathfrak{M}(\psi) \leqslant \mathfrak{M}_r(f)\sum n^{\beta-2} = B\mathfrak{M}_r(f) \leqslant \frac{B\mathfrak{M}_r(f, \rho^{\frac{1}{2}})}{(1-\rho)^\beta} \leqslant \frac{Bc(\rho^{\frac{1}{2}})}{(1-\rho)^\beta}. \tag{8.9}$$

The theorem now follows from (8.7), (8.8), and (8.9).

9. Theorem 8. *If $r > 1$, $s > 1$, $b < 1$, and $f(0) = 0$, then*

$$B \leqslant \frac{\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho}{\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(f')\,d\rho} \leqslant B. \tag{9.1}$$

In particular, when $s = r$,

$$B \leqslant \frac{\int_0^1\int_{-\pi}^{\pi} (1-\rho)^{-b}|f|^r\,d\rho d\theta}{\int_0^1\int_{-\pi}^{\pi} (1-\rho)^{r-b}|f'|^r\,d\rho d\theta} \leqslant B. \tag{9.2}$$

The inequalities are to be interpreted with the obvious conventions: if either integral is positive and finite, then the other is positive and finite and satisfies the inequalities. The condition $f(0) = 0$ is essential: the right-hand inequalities are obviously false, for example, when $f(z)$ is the constant 1.

The left-hand inequality (9.1) is a corollary of Theorem 3 and Lemma γ. Thus, by Theorem 3, with s for r and $c(\rho) = \mathfrak{M}_s(f, \rho)$,

$$\mathfrak{M}_s(f') \leqslant B\rho^{-\frac{1}{2}}(1-\rho)^{-1}\mathfrak{M}_s(f, \rho^{\frac{1}{2}}),$$

$$\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(f')\,d\rho \leqslant B\int_0^1 (1-\rho)^{-b}\rho^{-\frac{1}{2}r}\mathfrak{M}_s^r(f, \rho^{\frac{1}{2}})\,d\rho$$

$$= B\int_0^1 (1-\rho^2)^{-b}\rho^{1-r}\mathfrak{M}_s^r(f, \rho)\,d\rho \leqslant B\int_0^1 (1-\rho)^{-b}\rho^{1-r}\mathfrak{M}_s^r(f, \rho)\,d\rho,$$

the last integral being convergent, at 1 by hypothesis, and at 0 because $c_0 = 0$ and so $\mathfrak{M}_s(f,\rho) = O(\rho)$. It now follows from Lemma γ that

$$\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(f')\,d\rho \leqslant B\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho.$$

Passing to the right-hand inequality (9.1), we suppose first that $f(z)$ is regular for $\rho \leqslant 1$. Then, integrating by parts,*

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho = \frac{1}{1-b}\int_0^1 (1-\rho)^{1-b}\frac{d}{d\rho}\{\mathfrak{M}_s^r(f)\}\,d\rho.$$

But

$$\frac{d}{d\rho}\{\mathfrak{M}_s^r(f)\} = \frac{d}{d\rho}\{\mathfrak{M}_s^s(f)\}^{r/s} = \frac{r}{s}\{\mathfrak{M}_s^s(f)\}^{(r-s)/s}\frac{d}{d\rho}\{\mathfrak{M}_s^s(f)\}$$

$$= \frac{r}{s}\mathfrak{M}_s^{r-s}(f)\frac{d}{d\rho}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^s\,d\theta\right),$$

and

$$\left|\frac{d}{d\rho}|f|^s\right| \leqslant \left|\frac{d}{d\rho}f^s\right| = s|f|^{s-1}|f'|.$$

Hence

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho \leqslant B\int_0^1 (1-\rho)^{1-b}\mathfrak{M}_s^{r-s}(f)\,d\rho\int_{-\pi}^{\pi} |f|^{s-1}|f'|\,d\theta.$$

Also

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^{s-1}|f'|\,d\theta \leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^s\,d\theta\right)^{(s-1)/s}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f'|^s\,d\theta\right)^{1/s}$$

$$= \mathfrak{M}_s^{s-1}(f)\mathfrak{M}_s(f'),$$

by Hölder's inequality; and hence

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho \leqslant B\int_0^1 (1-\rho)^{1-b}\mathfrak{M}_s^{r-1}(f)\mathfrak{M}_s(f')\,d\rho$$

$$= B\int_0^1 \{(1-\rho)^{-b}\mathfrak{M}_s^r(f)\}^{(r-1)/r}\{(1-\rho)^{r-b}\mathfrak{M}_s^r(f')\}^{1/r}\,d\rho$$

$$\leqslant B\left\{\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho\right\}^{(r-1)/r}\left\{\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(f')\,d\rho\right\}^{1/r}$$

* And observing that $(1-\rho)^{1-b}\mathfrak{M}_s^r(f) = 0$ for $\rho = 0$ and $\rho = 1$.

(again by Hölder's inequality). Removing the common factor, and raising the result to the rth power, we obtain the right-hand inequality (9.1).

We have supposed $f(z)$ regular for $\rho \leqslant 1$. This condition is satisfied by $f(\sigma z)$, for any $\sigma < 1$. Hence, applying what we have proved to $f(\sigma z)$, we have

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r\{f(\sigma z)\}\, d\rho$$
$$\leqslant B\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r\{\sigma f'(\sigma z)\}\, d\rho \leqslant B\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r\{f'(z)\}\, d\rho;$$

and so (using 'Fatou's Lemma')

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r\{f(z)\}\, d\rho$$
$$\leqslant \varliminf_{\sigma\to 1}\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r\{f(\sigma z)\}\, d\rho \leqslant B\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r\{f'(z)\}\, d\rho$$

in the general case.

The right-hand inequality (9.1) can be sharpened a little, and we shall need the refinement in the next section.

THEOREM 9. *Under the conditions of Theorem* 8

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\, d\rho \leqslant B\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(zf')\, d\rho. \tag{9.3}$$

For

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\, d\rho \leqslant B\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(f')\, d\rho$$
$$= B\int_0^1 (1-\rho)^{r-b}\rho^{-b}\mathfrak{M}_s^r(zf')\, d\rho \leqslant B\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(zf')\, d\rho,$$

the first inequality following from Theorem 8 and the last from Lemma γ.

10. Our next theorem is an extension of Theorem 8 to derivatives of non-integral order.

THEOREM 10. *If the conditions of Theorem* 8 *are satisfied, and* $0 < \beta < 1$, *then*

$$B \leqslant \frac{\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\, d\rho}{\int_0^1 (1-\rho)^{\beta r-b}\mathfrak{M}_s^r(f^{\beta})\, d\rho} \leqslant B. \tag{10.1}$$

The left-hand inequality is a corollary of Theorem 7. For*

$$\mathfrak{M}_s^r(f^\beta) \leqslant \frac{B\mathfrak{M}_s^r(f, \rho^{\frac{1}{2}})}{(1-\rho)^{\beta r}},$$

by Theorem 7; and so

$$\int_0^1 (1-\rho)^{\beta r-b}\mathfrak{M}_s^r(f^\beta)\, d\rho \leqslant B\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f, \rho^{\frac{1}{2}})\, d\rho$$

$$= B\int_0^1 (1-\rho^2)^{-b}\rho\,\mathfrak{M}_s^r(f, \rho)\, d\rho \leqslant B\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\, d\rho.$$

To prove the right-hand inequality we observe that

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\, d\rho \leqslant B\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(zf')\, d\rho, \tag{10.2}$$

by Theorem 9. But

$$zf' = c_1 z+2c_2 z^2+\ldots = -if^1$$

and

$$f^1 = (f^\beta)^{1-\beta}.$$

Hence, by the left-hand inequality, with f^β in place of f, and $1-\beta$ in place of β, we have

$$\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(zf')\, d\rho \leqslant B\int_0^1 (1-\rho)^{\beta r-b}\mathfrak{M}_s^r(f^\beta)\, d\rho; \tag{10.3}$$

and the right-hand inequality follows from (10.2) and (10.3).

Another preliminary theorem

11. THEOREM 11.† *If*

$$1 < r < s, \qquad \alpha = \frac{1}{r}-\frac{1}{s}, \qquad l \geqslant r, \tag{11.1}$$

and $\mathfrak{M}_r(f) \leqslant C$, *then*

$$\int_0^1 (1-\rho)^{l\alpha-1}\mathfrak{M}_s^l(f, \rho)\, d\rho \leqslant BC. \tag{11.2}$$

We may suppose $C = 1$.

* It will be observed that the proof is a little simpler than the corresponding part of the proof of Theorem 8. This is because f^β usually begins with a term in z, f' with a constant term. The result is true with $f^{(\beta)}$ for f^β, but the proof involves small complications like those of the proof of Theorem 8.

† Hardy and Littlewood, **6**, 411–14 (Theorem 31). There the theorem is proved for $0 < r < s$. A corollary is $\mathfrak{M}_s(f) = o\{(1-\rho)^{-\alpha}\}$.

It is sufficient to prove the theorem in the special case $l = r$. For

$$\mathfrak{M}_s(f) \leqslant B(1-\rho)^{-\alpha},$$

by Theorem 1, and so (if the special case has been proved)

$$\int_0^1 (1-\rho)^{l\alpha-1}\mathfrak{M}_s^l(f)\,d\rho = \int_0^1 (1-\rho)^{r\alpha-1}\mathfrak{M}_s^r(f)\{(1-\rho)^\alpha\mathfrak{M}_s(f)\}^{l-r}\,d\rho$$

$$\leqslant B\int_0^1 (1-\rho)^{r\alpha-1}\mathfrak{M}_s^r(f)\,d\rho \leqslant B.$$

We may therefore suppose that $l = r$.

Next, we may simplify the theorem as we simplified Theorem 1 in § 4. We may suppose first that f has no zeros in $\rho < 1$, and then (by putting $f = \phi^{2/r}$) that $r = 2$, in which case $s = q > 2$. We have then

$$r = 2,\ s > 2,\ \alpha = \frac{1}{2}-\frac{1}{s}, \qquad \sum |c_n|^2 = \mathfrak{M}_2^2(f) \leqslant 1;$$

and our conclusion is to be that

$$\int_0^1 (1-\rho)^{-2/q}\mathfrak{M}_q^2(f)\,d\rho \leqslant B. \tag{11.3}$$

We write $$f = c_0+\sum_1^\infty c_n z^n = c_0+g.$$

Then $$\mathfrak{M}_q(f) \leqslant \mathfrak{M}_q(c_0)+\mathfrak{M}_q(g) = |c_0|+\mathfrak{M}_q(g)$$

and $$\mathfrak{M}_q^2(f) \leqslant 2|c_0|^2+2\mathfrak{M}_q^2(g). \tag{11.4}$$

Now $g(0) = 0$, and so

$$\int_0^1 (1-\rho)^{-2/q}\mathfrak{M}_q^2(g)\,d\rho \leqslant B\int_0^1 (1-\rho)^{2-2/q}\mathfrak{M}_q^2(g')\,d\rho,$$

by Theorem 8. Also

$$\mathfrak{M}_q(g',\rho) \leqslant B(1-\rho)^{\frac{1}{q}-\frac{1}{2}}\mathfrak{M}_2(g',\rho) \leqslant B(1-\rho)^{\frac{1}{q}-\frac{1}{2}}\mathfrak{M}_2(g',\rho^{\frac{1}{2}}),$$

by Theorem 1. Hence

$$\int_0^1 (1-\rho)^{-2/q}\mathfrak{M}_q^2(g)\,d\rho \leqslant B\int_0^1 (1-\rho)\mathfrak{M}_2^2(g',\rho^{\frac{1}{2}})\,d\rho$$

$$= B\int_0^1 (1-\rho)\Big(\sum_1^\infty n^2|c_n|^2\rho^{n-1}\Big)\,d\rho = B\sum_1^\infty n^2|c_n|^2\int_0^1 (1-\rho)\rho^{n-1}\,d\rho$$

$$= B\sum_1^\infty \frac{n}{n+1}|c_n|^2 \leqslant B\sum_1^\infty |c_n|^2 \leqslant B. \tag{11.5}$$

It then follows from (11.4) that

$$\int_0^1 (1-\rho)^{-2/q}\mathfrak{M}_q^2(f)\,d\rho \leqslant B|c_0|^2\frac{q}{q-2}+B \leqslant B,$$

which is (11.3); and this completes the proof.

The theorem '$r \to s$' and its extensions

12. The 'Hauptsatz' of our papers **3** and **6** was

THEOREM 12. *If*
$$r < s, \qquad \alpha = \frac{1}{r}-\frac{1}{s}, \tag{12.1}$$

and
$$f_\alpha(z) = \sum_1^\infty (in)^{-\alpha} c_n z^n, \tag{12.2}$$

then
$$\mathfrak{M}_s(f_\alpha) \leqslant B\mathfrak{M}_r(f). \tag{12.3}$$

We proved this theorem there for $0 < r < s$; but now we are concerned only with the range $1 < r < s$.

Our proof of Theorem 12, for the range $1 < r < s$, was not at all 'function-theoretic', but was based on certain 'elementary' inequalities which we had proved in collaboration with Pólya.* Its logic was roughly as follows. We began by a rather difficult, though 'elementary', proof of the inequality

$$\left|\sum\sum{}' \frac{a_m b_n}{|m-n|^\lambda}\right| \leqslant B(\sum |a_m|^R)^{1/R}(\sum |b_n|^S)^{1/S}, \tag{12.4}$$

where

$$R > 1, \qquad S > 1, \qquad \frac{1}{R}+\frac{1}{S} > 1, \qquad \lambda = 2-\frac{1}{R}-\frac{1}{S},$$

and the dash excludes equal values of m and n from the summation. From this we passed, by standard limiting processes, to the integral inequality

$$\left|\int\int \frac{f(x)g(y)}{|x-y|^\lambda}\,dxdy\right| \leqslant B\left(\int |f(x)|^R\,dx\right)^{1/R}\left(\int |g(y)|^S\,dy\right)^{1/S}; \tag{12.5}$$

and from (12.5) we deduced, by the 'converse of Hölder's inequality', that

$$\left(\int |f_\alpha(x)|^{S'}\,dx\right)^{1/S'} \leqslant B\left(\int |f(x)|^R\,dx\right)^{1/R}, \tag{12.6}$$

where
$$\frac{1}{S}+\frac{1}{S'} = 1, \qquad \alpha = \frac{1}{R}-\frac{1}{S'}, \tag{12.7}$$

* See Hardy, Littlewood, and Pólya, **8**, and **9**, ch. x; and Hardy and Littlewood, **3**, 568–76.

$f_\alpha(x)$, the integral of $f(x)$ of order α, is defined by

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t)\, db, \tag{12.8}$$

$0 < \alpha < 1$, and c may be $-\infty$ in certain circumstances.* Finally, we replaced R, S' by r, s and derived (12.3) from (12.6).†

The main theorems of our paper **7**, viz. Theorem 1 (which is proved) and Theorem 3 (which is stated without proof) are generalizations of Theorem 12. It seems desirable in any case to have a proof of Theorem 12 which is function-theoretic in character. Incidentally, we shall show (without going into details) how it would be possible to *deduce* (12.4) and (12.5) from (12.3). The only published proofs of these inequalities depend on difficult theorems concerning 'rearrangements'.‡

13. We begin by framing a generalization of Theorem 12. The function $f_\alpha(z)$ is substantially§ the Faltung of the two functions $f(z)$ and

$$g(z) = \sum n^{-\alpha} z^n; \tag{13.1}$$

and $g(z)$ satisfies

$$\mathfrak{M}_\sigma(g') \leqslant B(1-\rho)^{\alpha-1-1/\sigma'}, \tag{13.2}$$

where σ' is defined, as usual, by

$$\sigma' = \frac{\sigma}{\sigma-1}, \qquad \frac{1}{\sigma}+\frac{1}{\sigma'} = 1,$$

for any $\sigma \geqslant 1$.‖ It is therefore natural to replace the special function (13.1) by a general

$$g(z) = \sum_1^\infty b_n z^n \tag{13.3}$$

subject to (13.2) for some $\sigma \geqslant 1$.

* In particular when $f(x)$ has the period 2π and mean value 0 over a period. This is the case important in the theory of Fourier series.

† Afterwards extending it to the range $0 < r < s$ by 'function-theoretic' arguments. When $r \leqslant 1$ the result is essentially one about power-series, the analogue for general Fourier series (or harmonic functions) being false.

‡ See Hardy, Littlewood, and Pólya, **9**, ch. x, especially Theorems 371–3 and 379–82.

§ See § 15 for a precise statement.

‖ $1/\sigma' = 0$ when $\sigma = 1$. Since $|g'| \leqslant B|1-z|^{\alpha-2}$,

$$\mathfrak{M}_\sigma(g') \leqslant \left(B \int \frac{d\theta}{|1-z|^{\sigma(2-\alpha)}}\right)^{1/\sigma} \leqslant B(1-\rho)^{-2+\alpha+1/\sigma} = B(1-\rho)^{\alpha-1-1/\sigma'}.$$

We suppose then that
$$\mathfrak{M}_r(f) \leqslant 1 \tag{13.4}$$
and
$$\mathfrak{M}_\sigma(g') \leqslant (1-\rho)^{k-1}, \tag{13.5}$$
where
$$k = \alpha - \frac{1}{\sigma'}; \tag{13.6}$$
and write
$$h(z) = \sum_1^\infty b_n c_n z^n. \tag{13.7}$$
Analogy with Theorem 8 then suggests that
$$\mathfrak{M}_s(h) \leqslant B, \tag{13.8}$$
where
$$\frac{1}{s} = \frac{1}{r} - \alpha = \frac{1}{r} - k - \frac{1}{\sigma'}. \tag{13.9}$$
We must, however, impose some limitations on k. In the first place, we must suppose that
$$k < \frac{1}{r} - \frac{1}{\sigma'} \tag{13.10}$$
in order that s shall be positive. Secondly, as we shall see, we must suppose that $k \geqslant 0$. The cases $k > 0$ and $k = 0$ turn out to differ essentially, and we begin with the first.

14. Theorem 13. *If*
$$\sigma \geqslant 1, \qquad 0 < k < \frac{1}{r} - \frac{1}{\sigma'} \tag{14.1}$$
($1/\sigma'$ *being* 0 *if* $\sigma = 1$),
$$\frac{1}{s} = \frac{1}{r} - k - \frac{1}{\sigma'} \tag{14.2}$$
(*so that* $\quad 1 < r < s < \infty, \quad r < \sigma', \quad \sigma < s$);
$$f(z) = \sum_1^\infty c_n z^n, \qquad g(z) = \sum_1^\infty b_n z^n, \qquad h(z) = \sum_1^\infty b_n c_n z^n; \tag{14.3}$$
$$\mathfrak{M}_r(f) \leqslant 1, \qquad \mathfrak{M}_\sigma(g') \leqslant (1-\rho)^{k-1}; \tag{14.4}$$
then
$$\mathfrak{M}_s(h) \leqslant B. \tag{14.5}$$

It will be convenient to introduce a new parameter t defined by
$$\frac{1}{t} = \frac{1}{r} - k. \tag{14.6}$$
Then $r < t \leqslant s$ (and $t < s$ except when $\sigma = 1$).

We observe first that

$$h(\rho e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\{\rho^{\frac{1}{2}}e^{i(\theta-\phi)}\}g(\rho^{\frac{1}{2}}e^{i\phi})\,d\phi, \tag{14.7}$$

$$\rho e^{i\theta}h'(\rho e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\{\rho^{\frac{1}{2}}e^{i(\theta-\phi)}\}\rho^{\frac{1}{2}}e^{i\phi}g'(\rho^{\frac{1}{2}}e^{i\phi})\,d\phi; \tag{14.8}$$

and, if we take

$$\alpha = t, \qquad \beta = \sigma, \qquad \frac{1}{\gamma} = \frac{1}{t}+\frac{1}{\sigma}-1 = \frac{1}{r}-k-\frac{1}{\sigma'},$$

the conditions of Theorem A are satisfied. Hence

$$\begin{aligned}\rho^{\frac{1}{2}}\mathfrak{M}_s(h',\rho) &\leqslant \mathfrak{M}_t(f,\rho^{\frac{1}{2}})\mathfrak{M}_\sigma(g',\rho^{\frac{1}{2}})\\ &\leqslant (1-\rho^{\frac{1}{2}})^{k-1}\mathfrak{M}_t(f,\rho^{\frac{1}{2}}) \leqslant B(1-\rho)^{k-1}\mathfrak{M}_t(f,\rho^{\frac{1}{2}}).\end{aligned} \tag{14.9}$$

We must now distinguish three cases of the theorem, each needing a different proof. We suppose first that

$$1 < r < s \leqslant 2. \tag{14.10}$$

Case (a) of Theorem 13: $r < s \leqslant 2$

15. Since $s \leqslant 2$, we have

$$\mathfrak{M}_s^s(h) \leqslant B\int_0^1\int_{-\pi}^{\pi}(1-\rho)^{s-1}|h'|^s\,d\rho d\theta = B\int_0^1(1-\rho)^{s-1}\mathfrak{M}_s^s(h')\,d\rho, \tag{15.1}$$

by (2.13) of Theorem D. Hence, using (14.9) and Lemma γ, we obtain

$$\begin{aligned}\mathfrak{M}_s^s(h) &\leqslant B\int_0^1(1-\rho)^{sk-1}\rho^{-\frac{1}{2}s}\mathfrak{M}_t^s(f,\rho^{\frac{1}{2}})\,d\rho\\ &= B\int_0^1(1-\rho^2)^{sk-1}\rho^{1-s}\mathfrak{M}_t^s(f,\rho)\,d\rho \leqslant B\int_0^1(1-\rho)^{sk-1}\rho^{1-s}\mathfrak{M}_t^s(f)\,d\rho\\ &\leqslant B\int_0^1(1-\rho)^{sk-1}\mathfrak{M}_t^s(f)\,d\rho.\end{aligned} \tag{15.2}$$

Finally, $s > r$ and $t > r$, and therefore

$$\int_0^1(1-\rho)^{sk-1}\mathfrak{M}_t^s(f)\,d\rho \leqslant B,$$

by Theorem 11.

Case (b): $r \leqslant 2 \leqslant s$

16. We suppose next that $r \leqslant 2 \leqslant s$. Since (15.1) may now be false, we use Theorem E instead of Theorem D. We have

$$\mathfrak{M}_s^s(h) \leqslant B \int_{-\pi}^{\pi} d\theta \left\{\int_0^1 (1-\rho)|h'|^2\, d\rho\right\}^{\frac{1}{2}s}.$$

But

$$\left[\int_{-\pi}^{\pi} d\theta \left\{\int_0^1 (1-\rho)|h'|^2\, d\rho\right\}^{\frac{1}{2}s}\right]^{2/s} \leqslant \int_0^1 (1-\rho)\, d\rho \left(\int_{-\pi}^{\pi} |h'|^s\, d\theta\right)^{2/s},$$

by Minkowski's inequality.* Hence

$$\mathfrak{M}_s^2(h) \leqslant B \int_0^1 (1-\rho)\mathfrak{M}_s^2(h')\, d\rho \leqslant B \int_0^1 (1-\rho)^{2k-1}\rho^{-1}\mathfrak{M}_t^2(f,\rho^{\frac{1}{2}})\, d\rho$$

$$= B \int_0^1 (1-\rho^2)^{2k-1}\rho^{-1}\mathfrak{M}_t^2(f,\rho)\, d\rho \leqslant B \int_0^1 (1-\rho)^{2k-1}\rho^{-1}\mathfrak{M}_t^2(f,\rho)\, d\rho$$

$$\leqslant B \int_0^1 (1-\rho)^{2k-1}\mathfrak{M}_t^2(f,\rho)\, d\rho,$$

by (14.9) and Lemma γ. The conclusion now follows from Theorem 11, since $2 \geqslant r$ and $t > r$.

Case (c): $2 < r < s$

17. When finally $2 < r < s$, we have to use a quite different method. Actually, we deduce Case (*c*) from Case (*a*) by a 'conjugacy' argument, here of a very simple type.†

Suppose that
$$\psi(\theta) = \sum_{-N}^{N} \gamma_n e^{ni\theta} \tag{17.1}$$
is an arbitrary trigonometrical polynomial. It will be convenient to write

$$\psi = \sum_{-N}^{N} = \sum_{1}^{N} + \sum_{-N}^{0} = \psi_1 + \psi_2. \tag{17.2}$$

Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(\rho^2 e^{-i\theta})\psi(\theta)\, d\theta = \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} \psi(\theta)\, d\theta \int_{-\pi}^{\pi} f(\rho e^{-i\phi})g(\rho e^{i\phi-i\theta})\, d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho e^{-i\phi})H(\rho e^{i\phi})\, d\phi, \tag{17.3}$$

* In the form used in the proof of Lemma α (§ 3).

† There is a more difficult specimen in Littlewood and Paley, **11**, § 8.

where

$$H(\rho e^{i\phi}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(\theta)g(\rho e^{i\phi-i\theta})\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_1(\theta)g(\rho e^{i\phi-i\theta})\,d\theta = \sum_1^N b_n\gamma_n\rho^n e^{ni\phi}. \tag{17.4}$$

If we write for the moment

$$\mathfrak{f}(z) = \sum_1^N \gamma_n z^n, \qquad \mathfrak{g}(z) = g(z), \qquad \mathfrak{h}(z) = H(z),$$

so that

$$\mathfrak{f}(e^{i\theta}) = \psi_1(\theta),$$

then $\mathfrak{f}$, $\mathfrak{g}$, and $\mathfrak{h}$ are related as f, g, and h are related in the main theorem. Also, if

$$\frac{1}{r}+\frac{1}{r'} = \frac{1}{s}+\frac{1}{s'} = 1,$$

then

$$1 < s' < r' < 2$$

and

$$\frac{1}{s'}-\frac{1}{r'} = \frac{1}{r}-\frac{1}{s}.$$

We may therefore apply Case (a) of the theorem to $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$, with s', r' for r, s; and this gives

$$\mathfrak{M}_{r'}\{H(\rho e^{i\theta})\} \leqslant B\mathfrak{M}_{s'}\{\psi_1(\theta)\}.$$

But

$$\mathfrak{M}_{s'}\{\psi_1(\theta)\} \leqslant B\mathfrak{M}_{s'}\{\psi(\theta)\},$$

by Theorem F;* and so

$$\mathfrak{M}_{r'}\{H(\rho e^{i\theta})\} \leqslant B\mathfrak{M}_{s'}\{\psi(\theta)\}. \tag{17.5}$$

Also

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho e^{-i\phi})H(\rho e^{i\phi})\,d\phi\right| \leqslant \mathfrak{M}_r(f)\mathfrak{M}_{r'}(H) \leqslant \mathfrak{M}_{r'}(H), \tag{17.6}$$

by Hölder's inequality and (14.4). Hence, collecting our results from (17.3), (17.6), and (17.5), we obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} h(\rho^2 e^{-i\theta})\psi(\theta)\,d\theta\right| \leqslant B\mathfrak{M}_{s'}\{\psi(\theta)\}. \tag{17.7}$$

* In the form (2.22).

This is true for all trigonometrical polynomials ψ, and so for all ψ of $L^{s'}$; and therefore, by the converse of Hölder's inequality,*

$$\mathfrak{M}_s\{h(\rho^2 e^{i\theta})\} = \mathfrak{M}_s\{h(\rho^2 e^{-i\theta})\} \leqslant B.$$

This completes the proof of Theorem 13.†

In particular, if we take $g(z) = \sum n^{-\alpha} z^n$, we obtain Theorem 12.

18. We return for a moment to our remarks about Theorem 12 in § 12. We have now a proof of this theorem independent of the inequality (12.4) and of the theorems on 'rearrangements' on which that inequality was based. It is interesting to observe (without attempting to carry out the process in detail) how we could reverse the old argument and *deduce* (12.4) from Theorem 12.

We should begin by passing from

$$f_\alpha = \sum (in)^{-\alpha} c_n z^n$$

to the fractional integral of an arbitrary real function $f(x)$, defined as in (12.8). Here c might be finite (as with Riemann and Liouville) or $-\infty$ (as with Weyl). The deduction would involve only arguments similar to those of § 8, and processes of approximation of standard types. We should thus prove (12.6), and from this deduce that

$$\left|\iint |x-y|^{\alpha-1} f(x)g(y)\,dxdy\right| \leqslant B\left(\int |f(x)|^r\,dx\right)^{1/r}\left(\int |g(y)|^{s'}\,dy\right)^{1/s'}.$$

Since $$1-\alpha = 1-\frac{1}{r}+\frac{1}{s} = 2-\frac{1}{r}-\frac{1}{s'},$$

this is equivalent to (12.5); and the passage back to (12.4) is straightforward.

* 'If $k > 1$ and $\left|\int FG\,dx\right| \leqslant C\left(\int |G|^{k'}\,dx\right)^{1/k'}$

for all G of $L^{k'}$, then $\left(\int |F|^k\,dx\right)^{1/k} \leqslant C$.'

See, for example, Hardy, Littlewood, and Pólya, **9**, 142 (Theorem **191**): the theorem was first proved by F. Riesz. It remains true if G is confined to one of the standard classes of approximating functions (step-functions, polynomials, or trigonometrical polynomials).

† There is an analogue of Theorem 13 for harmonic functions which an experienced reader will be able to state and prove for himself. When $\sigma > 1$, the proof depends on Theorem F (and Theorem 13 itself). When $\sigma = 1$ we require an additional weapon: *if* $k < 1$ *and*

$$\mathfrak{M}_1(u) \leqslant (1-\rho)^{k-1},$$

then $$\mathfrak{M}_1(u_1) \leqslant B(1-\rho)^{k-1}, \qquad \mathfrak{M}_1(u_2) \leqslant B(1-\rho)^{k-1}.$$

The case $k = 0$

19. In Theorem 13 we supposed that $k > 0$, and this assumption was essential to the proof. Thus if $k = 0$, $t = r$, and the appeal to Theorem 11, at the end of § 15, is no longer justified. Actually, as we shall see, the theorem becomes false when $k = 0$ and $r < s < 2$ or $2 < r < s$, i.e. in cases (*a*) and (*c*).

Case (*b*) of the theorem, however, survives, as we proved in **7**. The theorem which follows is in fact Theorem 1 of **7**, but the proof which we give here is shorter.*

THEOREM 14. *If f, g, and h are defined as in Theorem* 13;

$$1 \leqslant r \leqslant 2 \leqslant s; \tag{19.1}$$

$$\sigma \geqslant 1, \qquad r < \sigma', \qquad \frac{1}{s} = \frac{1}{r} - \frac{1}{\sigma'}; \tag{19.2}$$

$$\mathfrak{M}_r(f) \leqslant 1, \qquad \mathfrak{M}_\sigma(g') \leqslant \frac{1}{1-\rho}; \tag{19.3}$$

then

$$\mathfrak{M}_s(h) \leqslant B. \tag{19.4}$$

Since we shall be concerned (in the statement and proof of the theorem) only with power-series, we have departed from our practice in the rest of the paper by including the case $r = 1$,† which is in fact particularly interesting.‡

20. We begin with a trivial simplification of the theorem, showing that (as the reader will readily believe) it is sufficient to prove it in the special case in which $c_1 = 0$ and $b_1 = 0$.

Suppose that it has been proved in this case, and let

$$F(z) = f(z) - c_1 z, \qquad G(z) = g(z) - b_1 z, \qquad H(z) = h(z) - b_1 c_1 z.$$

Then

$$\mathfrak{M}_r(F) \leqslant \mathfrak{M}_r(f) + |c_1|\rho \leqslant \mathfrak{M}_r(f) + \mathfrak{M}_1(f) \leqslant 2\mathfrak{M}_r(f) \leqslant 2$$

and

$$\mathfrak{M}_\sigma(G') \leqslant \mathfrak{M}_\sigma(g') + |b_1| \leqslant \mathfrak{M}_\sigma(g') + \mathfrak{M}_1(g') \leqslant 2\mathfrak{M}_\sigma(g') \leqslant \frac{2}{1-\rho}.$$

Hence, by our assumption,

$$\mathfrak{M}_s(H) \leqslant 4B = B,$$

* Though, as we stated in § 1, it depends on more difficult theorems.

† We used harmonic polynomials in the proof of case (*c*) of Theorem 13 (§ 17). Theorem 14, like Theorem 13, can be extended to harmonic functions when $r > 1$, but not when $r = 1$. See **7**, 162, footnote †.

‡ It includes the theorem of Paley referred to in **7**.

and therefore

$$\mathfrak{M}_s(h) \leqslant \mathfrak{M}_s(H) + |b_1||c_1|\rho \leqslant B + \mathfrak{M}_r(f)\mathfrak{M}_\sigma(g', 0) \leqslant B+1 = B;$$

so that the theorem is true generally.

We suppose then in what follows that $b_1 = 0$ and $c_1 = 0$. By Theorem E, we have

$$\lim_{\rho\to 1} \mathfrak{M}_s^s(h) \leqslant B \int_{-\pi}^{\pi} d\theta \left\{ \int_0^1 (1-\rho)|h'|^2\, d\rho \right\}^{\frac{1}{2}s};$$

and (using Minkowski's inequality as in § 16) we deduce

$$\lim_{\rho\to 1} \mathfrak{M}_s^2(h) \leqslant B \int_0^1 (1-\rho)\mathfrak{M}_s^2(h')\, d\rho. \tag{20.1}$$

If $0 < \tau < 1$ and

$$k(\zeta) = k(\tau e^{i\theta}) = h'(\rho\tau e^{i\theta}),$$

then $k(0) = 0$,* and we may apply (20.1) to $k(\zeta)$. We have thus

$$\lim_{\rho\to 1} \mathfrak{M}_s^2(k) \leqslant B \int_0^1 (1-\tau)\mathfrak{M}_s^2(k', \tau)\, d\tau,$$

i.e.

$$\mathfrak{M}_s^2(h', \rho) \leqslant B \int_0^1 (1-\tau)\mathfrak{M}_s^2\{\rho h''(\rho\tau e^{i\theta})\}\, d\tau$$

$$= B\rho^2 \int_0^1 (1-\tau)\mathfrak{M}_s^2\{h''(\rho\tau e^{i\theta})\}\, d\tau = B \int_0^\rho (\rho-\omega)\mathfrak{M}_s^2\{h''(\omega e^{i\theta})\}\, d\omega.$$

Substituting in (20.1), we obtain

$$\lim_{\rho\to 1} \mathfrak{M}_s^2(h) \leqslant B \int_0^1 (1-\rho)\, d\rho \int_0^\rho (\rho-\omega)\mathfrak{M}_s^2(h'', \omega)\, d\omega$$

$$= B \int_0^1 \mathfrak{M}_s^2(h'', \omega)\, d\omega \int_\omega^1 (1-\rho)(\rho-\omega)\, d\rho;$$

and so, performing the inner integration and then replacing ω by ρ,

$$\lim_{\rho\to 1} \mathfrak{M}_s^2(h) \leqslant B \int_0^1 (1-\rho)^3\mathfrak{M}_s^2(h'', \rho)\, d\rho. \tag{20.2}$$

If ϑ is the operator $\qquad \vartheta = z\dfrac{d}{dz},$

* Here we use $c_1 = 0$ (or $b_1 = 0$).

then

$$\vartheta f = \sum nc_n z^n, \qquad \vartheta g = \sum nb_n z^n, \qquad \vartheta^2 h = \sum n^2 b_n c_n z^n,$$

and
$$\vartheta^2 h(\rho e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \vartheta f(\rho^{\frac{1}{2}} e^{i\phi})\, \vartheta g(\rho^{\frac{1}{2}} e^{i\theta - i\phi})\, d\phi.$$

Hence, by Theorem A,

$$\mathfrak{M}_s(\vartheta^2 h, \rho) \leqslant \mathfrak{M}_r(\vartheta f, \rho^{\frac{1}{2}})\mathfrak{M}_\sigma(\vartheta g, \rho^{\frac{1}{2}}). \tag{20.3}$$

Also
$$\vartheta h(\rho e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho^{\frac{1}{2}} e^{i\phi})\, \vartheta g(\rho^{\frac{1}{2}} e^{i\theta - i\phi})\, d\phi$$

and
$$\mathfrak{M}_s(\vartheta h, \rho) \leqslant \mathfrak{M}_r(f, \rho^{\frac{1}{2}})\mathfrak{M}_\sigma(\vartheta g, \rho^{\frac{1}{2}}). \tag{20.4}$$

Now
$$\vartheta^2 h = z^2 h'' + zh, \qquad h'' = z^{-2}\vartheta^2 h - z^{-2}\vartheta h.$$

Hence
$$\mathfrak{M}_s^2(h'') \leqslant B\{\rho^{-4}\mathfrak{M}_s^2(\vartheta^2 h) + \rho^{-4}\mathfrak{M}_s^2(\vartheta h)\}$$

and, by (20.2),

$$\mathfrak{M}_s^2(h) \leqslant \lim_{\rho \to 1} \mathfrak{M}_s^2(h) \leqslant B(J_1 + J_2), \tag{20.5}$$

where
$$J_1 = \int_0^1 (1-\rho)^3 \rho^{-4}\mathfrak{M}_s^2(\vartheta^2 h)\, d\rho, \tag{20.6}$$

and
$$J_2 = \int_0^1 (1-\rho)^3 \rho^{-4}\mathfrak{M}_s^2(\vartheta h)\, d\rho. \tag{20.7}$$

The theorem will therefore be proved if we can show that $J_1 \leqslant B$ and $J_2 \leqslant B$.

21. Since $\vartheta^2 h = O(\rho^2)$ for small ρ, J_1 is convergent at the origin, and therefore

$$J_1 \leqslant B\int_0^1 (1-\rho)^3\mathfrak{M}_s^2(\vartheta^2 h)\, d\rho \leqslant B\int_0^1 (1-\rho)^3 \mathfrak{M}_r^2(\vartheta f, \rho^{\frac{1}{2}})\mathfrak{M}_\sigma^2(\vartheta g, \rho^{\frac{1}{2}})\, d\rho$$

$$= B\int_0^1 \rho(1-\rho^2)^3 \mathfrak{M}_r^2(\vartheta f, \rho)\mathfrak{M}_\sigma^2(\vartheta g, \rho)\, d\rho$$

$$\leqslant B\int_0^1 (1-\rho)\mathfrak{M}_r^2(f', \rho)\, d\rho, \tag{21.1}$$

by Lemma γ, (20.3) and (19.3). But

$$\left\{\int_0^1 (1-\rho)\mathfrak{M}_r^2(f')\,d\rho\right\}^{\frac{1}{2}r} = \left\{\int_0^1 (1-\rho)\,d\rho\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'|^r\,d\theta\right)^{2/r}\right\}^{\frac{1}{2}r}$$

$$\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\left\{\int_0^1 (1-\rho)|f'|^2\,d\theta\right\}^{\frac{1}{2}r} \leqslant B\mathfrak{M}_r(f) \leqslant B, \quad (21.2)$$

by Minkowski's inequality and Theorem E. From (21.1) and (21.2) it follows that $J_1 \leqslant B$.

There remains J_2.* Here we have

$$J_2 \leqslant B\int_0^1 (1-\rho)^3\mathfrak{M}_s^2(\vartheta h)\,d\rho \leqslant B\int_0^1 (1-\rho)^3\mathfrak{M}_r^2(f,\rho^{\frac{1}{2}})\mathfrak{M}_\sigma^2(\vartheta g,\rho^{\frac{1}{2}})\,d\rho$$

$$\leqslant B\int_0^1 (1-\rho)^3(1-\rho^{\frac{1}{2}})^{-2}\,d\rho = B,$$

by Lemma γ, (20.4), and (19.3); and this completes the proof.

The limitations of the theorems

22. The rest of the paper is negative: we construct examples to show that the restrictions which we have imposed on k, r, and s are essential for the truth of our theorems. We have supposed

(i) that $k \geqslant 0$,
(ii) that $2 \leqslant r \leqslant s$ when $k = 0$.

We shall now show

(1) that all our conclusions would be false were $k < 0$,
(2) that the conclusion of Theorem 14 (in which $k = 0$) would be false were $r \leqslant s < 2$ or $2 < r \leqslant s$.

23. To prove (1), we take

$$f(z) = \sum n^{-2}z^{2^n}, \qquad g(z) = \sum 2^{-kn}z^{2^n}, \qquad h(z) = \sum n^{-2}2^{-kn}z^{2^n} \quad (23.1)$$

(the summations being from 0 to ∞). Then $f(z)$ is continuous in $\rho \leqslant 1$, and

$$\mathfrak{M}_r(f) \leqslant A \quad (23.2)$$

for every r.

Next, we write $a = 1-k$ and $\rho = e^{-\delta}$. Then

$$zg'(z) = \sum 2^{an}z^{2^n}.$$

* Which is really trivial.

Now $g'(z)$ is bounded in $\rho \leqslant e^{-a}$, so that

$$|g'(z)| \leqslant B \leqslant B(1-\rho)^{-a} = B(1-\rho)^{k-1} \tag{23.3}$$

if $\rho \leqslant e^{-a}$, i.e. if $\delta \geqslant a$. On the other hand, if $\delta < a$, the function

$$\phi(x) = 2^{ax}e^{-2^x\delta},$$

where $x \geqslant 0$, has a single maximum, equal to

$$\left(\frac{a}{\delta}\right)^a e^{-a},$$

when $\qquad x = \xi, \qquad 2^\xi = a/\delta.$

Also $\qquad |zg'(z)| \leqslant \sum 2^{an}e^{-2^n\delta} = \sum \phi(n),$

and

$$\sum \phi(n) \leqslant 2\phi(\xi)+\int\limits_0^\infty \phi(x)\,dx = 2\left(\frac{a}{\delta}\right)^a e^{-a}+\int\limits_0^\infty 2^{ax}e^{-2^x\delta}\,dx$$

$$= 2\left(\frac{a}{\delta}\right)^a e^{-a}+A\int\limits_1^\infty y^{a-1}e^{-\delta y}\,dy < \{2a^ae^{-a}+A\Gamma(a)\}\delta^{-a}$$

$$= B\delta^{-a} \leqslant B(1-\rho)^{-a} = B(1-\rho)^{k-1};$$

so that

$$|g'(z)| < Be^a(1-\rho)^{k-1} = B(1-\rho)^{k-1} \tag{23.4}$$

if $\delta < a$.

It follows from (23.3) and (23.4) that

$$\mathfrak{M}_\sigma(g') \leqslant \mathfrak{M}(g') \leqslant B(1-\rho)^{k-1}$$

for $0 \leqslant \rho < 1$ and any σ. Thus f and g satisfy the requirements of Theorem 13. But the coefficients of h do not tend to 0, and so $\mathfrak{M}_s(h)$ cannot be bounded for any $s \geqslant 1$.

24. It is more difficult to prove assertion (2) of § 22, and we do not attempt so comprehensive a refutation of any extension of Theorem 14. We examine a particular, and representative case: even there, we cannot produce a definite 'Gegenbeispiel', though we can show that such exist.

The two cases, corresponding to cases (*a*) and (*c*) of Theorem 13, would stand or fall together (by the 'conjugacy' argument used in § 17). We may therefore suppose that $2 < r \leqslant s$, and we select the case

$$\sigma = 1, \qquad \sigma' = \infty, \qquad r = 4, \qquad s = 4.* \tag{24.1}$$

* $r = s$ if $k = 0$ and $\sigma = 1$. In Theorem 13, s is necessarily greater than r; and this is true in Theorem 14 also except when $r = s = 2$. But here the case $r = s$ is representative.

We shall prove that

$$\mathfrak{M}_4(f) \leqslant 1, \qquad \mathfrak{M}_1(g') \leqslant \frac{1}{1-\rho} \tag{24.2}$$

do not imply

$$\mathfrak{M}_4(h) \leqslant A. \tag{24.3}$$

It will plainly be enough to show that, given any Ω, we can find two A's and an f and g such that

$$\mathfrak{M}_4(f) \leqslant A, \qquad \mathfrak{M}_1(g') \leqslant A, \tag{24.4}$$

and

$$\mathfrak{M}_4(h) > \Omega. \tag{24.5}$$

We write

$$D_n(z) = D_n(\rho e^{i\theta}) = \sum_{N_n}^{N_n+n-1} z^p = z^{N_n}\frac{1-z^n}{1-z}, \tag{24.6}$$

where

$$N = 2^{2^n}. \tag{24.7}$$

There is plainly no overlapping of the powers of z in different D_n. We then define f and g by

$$f(z) = \sum_2^{\nu} n^{-\beta} D_n(ze^{i\alpha_n}), \tag{24.8}$$

$$g(z) = \sum_2^{\infty} \frac{1}{\log n} D_n(ze^{-i\alpha_n}); \tag{24.9}$$

so that

$$h(z) = \sum_2^{\nu} \frac{n^{-\beta}}{\log n} D_n(z). \tag{24.10}$$

Here

$$\beta = \tfrac{9}{8}, \tag{24.11}$$

the α_n are arbitrary, and ν is large; f and h are polynomials, and h is independent of the choice of the α_n.

25.* We prove first that

$$\mathfrak{M}_1(g') \leqslant \frac{A}{1-\rho}. \tag{25.1}$$

In proving this we may suppose that $\rho \geqslant \frac{15}{16}$.

Let

$$\rho_n = 1-\frac{1}{N_n}, \tag{25.2}$$

and suppose that

$$\rho_{m-1} \leqslant \rho < \rho_m, \tag{25.3}$$

* The only part of the argument of this section which is not really trivial is the proof of (25.12). It is there only that we use the $\log n$ in (24.10), or the distinction between $\mathfrak{M}_1$ and $\mathfrak{M}$.

where $m > 2$.* Then

$$g'(z) = \sum_{2}^{\infty} \frac{1}{\log n} \frac{d}{dz} D_n(ze^{-i\alpha_n}), \tag{25.4}$$

$$|g'(z)| \leqslant \sum_{2}^{\infty} \frac{1}{\log n} |D_n'(ze^{-i\alpha_n})| = \gamma_1(z)+\gamma_2(z)+\gamma_3(z), \tag{25.5}$$

where γ_1 contains the terms for which $n < m-1$ (if any), γ_2 those for which $n = m-1$ and $n = m$, and γ_3 those for which $n > m$.

It is plain that

$$|D_n'(ze^{-i\alpha_n})| \leqslant \sum_{p=0}^{n-1} (N_n+p)\rho^{N_n+p-1} \leqslant 2\rho^{N_n} \sum_{p=0}^{n-1} (N_n+p) \leqslant 4nN_n \rho^{N_n} < 4nN_n. \tag{25.6}$$

Hence, first

$$\gamma_1(z) \leqslant 4 \sum_{2}^{m-2} nN_n \leqslant 4m(m-2)2^{2^{m-2}} < 4.2^{2^{m-1}} = 4N_{m-1} = \frac{4}{1-\rho_{m-1}} \leqslant \frac{4}{1-\rho}.\dagger \tag{25.7}$$

Secondly

$$\gamma_3(z) \leqslant 4 \sum_{m+1}^{\infty} nN_n \rho^{N_n} < 4 \sum_{m+1}^{\infty} nN_n\left(1-\frac{1}{N_m}\right)^{N_n} < 4 \sum_{m+1}^{\infty} nN_n e^{-N_n/N_m}$$

$$= 4 \sum_{m+1}^{\infty} n2^{2^n} \exp(-2^{2^n-2^m}) < 4 \sum_{m+1}^{\infty} n2^{2^n} \exp(-2^{2^{n-1}}) < 4 \sum_{1}^{\infty} n2^{-2^n}\ddagger$$

$$= 4(2^{-2}+2.2^{-4}+3.2^{-8}+\ldots) < 4 < \frac{4}{1-\rho}. \tag{25.8}$$

From (25.7) and (25.8) it follows that

$$\mathfrak{M}_1(\gamma_1)+\mathfrak{M}_1(\gamma_3) \leqslant \mathfrak{M}(\gamma_1)+\mathfrak{M}(\gamma_3) < \frac{8}{1-\rho}. \tag{25.9}$$

As regards γ_2, we have

$$D_n'(ze^{-i\alpha_n}) = D_n'(\zeta) = N_n \zeta^{N_n-1}\frac{1-\zeta^n}{1-\zeta} + \zeta^{N_n}\frac{d}{d\zeta}\left(\frac{1-\zeta^n}{1-\zeta}\right) = P_n+Q_n,$$

* So that $\rho \geqslant \rho_2 = 1-\frac{1}{N_2} = \frac{15}{16}$.

† $m(m-2) < 2^{2^{m-2}}$ for $m \geqslant 2$.

‡ $e > 2$, $2^{2^n} > 2.2^{2^{n-1}}$, and $2^{2^{n-1}} > 2^{n+1}$, for $n > 3$.

say; and

$$|\gamma_2(z)| \leqslant \sum_{m-1}^{m} \frac{|P_n|}{\log n} + \sum_{m-1}^{m} \frac{|Q_n|}{\log n} = \lambda(z)+\mu(z), \tag{25.10}$$

say. Now

$$|Q_n| < 1+2+\ldots+(n-1) < n^2 \leqslant m^2 < N_{m-1} = \frac{1}{1-\rho_{m-1}} \leqslant \frac{1}{1-\rho},$$

and so
$$\mathfrak{M}_1(\mu) \leqslant \mathfrak{M}(\mu) < \frac{4}{1-\rho} \tag{25.11}$$

(since $\log 2 > \frac{1}{2}$). Also

$$\mathfrak{M}_1(P_n) = N_n \rho^{N_n-1} \int_{-\pi}^{\pi} \left|\frac{1-\zeta^n}{1-\zeta}\right| d\theta \leqslant AN_n \rho^{N_n} \log n,$$

by Lemma δ. The maximum of $x\rho^x$, for fixed $\rho < 1$ and positive x, is

$$\frac{A}{\log(1/\rho)} < \frac{A}{1-\rho};$$

and so (whether n be $m-1$ or m)

$$\frac{\mathfrak{M}_1(P_n)}{\log n} < \frac{1}{\log n} \,.\, A \log n \,.\, \frac{A}{1-\rho} = \frac{A}{1-\rho}.$$

Hence
$$\mathfrak{M}_1(\lambda) < \frac{A}{1-\rho}. \tag{25.12}$$

Finally, (25.1) follows from (25.5), (25.9), (25.10), (25.11), and (25.12).

26. Next, we prove that

$$\mathfrak{M}_4(h) > \Omega \tag{26.1}$$

if $z = e^{i\theta}$ and ν is sufficiently large. We write

$$\phi_{m,n}(z) = \frac{D_m(z)}{m^\beta \log m} \frac{D_n(z)}{n^\beta \log n} = \sum_k c_{m,n,k} z^k:$$

$\phi_{m,n}$ is a polynomial with positive coefficients. Then

$$h^2(z) = \sum_{m,n} \phi_{m,n}(z) = \sum_k c_k z^k,$$

where
$$c_k = \sum_{m,n} c_{m,n,k};$$

and
$$\mathfrak{M}_4^4(h) = \sum_k c_k^2.$$

Since

$$\sum_k c_k^2 = \sum_k \Big(\sum_{m,n} c_{m,n,k}\Big)^2 > \sum_{m,n,k} c_{m,n,k}^2 = \sum_{m,n} \mathfrak{M}_2^2(\phi_{m,n}),$$

we have

$$\mathfrak{M}_4^4(h) \geqslant \sum_{m,n} \left(\frac{1}{m^\beta \log m}\right)^2 \left(\frac{1}{n^\beta \log n}\right)^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_m(z) D_n(z)|^2 \, d\theta. \quad (26.2)$$

It will be sufficient to consider the sum over the range

$$1 < \tfrac{1}{2}n \leqslant m \leqslant n.$$

Since

$$D_m(z)D_n(z) = z^{N_m+N_n}(1+z+\ldots+z^{m-1})(1+z+\ldots+z^{n-1}),$$

and the product of the last two factors contains the terms

$$1+2z+3z^2+\ldots+mz^{m-1},$$

we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_m(z)D_n(z)|^2 \, d\theta \geqslant 1^2+2^2+\ldots+m^2 > \tfrac{1}{3}m^3.$$

Substituting in (26.2), we see that

$$\mathfrak{M}_4^4(h) \geqslant \frac{1}{3} \sum_{n=2}^{\nu} \frac{1}{n^{2\beta}(\log n)^2} \sum_{m=\frac{1}{2}n+1}^{n} \frac{m^3}{m^{2\beta}(\log m)^2}$$

$$\geqslant A \sum_{2}^{\nu} \frac{n^{4-4\beta}}{(\log n)^4} = A \sum_{2}^{\nu} \frac{n^{-1}}{(\log n)^4} > \Omega^4,$$

if ν is sufficiently large.

27. So far our results have been true however the α_n are chosen. We shall now prove that

$$\mathfrak{M}_4\{f(e^{i\theta})\} \leqslant A \quad (27.1)$$

for *some* choice of the α_n. We denote by $\mathfrak{A}_\alpha$ the operator

$$\left(\frac{1}{2\pi}\right)^{\nu-1} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \ldots d\alpha_2 \, d\alpha_3 \ldots d\alpha_\nu$$

(an average over all different values of the α_n). It is plain that, if

$$\mathfrak{A}_\alpha[\mathfrak{M}_4^4\{f(e^{i\theta})\}] \leqslant A^4, \quad (27.2)$$

then (27.1) must be true for some set of α_n.

We shall now prove (27.2). We shall suppose throughout that z lies on the unit circle (as we may, since f is a polynomial); and f will mean $f(e^{i\theta})$.

We write $\quad D_n(ze^{i\alpha_n}) = D_n\{e^{i(\theta+\alpha_n)}\} = \Delta_n(\theta+\alpha_n), \quad (27.3)$

and $\bar{\Delta}_n$ for the conjugate of Δ_n. Then

$$\mathfrak{M}_4^4(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^4\,d\theta$$

$$= \sum_{m,n,p,q} (mnpq)^{-\beta}\frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta_m(\theta+\alpha_m)\Delta_n(\theta+\alpha_n)\bar{\Delta}_p(\theta+\alpha_p)\bar{\Delta}_q(\theta+\alpha_q)\,d\theta. \tag{27.4}$$

Since

$$\Delta_m(\theta+\alpha_m) = e^{iN_m(\theta+\alpha_m)}\{1+e^{i(\theta+\alpha_m)}+\ldots+e^{i(m-1)(\theta+\alpha_m)}\},$$

$$\bar{\Delta}_p(\theta+\alpha_p) = e^{-iN_p(\theta+\alpha_p)}\{1+e^{-i(\theta+\alpha_p)}+\ldots+e^{-i(p-1)(\theta+\alpha_p)}\},$$

integration over the α destroys all the terms in (27.4) except those for which either $p = m$ and $q = n$, or else $p = n$ and $q = m$: consequently

$$\mathfrak{A}_\alpha\{\mathfrak{M}_4^4(f)\}$$
$$\leqslant 2\sum_{m,n}(mn)^{-2\beta}\mathfrak{A}_\alpha\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_m(\theta+\alpha_m)|^2|\Delta_n(\theta+\alpha_n)|^2\,d\theta\right\} = 2(S_1+S_2), \tag{27.5}$$

where S_1 contains the terms for which $m = n$ and S_2 the remainder.

First, $$S_1 = \sum n^{-4\beta}\mathfrak{A}_\alpha\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_n(\theta+\alpha_n)^4|\,d\theta\right\}.$$

But

$$\mathfrak{A}_\alpha\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_n(\theta+\alpha_n)|^4\,d\theta\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha_n\,\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_n(\theta+\alpha_n)|^4\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_n(\psi)|^4\,d\psi$$

(since the inner integral has this value for every value of α_n). Also

$$|\Delta_n(\psi)| \leqslant n;$$

and so $$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_n(\psi)|^4\,d\psi \leqslant \frac{n^2}{2\pi}\int_{-\pi}^{\pi} |\Delta_n(\psi)|^2\,d\psi = n^3.$$

Hence $$\mathfrak{A}_\alpha\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_n(\theta+\alpha_n)|^4\,d\theta\right\} \leqslant n^3,$$

and $$S_1 \leqslant \sum n^{3-4\beta} = \sum n^{-\frac{5}{4}} = A. \tag{27.6}$$

On the other hand

$$S_2 = \sum_{m \neq n} (mn)^{-2\beta} \mathfrak{A}_\alpha \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_m(\theta+\alpha_m)|^2 |\Delta_n(\theta+\alpha_n)|^2 \, d\theta \right\}.$$

But

$$\mathfrak{A}_\alpha \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_m(\theta+\alpha_m)|^2 |\Delta_n(\theta+\alpha_n)|^2 \, d\theta \right\}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\alpha_m \frac{1}{2\pi} \int_{-\pi}^{\pi} d\alpha_n \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_m(\theta+\alpha_m)|^2 |\Delta_n(\theta+\alpha_n)|^2 \, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_m(\theta+\alpha_m)|^2 \, d\alpha_m \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_n(\theta+\alpha_n)|^2 \, d\alpha_n \right\}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_m(\psi)|^2 \, d\psi \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_n(\psi)|^2 \, d\psi = mn.$$

Hence $$S_2 \leqslant \sum_{m \neq n} (mn)^{1-2\beta} \leqslant (\sum m^{-\frac{4}{3}})^2 = A. \tag{27.7}$$

Finally, it follows from (27.5), (27.6), and (27.7) that

$$\mathfrak{A}_\alpha\{\mathfrak{M}_4^4(f)\} \leqslant A \leqslant A^4,$$

which is (27.2).

We have thus proved what we stated in § 24, though we have not produced a definite f and g. It will be observed that we have made no use of the Littlewood-Paley theorems in this part of our work.

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CORRECTIONS

p. 222, *line* 5 *of statement of Theorem A.* For $\alpha > 1, \beta > 1$ read $\alpha \geqslant 1, \beta \geqslant 1$. (The case $\alpha = 1$ is required in the paper.)

p. 228, *line* 17. For $D(\phi, \rho)$ read $D(\theta, \rho)$.

p. 233, *lines* 4 *and* 5 *from below.* For Theorem 3 read Theorem 4.

p. 234, *line* 9. For $\frac{d}{d\rho}\{\mathfrak{M}_s^r(f)\}^{r/s}$ read $\frac{d}{d\rho}\{\mathfrak{M}_s^s(f)\}^{r/s}$.

p. 235, *line* 3 *from end of* § 9. The ρ^{-b} in $B\int_0^1 (1-\rho)^{r-b}\rho^{-b}\mathfrak{M}_s^r(zf')\,d\rho$ should be $\rho^{-r/s}$.

p. 236, *equation* (11.2). For BC read BC^l.

p. 239, *equation* (12.8). For $\int \dots db$ read $\int \dots dt$.

p. 239, *footnote* §. For § 15 read § 14.

p. 240, *equation* (14.1). Add the condition $r > 1$.

p. 254, *line* 13. Transpose $^4|$.

COMMENTS

p. 230, *line* 14. See the comment on 1932, 4, p. 409.

p. 236, *Theorem* 11. The proof given here is valid for $0 < r < s$. The result holds also for $0 < r < s = \infty$ (see 1932, 4, Theorem 31).

p. 240. Generalizations of Theorems 13 and 14 have been given by T. M. Flett, *Pac. J. Math.* 25 (1968), 463–94.

ARRANGEMENT OF THE VOLUMES

VOLUME VI

VI. Theory of series

VOLUME VII

VII. 1 Integral equations and integral transforms

VII. 2 Miscellaneous papers

VII. 3 Questions from the *Educational Times*

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VII. 5 List of other writings

LIST OF PAPERS BY G. H. HARDY

Abbreviations

N.I.C. Notes on some points in the integral calculus
D.A. Some problems of Diophantine approximation
P.N. Some problems of 'Partitio Numerorum'
N.S. Notes on the theory of series

1899

1. Question 13848, *Educational Times*, 70, 43.
2. Question 13917, *Educational Times*, 70, 78–79.
3. Question 14124, *Educational Times*, 71, 100–101.
4. Question 14005, *Educational Times*, 71, 111–112.

} VII. 3

1900

1. On a class of definite integrals containing hyperbolic functions, *Messenger of Mathematics*, 29, 25–42. V
2. Question 14243, *Educational Times*, 72, 80–81.
3. Question 14271, *Educational Times*, 73, 36–37.
4. Question 14179, *Educational Times*, 73, 53–54.
5. Question 14317, *Educational Times*, 73, 61–63.

} VII. 3

1901

1. On differentiation and integration under the integral sign, *Quarterly Journal of Mathematics*, 32, 66–140. (Corrected in 1915, 2.) V
2. General theorems in contour integration, with some applications, *Quarterly Journal of Mathematics*, 32, 369–384. V
3. N.I.C. I: On the formula for integration by parts, *Messenger of Mathematics*, 30, 185–187. V
4. N.I.C. II: Two general convergence theorems, *Messenger of Mathematics*, 30, 187–190. V
5. Question 14496, *Educational Times*, 74, 37–38.
6. Question 14447, *Educational Times*, 74, 98–100.
7. Question 14467, *Educational Times*, 74, 111–112.
8. Question 14028, *Educational Times*, 74, 122–123.
9. Question 14369, *Educational Times*, 75, 135–136.

} VII. 3

1902

1. The elementary theory of Cauchy's principal values, *Proceedings of the London Mathematical Society*, (1) 34, 16–40. V
2. The theory of Cauchy's principal values (Second Paper: the use of principal values in some of the double limit problems of the integral calculus), *Proceedings of the London Mathematical Society*, (1) 34, 55–91. V

1902 (*cont.*)

3. On the Frullanian integral

$$\int_0^\infty \frac{\phi(ax^m)-\psi(bx^n)}{x}(\log x)^p\,dx,$$

Quarterly Journal of Mathematics, 33, 113–144. V

4. N.I.C. III: On the logarithmic criteria for the absolute convergence of an integral whose upper limit is ∞, *Messenger of Mathematics*, 31, 1–6. V

5. N.I.C. IV: On the integral $\int^\infty \sin x\psi(x)\,dx$, *Messenger of Mathematics*, 31, 6–8. V

6. A new proof of Kummer's series for $\log\Gamma(a)$, *Messenger of Mathematics*, 31, 31–33. IV. 1 (*d*)

7. N.I.C. V: On absolutely convergent integrals of functions which are infinitely often infinite, *Messenger of Mathematics*, 31, 73–76. V

8. N.I.C. VI: Absolute convergence of infinite multiple integrals, *Messenger of Mathematics*, 31, 125–128. V

9. N.I.C. VII: On differentiation under the integral sign, *Messenger of Mathematics*, 31, 132–134. V

10. N.I.C. VIII: Absolutely convergent integrals of irregular types, *Messenger of Mathematics*, 31, 177–183. V

11. On the zeroes of the integral function

$$x-\sin x=\sum_1^\infty(-)^{n-1}\frac{x^{2n+1}}{2n+1!},$$

Messenger of Mathematics, 31, 161–165. IV. 1 (*a*)

12. Questions 1423, 2316, 3941, 4794, *Educational Times*, (2) 1, 25.
13. Question 14851, *Educational Times*, (2) 1, 58–59.
14. Question 14055, *Educational Times*, (2) 2, 41–42.
} VII. 3

1903

1. The theory of Cauchy's principal values (Third Paper: differentiation and integration of principal values), *Proceedings of the London Mathematical Society*, (1) 35, 81–107. V

2. On the continuity and discontinuity of definite integrals which contain a continuous parameter, *Quarterly Journal of Mathematics*, 34, 28–53. V

3. Note on the limiting values of the elliptic modular functions, *Quarterly Journal of Mathematics*, 34, 76–86. IV. 1 (*d*)

4. N.I.C. IX: On the integral $\int^\infty \{A-\phi(\sin^2 x)\}\psi(x)\,dx$, *Messenger of Mathematics*, 32, 1–3. V

5. On the zeroes of certain integral functions, *Messenger of Mathematics*, 32, 36–45. IV. 1 (*a*)

6. On the integral $\displaystyle\int_{-\infty}^{\infty}\frac{\log(ax^2+2bx+c)^2}{\alpha x^2+2\beta x+\gamma}\,dx$, *Messenger of Mathematics*, 32, 45–50. V

7. N.I.C. X: On conditionally convergent infinite multiple integrals, *Messenger of Mathematics*, 32, 92–97. V

8. N.I.C. XI: Some conditionally convergent infinite double integrals, *Messenger of Mathematics*, 32, 159–165. V

9. N.I.C. XII: On the operation which is the inverse of double integration, *Messenger of Mathematics*, 32, 187–192. V

10. Question 14988, *Educational Times*, (2) 3, 94–95.
11. Question 14989, *Educational Times*, (2) 4, 69–70.
12. Question 15019, *Educational Times*, (2) 4, 75.
13. Question 15265, *Educational Times*, (2) 4, 109–110.
} VII. 3

COMPLETE LIST OF HARDY'S MATHEMATICAL PAPERS

1904

1. On the convergence of certain multiple series, *Proceedings of the London Mathematical Society*, (2) 1, 124–128. VI
2. A general theorem concerning absolutely convergent series, *Proceedings of the London Mathematical Society*, (2) 1, 285–90. VII. 2
3. On differentiation and integration of divergent series, *Transactions of the Cambridge Philosophical Society*, 19, 297–321. VI
4. Researches in the theory of divergent series and divergent integrals, *Quarterly Journal of Mathematics*, 35, 22–66. VI
5. A theorem concerning the infinite cardinal numbers, *Quarterly Journal of Mathematics*, 35, 87–94. VII. 2
6. Note on the function $\int_x^\infty e^{\frac{1}{2}(x^2-t^2)}dt$, *Quarterly Journal of Mathematics*, 35, 193–207. VII. 1
7. The asymptotic solution of certain transcendental equations, *Quarterly Journal of Mathematics*, 35, 261–282. IV. 1 (*a*)
8. N.I.C. XIII: On differentiation under the integral sign (continued), *Messenger of Mathematics*, 33, 62–67. V
9. The cardinal number of a closed set of points, *Messenger of Mathematics*, 33, 67–69. VII. 2
10. N.I.C. XIV: Integrals whose discontinuities are everywhere dense, *Messenger of Mathematics*, 33, 80–85. V
11. Note on divergent Fourier series, *Messenger of Mathematics*, 33, 137–144. III. 1 (*b*)
12. On the zeroes of two classes of Taylor series, *British Association Report*, 441–443. IV. 1 (*a*)
13. Question 15300, *Educational Times*, (2) 5, 61.
14. Additional note on Question 15282, *Educational Times*, (2) 5, 113–114.
15. Question 15361, *Educational Times*, (2) 5, 118.
16. Question 15125, *Educational Times*, (2) 6, 31.

(13–16) VII. 3

1905

1. On the roots of the equation $\frac{1}{\Gamma(x+1)} = c$, *Proceedings of the London Mathematical Society*, (2) 2, 1–7. IV. 1 (*a*)
2. (With T. J. I'A. Bromwich.) Some extensions to multiple series of Abel's theorem on the continuity of power series, *Proceedings of the London Mathematical Society*, (2) 2, 161–189. VI
3. Note in addition to a former paper on conditionally convergent multiple series. *Proceedings of the London Mathematical Society*, (2) 2, 190–191. VI
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7. On a class of analytic functions, *Proceedings of the London Mathematical Society*, (2) 3, 441–460. IV. 1 (*b*)

1905 (*cont.*)

8. On certain series of discontinuous functions connected with the modular functions, *Quarterly Journal of Mathematics*, 36, 93–123. IV. 1 (*d*)
9. Note on an integral function, *Messenger of Mathematics*, 34, 1–2. IV. 1 (*a*)
10. N.I.C. XV: On upper and lower integration, *Messenger of Mathematics*, 34, 3–6. V
11. N.I.C. XVI: A class of conditionally convergent infinite multiple integrals, *Messenger of Mathematics*, 34, 6–10. V
12. A generalization of Frullani's integral, *Messenger of Mathematics*, 34, 11–18, and note, p. 102. V
13. On the zeroes of a class of integral functions, *Messenger of Mathematics*. 34, 97–101. IV. 1 (*a*)
14. On certain conditionally convergent multiple series connected with the elliptic functions, *Messenger of Mathematics*, 34, 146–153. VI
15. Question 15686, *Educational Times*, (2) 8, 74. VII. 3
16. The expression of the double zeta function and double gamma function in terms of elliptic functions, *Transactions of the Cambridge Philosophical Society*, 20, 1–35. IV. 1 (*d*)

1906

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3. On the function $P_\rho(x)$, *Quarterly Journal of Mathematics*, 37, 146–172 (correction at end of 1906, 5). IV. 1 (*a*)
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6. N.I.C. XVII: On the integration of series, *Messenger of Mathematics*, 35, 126–130. V
7. A formula for the prime factors of any number, *Messenger of Mathematics*, 35, 145–146. II. 1
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9. Some notes on certain theorems in higher trigonometry, *Mathematical Gazette*, 3, 284–288. V

1907

1. The continuum and the second number class, *Proceedings of the London Mathematical Society*, (2) 4, 10–17. VII. 2
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3. On the singularities of functions defined by Taylor's series (Remarks in addition to a former paper), *Proceedings of the London Mathematical Society*, (2) 5, 197–205. IV. 1 (*b*)
4. The singular points of certain classes of functions of several variables, *Proceedings of the London Mathematical Society*, (2) 5, 342–360. IV. 1 (*b*)
5. On certain oscillating series, *Quarterly Journal of Mathematics*, 38, 269–288. VI
6. Some theorems concerning infinite series, *Mathematische Annalen*, 64, 77–94. VI
7. N.I.C. XIX: On Abel's lemma and the second theorem of the mean, *Messenger of Mathematics*, 36, 10–13. V
8. Higher trigonometry, *Mathematical Gazette*, 4, 13–14. VII. 2
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3. Further researches in the theory of divergent series and integrals, *Transactions of the Cambridge Philosophical Society*, 21, 1–48. VI
4. (With T. J. I'A. Bromwich.) The definition of an infinite integral as the limit of a finite or infinite series, *Quarterly Journal of Mathematics*, 39, 222–240. V
5. Some multiple integrals, *Quarterly Journal of Mathematics*, 39, 357–375. V
6. N.I.C. XX: On double Frullanian integrals, *Messenger of Mathematics*, 37, 96–103. V
7. N.I.C. XXI: On a conditionally convergent multiple integral, *Messenger of Mathematics*, 37, 127–130. V
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9. Question 16257, *Educational Times*, (2) 13, 79–80. VII. 3
10. Mendelian proportions in a mixed population, *Science* (American Association for the Advancement of Science), new series 28, 49–50. VII. 2

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4. N.I.C. XXIII: On certain oscillating cases of Dirichlet's integral, *Messenger of Mathematics*, 38, 1–8. V
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2. The ordinal relations of the terms of a convergent sequence, *Proceedings of the London Mathematical Society*, (2) 8, 295–300. VII. 2
3. Theorems relating to the summability and convergence of slowly oscillating series, *Proceedings of the London Mathematical Society*, (2) 8, 301–320. VI
4. The maximum modulus of an integral function, *Quarterly Journal of Mathematics*, 41, 1–9. IV. 2
5. On certain definite integrals considered by Airy and by Stokes, *Quarterly Journal of Mathematics*, 41, 226–240. IV. 1 (*d*)
6. N.I.C. XXV: Absolutely convergent integrals of irregular types (cont.), *Messenger of Mathematics*, 39, 28–32. V
7. The zeroes of the integral function $\sum \frac{x^{n^2}}{n^2!}$, and of some similar functions, *Messenger of Mathematics*, 39, 88–96. IV. 1 (*a*)

1910 (*cont.*)

8. N.I.C. XXVI: On a case of term-by-term integration of an infinite series, *Messenger of Mathematics*, 39, 136–139. V

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1911

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5. N.I.C. XXVIII: A conditionally convergent double integral, *Messenger of Mathematics*, 40, 62–69. V

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8. N.I.C. XXXI: The uniform convergence of Borel's integral, *Messenger of Mathematics*, 40, 161–165. VI

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1912

1. Properties of logarithmico-exponential functions, *Proceedings of the London Mathematical Society*, (2) 10, 54–90. IV. 1 (*c*)

2. On the multiplication of Dirichlet's series, *Proceedings of the London Mathematical Society*, (2) 10, 396–405. VI

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7. Note on a theorem of Cesàro, *Messenger of Mathematics*, 41, 17–22. VI

8. N.I.C. XXXII: On double series and double integrals, *Messenger of Mathematics*, 41, 44–48. V

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1913

1. (With J. E. L.) The relations between Borel's and Cesàro's methods of summation, *Proceedings of the London Mathematical Society*, (2) 11, 1–16. VI

2. (With J. E. L.) Contributions to the arithmetic theory of series, *Proceedings of the London Mathematical Society*, (2) 11, 411–478. VI

3. An extension of a theorem on oscillating series, *Proceedings of the London Mathematical Society*, (2) 12, 174–180. VI

1913 (*cont.*)

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6. A theorem concerning Taylor's series, *Quarterly Journal of Mathematics*, 44, 147–60. IV. 2
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8. N.I.C. XXXIV: Absolutely convergent integrals of irregular types (cont.), *Messenger of Mathematics*, 42, 13–18. V
9. N.I.C. XXXV: On an integral equation, *Messenger of Mathematics*, 42, 89–93. VII. 1
10. (With J. E. L.) Tauberian theorems concerning series of positive terms, *Messenger of Mathematics*, 42, 191–192. VI
11. (With J. E. L.) Sur la série de Fourier d'une fonction à carré sommable, *Comptes Rendus*, 156, 1307–1309. III. 1 (*b*)

1914

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2. (With A. E. Ingham and G. Pólya) Notes on moduli and mean values, *Proceedings of the London Mathematical Society*, (2) 27, 401–409. IV. 2
3. (With J. E. L.) N.S. VII: On Young's convergence criterion for Fourier series, *Proceedings of the London Mathematical Society*, (2) 28, 301–311. III. 1 (*b*)
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7. N.I.C. LXVI: The arithmetic mean of a Fourier constant, *Messenger of Mathematics*, 58, 50–52. III. 1 (*e*)
8. N.I.C. LXVII: On the repeated integral which occurs in the theory of conjugate functions, *Messenger of Mathematics*, 58, 53–58. VII. 1
9. N.I.C. LXVIII: The limit of an integral mean value, *Messenger of Mathematics*, 58, 115–120. II. 3
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3. (With J. E. L.) D.A.: A series of cosecants, *Bulletin of the Calcutta Mathematical Society*, 20, 251–266. I. 1
4. (With J. E. L.) N.S. XI: On Tauberian theorems, *Proceedings of the London Mathematical Society*, (2) 30, 23–37. VI
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6. (With E. C. Titchmarsh) A note on Parseval's theorem for Fourier transforms, *Journal of the London Mathematical Society*, 6, 44–48. VII. 1
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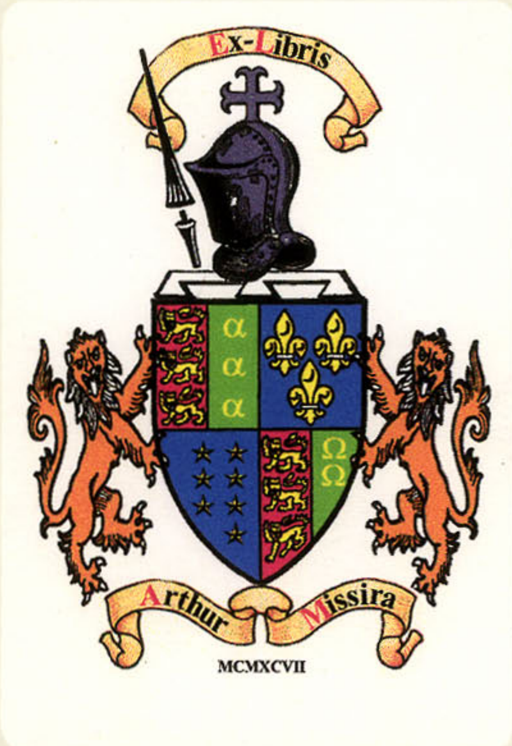
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